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# Modular irregularity strength of vertex amalgamation and comb product path with cycle related graphs

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## Abstract

Consider a graph $G = (V(G), E(G))$, where $V(G)$ is a nonempty set of vertices and $E(G)$ is a set of edges. Let $Z_n$ be the group of integers modulo $n$, and let $k$ be a positive integer. A modular irregular labeling of a graph $G$ of order $n$ is a $k$-edge labeling $\phi : E(G) \to \{1, 2, ..., k\}$, such that an induced weight function $wt_\phi : V(G) \to Z_n$ is bijective. The weight function is defined as follows: $wt_\phi(u) = \sum_{u \in N(v)} \phi(uv) \pmod n$ for all vertices $v$ in $V(G)$. The minimum value of $k$ is called the modular irregularity strength of $G$, denoted as $ms(G)$. Suppose $G$ and $H$ are two connected graphs, with $G$ has order $n$. Vertex amalgamation of graphs $G$ and $H$ is a graph obtained by identifying one vertex from each graph. Suppose that $o$ is a given vertex of $H$. The comb product of $G \triangleright H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$ and then attaching the vertex $o$ of the $i$-th copy of $H$ to the $i$-th vertex of $G$. In this paper, we discuss on the exact values of the modular irregularity strength for several graphs such as: vertex amalgamation of cycles; comb product path (or cycle) and cycle and comb product path (or cycle) and regular graphs.

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## 1. Introduction

Given graph $G$ denoted by $G = (V(G), E(G))$ where $V(G)$ is a non-empty set of vertices and $E(G)$ is a set of edges. The degree of a vertex $v$, indicated by $deg(v)$, is the number of edges incident to $v$. The maximum degree of all vertices $G$ is denoted by $\Delta(G)$. A graph is said to be regular if every vertex in the graph has the same degree, and is said to be irregular if all vertices have distinct degree. Two vertices $u, v \in V(G)$ are said to be adjacent if they are connected by an edge in $G$ [8].

Irregular labeling was first introduced by Chartrand *et al.* in [6]. An irregular labeling is $k$-edge labeling $\alpha : E(G) \to \{1, 2, \ldots, k\}$ such that the sum of the labels of the edges incident to a vertex is distinct for all vertices, henceforth called the vertex weight. The weight function of vertex $u$ is defined as $wt_\alpha(u) = \sum_{u \in N(v)} \alpha(uv)$, where $N(v)$ is the set of all vertices $u$ adjacent to $v$. An interesting discussion on this kind of labeling can be found in [1, 3, 4, 18]. The minimum number $k$ for which an irregular labeling exist in $G$ is called the irregularity strength, denoted by $s(G)$. A survey on irregularity strength can be found in [1] and a general survey on graph labeling can be found in [9]. Given a connected graph $G$ of order $n \geq 3$ containing $n_i$ vertices of degree $i$, the lower bound of the irregularity strength of the graph $G$ is expressed as follows.

**Lemma 1.1.** *[6] Let $G$ be a connected graph of order $n \geq 3$ and let $G$ have $n_i$ vertices of degree $i$. Then,*

$$s(G) \geq \max_{1 \leq i \leq \Delta(G)} \left\{ \frac{n_i + i - 1}{i} \right\}.$$

In 2020, Bača *et al.* introduced a variation of irregular labeling called modular irregular labeling [2]. Given a graph $G$ of order $n$ and the group of integers modulo $n$, $Z_n$, a modular irregular labeling of $G$ is a $k$-edge labeling $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that there is a bijective weight function $\Phi : V(G) \to Z_n$. The modular weight of a vertex $u \in V(G)$ is defined as $wt_\phi(u) = \sum_{u \in N(u)} \phi(uv) \pmod{n}$. The modular irregularity strength, denoted by $ms(G)$, is the minimum number $k$ for which the graph $G$ has a modular irregular labeling. Interesting results on this kind of labeling can be found in the following works:[2, 3, 4, 16, 18, 7, 12, 13, 15, 10, 14, 5, 17, 11]. Some known theorems on modular irregular labeling are as follows.

**Theorem 1.1.** *.[2] Let $G$ be a graph without components of order $\leq 2$. Then,*

$$s(G) \leq ms(G).$$

**Theorem 1.2.** *.[2] If $G$ is a graph of order $n$, $n \equiv 2 \pmod{4}$, then $G$ does not have modular irregular $k$-labeling, i.e., ms$(G) = \infty$.*

Suppose $G$ and $H$ are two connected graphs and $G$ has order $n$. The vertex amalgamation of graphs $G$ and $H$ is a graph obtained by identifying one vertex from each graph. Let $o$ be a given vertex of $H$. The comb product of $G \triangleright H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$ and then attaching the vertex $o$ of the $i$-th copy of $H$ to the $i$-th vertex of $G$. By the definition of the comb product, the vertex set is $V(G \triangleright H) = \{(a,u)|a \in V(G), u \in V(H)\}$. Two vertices $(a,u)(b,v)$ are adjacent if either of the following condition holds: $a = b$ and $uv \in E(H)$, or when $ab \in E(G)$ and $u = v = o$. In this paper, we discuss the exact values of the modular irregularity strength for the vertex amalgamation of cycles and the comb product of path and cycle related graphs.

## 2. Result

### 2.1. Vertex Amalgamation Product of Cycles

An example of vertex amalgamation product of cycles is called a Dutch Windmill graph. The Dutch windmill graph $D_3^m$, often called a friendship graph, is the graph obtained by taking $m$ copies of $C_3$ and identifying one common vertex. This definition can be extended to $D_4^m$, which is composed of $m$ copies of graph $C_4$ joined at a single common vertex[19].

**Theorem 2.1.** *Let $D_4^m$ be a dutch windmill graph with $m \geq 2$. Then,*

$$ms(D_4^m) = \begin{cases} \infty, & m \equiv 3 \pmod 4, \\ \left\lceil \dfrac{3m+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

*and*

$$s(D_4^m) = \left\lceil \frac{3m+1}{2} \right\rceil.$$

*Proof.* Let $D_4^m$ be a dutch windmill graph with $m \geq 2$. The graph $D_4^m$ has $3m$ vertices of degree two and one vertex with degree $2m$. Using Lemma 1.1, we have

$$s(G) \geq \max_{1 \leq i \leq \Delta(G)} \left\{ \frac{p_i - 1}{i} + 1 \right\} \geq \max_{1 \leq i \leq 2m} \left\{ \frac{3m-1}{2} + 1, \frac{1-1}{2m} + 1 \right\} \geq \left\lceil \frac{3m+1}{2} \right\rceil.$$

Thus, $ms(D_4^m) \geq s(D_4^m) \geq \left\lceil \frac{3m+1}{2} \right\rceil$.

Let the order of graph $D_4^m$ be $3m+1$ with $m \equiv 3 \pmod 4$. Then, we can write $m = 4k+3$, for $k = 0, 1, 2, \ldots$. Substituting this into the order, we have $3m+1 = 3(4k+3)+1 \equiv 2 \pmod 4$. According to Theorem 1.2, if the order is $n \equiv 1 \pmod 4$, the graph has the modular irregularity strength $ms(D_4^m) = \infty$.

We denote the center(common) vertex as $c$, the vertices in the $i$-th cycle as $x_i, y_i$ and $z_i$, usually following a clockwise direction, for $i = 1, 2, ..., m$.

Suppose that $m \not\equiv 3 \pmod 4$. Based on the pattern of the edge labels, we divide the proof into four cases concerning the value of $m$, which are $m \equiv 1, 3 \pmod 4$ and $m \equiv 1, 5 \pmod 8$.

**Case 1.** For $m \equiv 0 \pmod 4$. Define the edge label $\psi : E(D_4^m) \rightarrow \{1, 2, , \ldots, \lceil \frac{3m+1}{2} \rceil\}$ as follows.

$$\psi(x_i y_i) = \begin{cases} \dfrac{3i-1}{2}, & \text{for } i \text{ is odd and } 1 \leq i < \dfrac{m}{2}, \\ \dfrac{3i+1}{2}, & \text{for } i \text{ is odd and } \dfrac{m}{2} < i < m, \\ \dfrac{3i}{2}, & \text{for } i \text{ is even and } 1 < i \leq m. \end{cases}$$

$$\psi(y_i z_i) = \begin{cases} \dfrac{3i+1}{2}, & \text{for } i \text{ is odd and } 1 \leq i < m, \\ \dfrac{3i}{2}, & \text{for } i \text{ is even and } 1 < i < \dfrac{m}{2}, \\ \dfrac{3i+2}{2}, & \text{for } i \text{ is even and } \dfrac{m}{2} \leq i \leq m. \end{cases}$$

$$\psi(x_i c) = \begin{cases} \dfrac{3i-1}{2}, & \text{for } i \text{ is odd and } 1 \leq i < m, \\ \dfrac{3i-2}{2}, & \text{for } i \text{ is even and } 1 < i \leq \dfrac{m}{2}, \\ \dfrac{3i}{2}, & \text{for } i \text{ is even and } \dfrac{m}{2} < i \leq m. \end{cases}$$

$$\psi(z_i c) = \begin{cases} \dfrac{3i+1}{2}, & \text{for } i \text{ is odd and } 1 \leq i < \dfrac{m}{2}, \\ \dfrac{3i+3}{2}, & \text{for } i \text{ is odd and } \dfrac{m}{2} < i < m, \\ \dfrac{3i+2}{2}, & \text{for } i \text{ is even and } 1 < i \leq m. \end{cases}$$

From the definition of edge labels $\psi$, we find that the maximum value of the edge label is $\lceil \frac{3m+1}{2} \rceil$. Thus, $\psi$ is a $\lceil \frac{3m+1}{2} \rceil$-labeling.

The weight of every vertex in $D_4^m$ is as follows.

$$w_\psi(x_i) = \begin{cases} 3i - 1 \pmod{(3m+1)}, & \text{for } 1 \leq i \leq \dfrac{m}{2}, \\ 3i \pmod{(3m+1)}, & \text{for } \dfrac{m}{2} \leq m. \end{cases}$$

$$w_\psi(y_i) = \begin{cases} 3i \pmod{(3m+1)}, & \text{for } i < \dfrac{m}{2}, \\ (3i+1) \pmod{(3m+1)}, & \text{for } \dfrac{m}{2} \leq i \leq m. \end{cases}$$

$$w_\psi(z_i) = \begin{cases} (3i+1) \pmod{(3m+1)}, & \text{for } 1 \leq i < \dfrac{m}{2}, \\ (3i+2) \pmod{(3m+1)}, & \text{for } \dfrac{m}{2} \leq i \leq m. \end{cases}$$

$$w_\psi(c) = \frac{(3m+1)m}{2} + \frac{3m}{2} \equiv \frac{3m}{2} \pmod{3m+1}.$$

Thus, the vertex weight set is $\{w_\psi(v) : v \in V(D_4^m)\} = \{w_\psi(x_i) : i = 1, 2, \dots, m\} \cup \{w_\psi(y_i) : i = 1, 2, \dots, m\} \cup \{w_\psi(z_i) : i = 1, 2, \dots, m\} \cup \{w_\psi(c)\}\{0, 1, 2, \dots, 3m-1, 3m\} = Z_{3m+1}$. Then, we conclude that $\psi$ is an irregular modular $\lceil\frac{3m+1}{2}\rceil$-labeling for $D_4^m$ with $m \equiv 0 \pmod 4$.

**Case 2.** For $m \equiv 2 \pmod 4$. Define an edge label $\psi : E(D_4^m) \to \{1, 2, ..., k\}$ as follows.

$$\psi(x_i y_i) = \begin{cases} \frac{3i-1}{2}, & \text{for } i \text{ odd and } 1 \le i \le \frac{m}{2}, \\ \frac{3i+1}{2}, & \text{for } i \text{ odd and } \frac{m}{2} < i < m, \\ \frac{3i}{2}, & \text{for } i \text{ even and } 1 < i \le m. \end{cases}$$

$$\psi(y_i z_i) = \begin{cases} \frac{3i+1}{2}, & \text{for } i \text{ odd and } 1 \le i < m, i \ne \frac{m}{2}, \\ \frac{3i+3}{2}, & \text{for } i = \frac{m}{2}, \\ \frac{3i}{2}, & \text{for } i \text{ even and } 1 < i < \frac{m}{2}, \\ \frac{3i+2}{2}, & \text{for } i \text{ even and } \frac{m}{2} < i \le m. \end{cases}$$

$$\psi(x_i c) = \begin{cases} \frac{3i-1}{2}, & \text{for } i \text{ odd and } 1 \le i < m, \\ \frac{3i-2}{2}, & \text{for } i \text{ even and } 1 < i < \frac{m}{2}, \\ \frac{3i}{2}, & \text{for } i \text{ even and } \frac{m}{2} < i \le m. \end{cases}$$

$$\psi(z_i c) = \begin{cases} \frac{3i+1}{2}, & \text{for } i \text{ odd and } 1 \le i \le \frac{m}{2}, \\ \frac{3i+3}{2}, & \text{for } i \text{ odd and } \frac{m}{2} < i < m, \\ \frac{3i+2}{2}, & \text{for } i \text{ even and } 1 < i \le m. \end{cases}$$

From the definition of edge labels $\psi$, we find that the maximum value of the edge label is $\lceil\frac{3m+1}{2}\rceil$. Thus, the $\psi$ is a $\lceil\frac{3m+1}{2}\rceil$-labeling.

The weight of each vertex in $D_4^m$ is as follows.

$$w_\psi(x_i) = \begin{cases} (3i-1) \pmod{(3m+1)}, & \text{for } 1 \le i \le \frac{m}{2}, \\ 3i \pmod{(3m+1)}, & \text{for } \frac{m}{2} < i \le m. \end{cases}$$

$$w_\psi(y_i) = \begin{cases} 3i \pmod{(3m+1)}, & \text{for } i < \frac{m}{2}, \\ 3i+1 \pmod{(3m+1)}, & \text{for } i \ge \frac{m}{2}. \end{cases}$$

$$w_\psi(z_i) = \begin{cases} 3i+1 \pmod{(3m+1)}, & \text{for } 1 \le i < \dfrac{m}{2}, \\ 3i+2 \pmod{(3m+1)}, & \text{for } \dfrac{m}{2} \le i \le m. \end{cases}$$

$$w_\psi(c) = \frac{(3m+1)m}{2} + \frac{3m}{2} \equiv \frac{3m}{2} \pmod{3m+1}.$$

Thus, the vertex weight set is $\{w_\psi(v) : v \in V(D_4^m)\} = \{w_\psi(x_i) : i = 1, 2, \ldots, m\} \cup \{w_\psi(y_i) : i = 1, 2, \ldots, m\} \cup \{w_\psi(z_i) : i = 1, 2, \ldots, m\} \cup \{w_\psi(c) : i = 1, 2, \ldots, m\} = \{0, 1, 2, \ldots, 3m-1, 3m\} = Z_{3m+1}$.

Then, we conclude that $\psi$ is an irregular modular $\lceil \frac{3m+1}{2} \rceil$-labeling for $D_4^m$ with $m \equiv 2 \pmod{4}$.

**Case 3.** For $m \equiv 1 \pmod{8}$. Define the edge labeling as follows.

$$\psi(x_iy_i) = \begin{cases} \dfrac{i+1}{2}, & \text{for } i \text{ odd and } 1 \le i < m, \\ \dfrac{i+2}{2}, & \text{for } i \text{ even and } 1 < i < m, \\ \dfrac{3m-3}{4}, & i = m. \end{cases}$$

$$\psi(y_iz_i) = \begin{cases} \dfrac{i-1}{2} + m, & \text{for } i \text{ odd and } 1 \le i < \dfrac{m+1}{2}, \\ \dfrac{i+1}{2} + m, & \text{for } i \text{ odd and } \dfrac{m+1}{2} \le i \le m, \\ \dfrac{i-2}{2} + m, & \text{for } i \text{ even and } 1 < i < \dfrac{m+1}{2}, \\ \dfrac{i}{2} + m, & \text{for } i \text{ even and } \dfrac{m+1}{2} < i < m. \end{cases}$$

$$\psi(x_ic) = \begin{cases} \dfrac{i+1}{2} + m, & \text{for } i \text{ odd and } 1 \le i < m, \\ \dfrac{i}{2} + m, & \text{for } i \text{ even and } 1 < i < m, \\ \dfrac{3m+5}{4} + m, & i = m. \end{cases}$$

$$\psi(z_ic) = \begin{cases} \dfrac{i+1}{2} + m, & \text{for } i \text{ odd and } 1 \le i < \dfrac{m-1}{4} \text{ or} \dfrac{m-1}{2} < i \le m, \\ \dfrac{i+3}{2} + m, & \text{for } i \text{ odd and } \dfrac{m-1}{4} < i < \dfrac{m-1}{2}, \\ \dfrac{i+2}{2} + m, & \text{for } i \text{ even and } 1 < i < \dfrac{m-1}{4} \text{ or } \dfrac{m-1}{2} < i < m, \\ \dfrac{i+4}{2} + m, & \text{for } i \text{ even and } .\dfrac{m-1}{4} \le i \le \dfrac{m-1}{2}. \end{cases}$$

From the definition of edge labels $\psi$, we find that the maximum value of the edge label is $\lceil \frac{3m+1}{2} \rceil$. Thus, $\psi$ is a $\lceil \frac{3m+1}{2} \rceil$-labeling.

The weight of every vertex in $D_4^m$ is as follows.

$$w_\psi(x_i) = \begin{cases} (i+1), & \text{for } 1 \leq i < m, \\ \dfrac{3m+1}{2}, & \text{for } i = m. \end{cases}$$

$$w_\psi(y_i) = \begin{cases} i+m, & \text{for } 1 \leq i < \dfrac{m+1}{2}, \\ i+m+1, & \text{for } \dfrac{m+1}{2} \leq i < m, \\ \dfrac{9m-1}{4}, & \text{for } i = m. \end{cases}$$

$$w_\psi(z_i) = \begin{cases} i+2m, & \text{for} 1 \leq i < \dfrac{m-1}{4}, \\ i+2m+1, & \text{for} \dfrac{m-1}{4} \leq i \leq m. \end{cases}$$

$$w_\psi(c) = (3m+1)((m+1)/2) + 1 \equiv 1 \pmod{3m+1}.$$

Thus, the vertex weight set is $\{w_\psi(v) : v \in V(D_4^m)\} = \{w_\psi(x_i) : i = 1, ..., m\} \cup \{w_\psi(y_i) : i = 1, ..., m\} \cup \{w_\psi(z_i) : i = 1, ..., m\} \cup \{w_\psi(c)\} = \{0, 1, 2, \ldots, 3m-1, 3m\} = Z_{3m+1}$.

Then, we conclude that $\Psi$ is an irregular modular $\lceil \frac{3m+1}{2} \rceil$-labeling for $D_4^m$ with $m \equiv 1 \pmod 8$.

**Case 4.** For $m \equiv 5 \pmod 8$. Consider the following edge labeling.

$$\psi(x_iy_i) = \begin{cases} \dfrac{i+1}{2}, & \text{for } i \text{ odd and } 1 \leq i < m, \\ \dfrac{i}{2}+1, & \text{for } i \text{ even and } 1 < i < m, \\ \dfrac{3m-3}{4}, & i = m. \end{cases}$$

$$\psi(y_iz_i) = \begin{cases} \dfrac{i-1}{2}+m, & \text{for } i \text{ odd and } 1 \leq i < \dfrac{m-1}{2}, \\ \dfrac{i+1}{2}+m, & \text{for } i \text{ odd and } \dfrac{m-1}{2} < i \leq m, \\ \dfrac{i-2}{2}, & \text{for } i \text{ even and } 1 < i \leq (m-1)/2, \\ \dfrac{i}{2}+m, & \text{for } i \text{ even and } \dfrac{m-1}{2} < i < m. \end{cases}$$

$$\psi(x_ic) = \begin{cases} \dfrac{i+1}{2}, & \text{for } i \text{ odd and } 1 \leq i < m, \\ \dfrac{i}{2}, & \text{for } i \text{ even and } 1 < i < m, \\ \dfrac{3m+5}{4}, & i = m. \end{cases}$$

$$\psi(z_i c) = \begin{cases} \dfrac{i+1}{2} + m, & \text{for } i \text{ odd and } 1 \leq i < \dfrac{m-1}{4} \text{ or } \dfrac{m-1}{2} < i \leq m, \\ \dfrac{i+3}{2} + m, & \text{for } i \text{ odd and } \dfrac{m-1}{4} \leq i < \dfrac{m-1}{2}, \\ \dfrac{i+2}{2} + m, & \text{for } i \text{ even and } 1 < i < \dfrac{m-1}{4} \text{ or } \dfrac{m-1}{2} < i < m, \\ \dfrac{i+4}{2} + m, & \text{for } i \text{ even and } \dfrac{m-1}{4} < i \leq\leq \dfrac{m-1}{2}. \end{cases}$$

From the definition of edge labels $\Psi$, we find that the maximum value of the edge label is $\lceil \frac{3m+1}{2} \rceil$. Thus, $\Psi$ is a $\lceil \frac{3m+1}{2} \rceil$-labeling.

The weight of every vertex in $D_4^m$ is as follows.

$$w_\psi(x_i) = \begin{cases} i+1, & \text{for } 1 \leq i < m, \\ \dfrac{3m+1}{2}, & \text{for } i = m. \end{cases}$$

$$w_\psi(y_i) = \begin{cases} i+m, & \text{for } 1 \leq i \leq \dfrac{m-1}{2}, \\ i+m+1, & \text{for } \dfrac{m-1}{2} < i < m, \\ \dfrac{9m-1}{4}, & \text{for } i = m. \end{cases}$$

$$w_\psi(z_i) = \begin{cases} i+2m, & \text{for } 1 \leq i < (\dfrac{m-1}{4}, \\ i+2m+1, & \text{for } \dfrac{m-1}{4} \leq i \leq m. \end{cases}$$

$$w_\psi(c) = (3m+1)((m+1)/2) + 1 \equiv 1 \pmod{(3m+1)}.$$

The set of vertex weight of $D_4^m$ is as follows. $\{w_\psi(v) : v \in V(D_4^m)\} = \{w_\psi(x_i) : i = 1, ..., m\} \cup \{w_\psi(y_i) : i = 1, ..., m\} \cup \{w_\psi(z_i) : i = 1, ..., m\} \cup \{w_\psi(c)\} = \{0, 1, 2, \ldots, 3m-1, 3m\} = Z_{3m+1}$.

Then, we see that $\Psi$ is an irregular modular $\lceil \frac{3m+1}{2} \rceil$-labeling for $D_4^m$ with $m \equiv 5 \pmod 8$.

For all cases, we conclude that

$$ms(D_4^m) = \lceil \frac{3m+1}{2} \rceil.$$

The dutch windmill graph $D_4^m$ has $3m$ vertices with degree 3, and 1 vertex with degree $2m$. Using Lemma 1.1, we obtain $s(D_4^m) \geq \frac{3m+1}{2}$. Since $s(G)$ is an integer, we have

$$s(D_4^m) \geq \lceil \frac{3m+1}{2} \rceil.$$

According to Theorems 1.2 and 2.1, we have

$$s(D_4^m) \leq ms(D_4^m).$$

and

$$s(D_4^m) \leq \lceil \frac{3m+1}{2} \rceil.$$

Thus, we can conclude that $s(D_4^m) = \lceil \frac{3m+1}{2} \rceil$.

$\square$

In Figure 1, we give an example of modular irregular labeling of $D_4^4$,

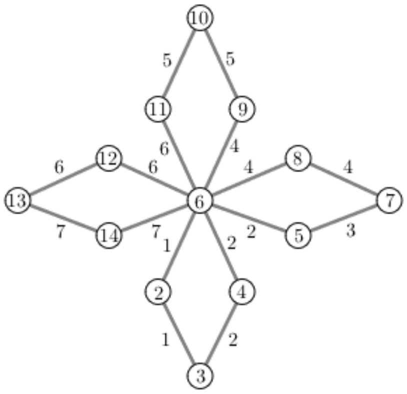

Figure 1. Example of Modular Irregular Labeling $D_4^4$ with $\mathrm{ms}(D_4^4) = 7$

### 2.2. *Comb Product of Path, Cycles and Complete Related Graphs*

The comb product of a path graph $P_m$ (with $m$ vertices) and a cycle graph $C_n$ (with $n$ vertices), denoted by $P_m \rhd C_n$, is the graph formed by taking one copy of the path graph $P_m$ and $m$ copies of the graph $C_n$, then attaching a given vertex, say $x_1^j$, of the $j-$th copy of $C_n$ to the $j$th vertex of the graph $P_m$, for all $j$, where $1 \leq j \leq m$.

**Theorem 2.2.** *Let $P_m \rhd C_n$ be a comb product path and cycle graph with $m, n \geq 3$. Then,*

$$ms(P_m \rhd C_n) = \begin{cases} \infty, & m \equiv 1,3 \pmod 4 \text{ and } n \equiv 2 \pmod 4, \\ \infty, & m \equiv 2 \pmod 4 \text{ and } n \equiv 1,3 \pmod 4, \\ \left\lceil \dfrac{(n-1)m+1}{2} \right\rceil, & \text{otherwise}. \end{cases}$$

*Proof.* $P_m \rhd C_n$ has $mn$ vertices with vertex set $V(P_m \rhd C_n) = \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(P_m \rhd C_n) = \{x_1^j x_1^{j+1} : 1 \leq j \leq m-1\} \cup \{x_i^j x_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq m\}$. To simplify labeling, we change the edge notation $x_n^j x_{n+1}^j$ as $x_n^j x_1^j$.

For $m \equiv 1,3 \pmod 4$ and $n \equiv 2 \pmod 4$ or $m \equiv 2 \pmod 4$ and $n \equiv 1,3 \pmod 4$, $P_m \rhd C_n$ have order $mn \equiv 2 \pmod 4$. According to Theorem 1.2, the $ms(P_m \rhd C_n) = \infty$.

For other cases, based on the pattern of the edge labels, we divided the proof of $mn \not\equiv 2 \pmod 4$ into five cases, each case we label its edge by $\alpha_k$, for $k = 1, 2, ..., 5$.

**Case 1.** For $m$ even and $n$ odd, define the edge labeling $\alpha_1 : E(P_m \triangleright C_n) \to \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\alpha_1(x_i^j x_{i+1}^j) = \begin{cases} \dfrac{(n-1)j-n+3}{2} + \left\lfloor \dfrac{i-1}{2} \right\rfloor, & 1 \le i \le n,\ 1 \le j \le \dfrac{m}{2}, \\ \dfrac{(n-1)j-n+3}{2} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ \dfrac{m}{2} < j \le m. \end{cases}$$

$$\alpha_1(x_1^j x_1^{j+1}) = \begin{cases} \left\lfloor \dfrac{(n-1)m-(n-2)j}{2} \right\rfloor, & 1 \le j \le \dfrac{m}{2}, \\ \dfrac{(n-2)m-(n-4)j-1}{2}, & \dfrac{m+2}{2} \le j \le m-1,\ j \text{ odd}, \\ \dfrac{n(m-j)}{2}, & \dfrac{m+2}{2} \le j \le m-1,\ j \text{ even}. \end{cases}$$

We have the vertex weight

$$wt_{\alpha_1}(x_i^j) = \begin{cases} (n-1)j-n+i+1, & 2 \le i \le n,\ 1 \le j \le \dfrac{m}{2}, \\ (n-1)j-n+i+2, & 2 \le i \le n,\ \dfrac{m+2}{2} \le j \le m. \end{cases}$$

$$wt_{\alpha_1}(x_1^j) = \begin{cases} (n-1)m, & j = 1, \\ (n-1)m+j+1, & 1 \le j \le \dfrac{m}{2}, \\ (n-1)m+\dfrac{m}{2}+3, & j = \dfrac{m+2}{2}, \\ (n-1)m+j+2, & \dfrac{m+4}{2} \le j \le m,\ j \text{ odd}, \\ (n-1)m+j, & \dfrac{m+4}{2} \le j \le m,\ j \text{ even}. \end{cases}$$

**Case 2.** For $m, n$ even, define edge labeling $\alpha_2 : E(P_m \triangleright C_n) \to \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\alpha_2(x_i^j x_{i+1}^j) = \alpha_1(x_i^j x_{i+1}^j), \quad 1 \le i \le n, 1 \le j \le \frac{m-2}{2} \text{ and } \frac{m+2}{2} \le j \le m, j \text{ is odd.}$$

$$\alpha_2(x_i^j x_{i+1}^j) = \begin{cases} \dfrac{(n-1)j-n+2}{2} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ 1 < j \le \dfrac{m-2}{2},\ j \text{ is even} \\ \dfrac{(n-1)m-2n+4}{4} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ j = \dfrac{m}{2},\ m \equiv 0 \pmod 4, \\ \dfrac{(n-1)m-2n+6}{4} + \left\lfloor \dfrac{i-1}{2} \right\rfloor, & 1 \le i \le n,\ j = \dfrac{m}{2},\ m \equiv 2 \pmod 4, \\ \dfrac{(n-1)j-n+4}{2} + \left\lfloor \dfrac{i-1}{2} \right\rfloor, & 1 \le i \le n,\ \dfrac{m+2}{2} \le j \le m,\ j \text{ is even.} \end{cases}$$

$$\alpha_2(x_1^j x_1^{j+1}) = \begin{cases} \dfrac{(n-1)m-(n-2)j}{2}, & 1 \le j \le \dfrac{m}{2}, \\ \dfrac{(n-2)m-(n-4)j-2}{2}, & \dfrac{m+2}{2} \le j \le m-1,\ j \text{ is odd}, \\ \alpha_1(x_1^j x_1^{j+1}), & \dfrac{m+2}{2} \le j \le m-1,\ j \text{ is even}. \end{cases}$$

We have the vertex weight $wt_{\alpha_2}(x_i^j)$ for $1 \le i \le n$ and $1 \le j \le m$:

$$wt_{\alpha_2}(x_i^j) = wt_{\alpha_1}(x_i^j),\ i \ne 1,\ j \ne \frac{m+2}{2}$$

$$wt_{\alpha_2}(x_1^{(m+2)/2}) = \begin{cases} wt_{\alpha_1}(x_1^{(m+2)/2}), & m \equiv 0 \pmod 4, \\ (n-1)m + \dfrac{m}{2} + 2, & m \equiv 2 \pmod 4. \end{cases}$$

**Case 3.** For $m$ odd, $n$ even, define an edge labeling $\alpha_3 : E(P_m \triangleright C_n) \to \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\alpha_3(x_i^1 x_{i+1}^1) = \begin{cases} \dfrac{(n-1)m-n+5}{4} + \left\lfloor \dfrac{i-1}{2} \right\rfloor, & 1 \le i \le n,\ m \equiv 1 \pmod 4, \\ \dfrac{(n-1)m-n+3}{4} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ m \equiv 3 \pmod 4. \end{cases}$$

$$\alpha_3(x_i^j x_{i+1}^j) = \begin{cases} \dfrac{(n-1)j-2n+4}{2} + \left\lfloor \dfrac{i-1}{2} \right\rfloor, & 1 \le i \le n,\ 2 \le j \le \dfrac{m-1}{2},\ j \text{ even}, \\ \dfrac{(n-1)j-2n+3}{2} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ 2 \le j \le \dfrac{m-1}{2},\ j \text{ odd}, \\ \dfrac{(n-1)j-n+2}{2} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ \dfrac{m+3}{2} \le j \le m,\ j \text{ is even}, \\ \dfrac{(n-1)j-n+3}{2} + \left\lfloor \dfrac{i-1}{2} \right\rfloor, & 1 \le i \le n,\ \dfrac{m+3}{2} \le j \le m,\ j \text{ is odd}. \end{cases}$$

$$\alpha_3(x_i^{(m+1)/2} x_{i+1}^{(m+1)/2}) = \begin{cases} \dfrac{(n-1)m-n+5}{4} - 2\left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le \dfrac{n}{2},\ m \equiv 1 \pmod 4, \\ \dfrac{(n-1)m-n+7}{4} - 2\left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le \dfrac{n}{2},\ m \equiv 3 \pmod 4, \\ \dfrac{(n-1)m-5n+5}{4} + i, & \dfrac{n+2}{2} \le i \le n,\ m \equiv 1 \pmod 4, \\ \dfrac{(n-1)m-5n+3}{4} + i, & \dfrac{n+2}{2} \le i \le n,\ m \equiv 3 \pmod 4. \end{cases}$$

$$\alpha_3(x_1^j x_1^{j+1}) = \begin{cases} \dfrac{(n-1)(m+1)-(n-2)j}{2}, & 1 \le j \le \dfrac{m-1}{2} \\ \dfrac{n(m-j)}{2}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is odd} \\ \dfrac{(n-2)m-(n-4)j}{2}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is even.} \end{cases}$$

We have vertex weight

$$wt_{\alpha_3}(x_i^j) = \begin{cases} \dfrac{(n-1)m-n+1}{2}+i, & 2 \le i \le n,\ j=1, \\ (n-1)j-2n+i+2, & 2 \le i \le n,\ 2 \le j \le \dfrac{m-1}{2}, \\ \dfrac{(n-1)m-n+5}{2}-2(i-1), & 2 \le i \le \dfrac{n}{2},\ j=\dfrac{m+1}{2} \\ \dfrac{(n-1)m-3n+7}{2}, & i=\dfrac{n+2}{2},\ j=\dfrac{m+1}{2} \\ \dfrac{(n-1)m-5n+3}{2}+2i, & \dfrac{n+4}{2} \le i \le n,\ j=\dfrac{m+1}{2},\ m \equiv 1 \pmod 4, \\ \dfrac{(n-1)m-5n+1}{2}+2i, & \dfrac{n+4}{2} \le i \le n,\ j=\dfrac{m+1}{2}, m \equiv 3 \pmod 4, \\ (n-1)j-n+i+1, & 2 \le i \le n,\ \dfrac{m+3}{2} \le j \le m. \end{cases}$$

$$wt_{\alpha_3}(x_1^j) = \begin{cases} (n-1)m+2, & j=1, \\ wt_{\alpha_1}(x_1^j), & 2 \le j \le \dfrac{m-1}{2}, \\ \dfrac{(2n-1)m+3}{2}, & j=\dfrac{m+1}{2},\ m \equiv 1 \pmod 4, \\ \dfrac{(2n-1)m+5}{2}, & j=\dfrac{m+1}{2},\ m \equiv 3 \pmod 4, \\ (n-1)m+j, & \dfrac{m+3}{2} \le j \le m,\ j \text{ is odd,} \\ (n-1)m+j+2, & \dfrac{m+3}{2} \le j \le m,\ j \text{ is even.} \end{cases}$$

**Case 4.** For $m,n$ odd, $m=n$, define edge labeling $\alpha_4 : E(P_m \triangleright C_n) \to \{1,2,...,\left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\alpha_4(x_i^j x_{i+1}^j) = \begin{cases} \dfrac{n+1}{2} - 2\left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le \dfrac{n-1}{2},\ j=1, \\ \dfrac{1-n}{2}+i, & \dfrac{n+1}{2} \le i \le n,\ j=1. \end{cases}$$

$$\alpha_4(x_i^j x_{i+1}^j) = \alpha_1(x_i^j x_{i+1}^j),\quad 1 \le i \le n,\ 2 \le j \le \frac{m+1}{2} \text{ and } \frac{m+3}{2} \le j \le m-1.$$

$$\alpha_4(x_1^jx_1^{j+1})=\begin{cases}\alpha_1(x_i^jx_{i+1}^j), & 1\le j\le \dfrac{m-1}{2},\ m\equiv 1 \pmod 4\\ \left\lfloor\dfrac{(n-1)m-(n-2)j-1}{2}\right\rfloor, & 1\le j\le \dfrac{m-1}{2},\ m\equiv 3 \pmod 4\\ \dfrac{n(m-j)}{2}, & \dfrac{m+1}{2}\le j\le m-1,\ j \text{ is odd},\\ \dfrac{(n-2)m-(n-4)j-1}{2}, & \dfrac{m+1}{2}\le j\le m-1,\ j \text{ is even}.\end{cases}$$

We have the vertex weight

$$wt_{\alpha_4}(x_i^j)=\begin{cases}n+1-2(i-1), & 2\le i\le \dfrac{n-1}{2},\ j=1,\\ 2, & i=\dfrac{n+1}{2},\ j=1,\\ 2i-n, & \dfrac{n+3}{2}\le i\le n,\ j=1,\\ wt_{\alpha_1}(x_i^j), & 2\le i\le n,\ 2\le j\le \dfrac{m+1}{2} \text{ and } \dfrac{m+3}{2}\le j\le m.\end{cases}$$

$$wt_{\alpha_4}(x_1^j)=\begin{cases}\dfrac{(n-1)m+n+3}{2}, & j=1,\\ wt_{\alpha_1}(x_1^j), & 2\le j\le \dfrac{m-1}{2} \text{ and } \dfrac{m+3}{2}\le j\le m,\\ mn-\dfrac{m-3}{2}, & j=\dfrac{m+1}{2},\ m\equiv 1 \pmod 4),\\ mn-\dfrac{m-5}{2}, & j=\dfrac{m+1}{2},\ m\equiv 3 \pmod 4).\end{cases}$$

**Case 5.** For $m, n$ odd, $m\neq n$, define edge labeling $\alpha_5 : E(P_m \triangleright C_n)\to\{1,2,...,\left\lceil\frac{(n-1)m+1}{2}\right\rceil\}$ as follows.

$$\alpha_5(x_i^jx_{i+1}^j)=\begin{cases}\dfrac{(n-1)m-n+5}{4}+\left\lceil\dfrac{i-1}{2}\right\rceil, & 1\le i\le n,\ j=1,\\ \dfrac{(n-1)j-2n+4}{2}+\left\lceil\dfrac{i-1}{2}\right\rceil, & 1\le i\le n,\ 2\le j\le \dfrac{m-1}{2},\\ \dfrac{(n-1)j-n+3}{2}+\left\lceil\dfrac{i-1}{2}\right\rceil, & 1\le i\le n,\ \dfrac{m+3}{2}\le j\le m.\end{cases}$$

$$\alpha_5(x_i^{(m+1)/2}x_{i+1}^{(m+1)/2})=\begin{cases}\dfrac{(n-1)(m-1)}{4}+2-2\left\lceil\dfrac{i-1}{2}\right\rceil, & 1\le i\le \dfrac{n-1}{2},\ m\equiv 1 \pmod 4),\\ \dfrac{(n-1)(m-1)}{4}+1-2\left\lfloor\dfrac{i-1}{2}\right\rfloor, & 1\le i\le \dfrac{n-1}{2},\ m\equiv 3 \pmod 4),\\ \dfrac{(n-1)(m-5)}{4}+i, & \dfrac{n+1}{2}\le i\le n.\end{cases}$$

$$\alpha_5(x_1^j x_1^{j+1}) = \begin{cases} \left\lceil \dfrac{(n-1)m-(n-2)(j-1)+1}{2} \right\rceil, & 1 \le j \le \dfrac{m-1}{2} \\ \dfrac{n(m-j)}{2}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is odd} \\ \dfrac{(n-2)m-(n-4)j+1}{2}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is even.} \end{cases}$$

We have vertex weight

$$wt_{\alpha_5}(x_i^j) = \begin{cases} wt_{\alpha_3}(x_i^j)+1, & 2 \le i \le n,\ 1 \le j \le \dfrac{m-1}{2} \text{ and } \dfrac{m+3}{2} \le j \le m, \\ wt_{\alpha_3}(x_i^j)+2, & 2 \le i \le \dfrac{n+1}{2},\ j = \dfrac{m+1}{2} \\ \dfrac{(n-1)(m-5)}{2}+2i-1, & \dfrac{n+3}{2} \le i \le n,\ j = \dfrac{m+1}{2}. \end{cases}$$

$$wt_{\alpha_5}(x_1^j) = wt_{\alpha_3}(x_1^j)+1, \quad 1 \le j \le m.$$

For all cases, it can be checked that the edge labeling $\alpha_k$ is the $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$-labeling for $k = 1, 2, ..., 5$. Then, under the labeling $\alpha_k$, the set weights of the vertices $\{wt_{\alpha_k}(x_i^j) : 1 \le j \le m,\ 1 \le i \le n,\ 1 \le k \le 4\}$ attain the values $\{2, 3, ..., mn, mn+1\}$ and for $\{wt_{\alpha_5}(x_i^j) : 1 \le j \le m,\ 1 \le i \le n\}$ attain the values $\{3, 4, ..., mn, mn+1, mn+2\}$. Thus, the set of modular weights $\{wt_{\alpha_k}(x_i^j) \pmod{mn} : 1 \le j \le m,\ 1 \le i \le n,\ 1 \le k \le 5\}$ is equal to $\{0, 1, 2, ..., mn-1\}$, which implies that $ms(P_m \triangleright C_n) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil$. □

An example of modular irregular labeling of $P_4 \triangleright C_8$is given in Figure 2.

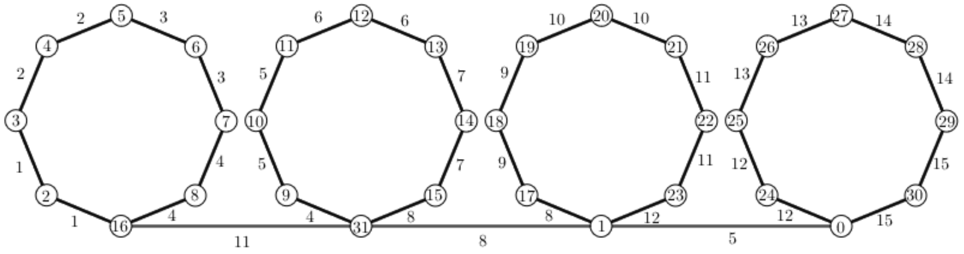

Figure 2. Example of Modular Irregular Labeling $P_4 \triangleright C_8$ with $\mathrm{ms}(P_4 \triangleright C_8) = 15$

If we add $k$ chords to the cycle $C_n$, we denote the resulting graph as $C_{n,k}$. In the following theorem, we consider the $r$-regular $C_{n,k}$ graphs, denoted as $r - C_{n,k}$ In these graphs, the added chords for $r-2$ edge disjoint 1-factor (perfect matching) of $C_{n,k}$.

**Theorem 2.3.** *Let $P_m$ be a path graph of order $m$ and $r - C_{n,k}$ be a $r$ regular cycle graph of order $n$ containing $k$ chords. Then,*

$$ms(P_m \triangleright r - C_{n,k}) = \begin{cases} \infty, & m \equiv 1, 3 \pmod 4,\ n \equiv 2 \pmod 4, \\ \infty, & m \equiv 2 \pmod 4,\ n \equiv 1, 3 \pmod 4, \\ \left\lceil \dfrac{(n-1)m+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

*Proof.* Let $P_m$ be a path graph of order $m$ and $r - C_{n,k}$ be a $r$-regular graph with cycle of order $n$ containing $k$ chords that have 1-factor $M(r - C_{n,k})$. Thus, $n$ is even. $P_m \triangleright r - C_{n,k}$ have $mn$ vertices with a vertex set $V(P_m \triangleright r - C_{n,k}) = \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and an edge set $E(P_m \triangleright r - C_{n,k}) = \{x_1^j x_1^{j+1} : 1 \leq j \leq m-1\} \cup \{x_i^j x_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{e : e \in M(r - C_{n,k}), e \notin x_i^j x_{i+1}^j\}$. To simplify labeling, we change the edge notation $x_n^j x_{n+1}^j = x_n^j x_1^j$.

For $m \equiv 1, 3 \pmod 4$ and $n \equiv 2 \pmod 4$ or $m \equiv 2 \pmod 4$ and $n \equiv 1, 3 \pmod 4$, $P_m \triangleright r - C_{n,k}$ have order $mn \equiv 2 \pmod 4$. According to Theorem 1.2, the $ms(P_m \triangleright r - C_{n,k}) = \infty$. Based on the pattern of the edge labels, we divided the proof of $mn \not\equiv 2 \pmod 4$ into two cases, each case we labeling its edge by $\beta_k$, for $k = 2, 3$.

For $k = 2, 3$, define the edge labeling $\beta_k : E(P_m \triangleright r - C_{n,k}) \rightarrow \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\begin{aligned} \beta_k(x_i^j x_{i+1}^j) &= \alpha_k(x_i^j x_{i+1}^j), & 1 \leq i \leq n,\ 1 \leq j \leq m, \\ \beta_k(e) &= 1, & e \in M(r - C_{n,k}), e \notin \beta_k(x_i^j x_{i+1}^j), \\ \beta_k(x_1^j x_1^{j+1}) &= \alpha_k(x_1^j x_1^{j+1}), & 1 \leq j \leq m-1. \end{aligned}$$

We have the vertex weight

$$\begin{aligned} wt_{\beta_k}(x_i^j) &= wt_{\alpha_k}(x_i^j) + r - 2, & 2 \leq i \leq n,\ 1 \leq j \leq m, \\ wt_{\beta_k}(x_1^j) &= wt_{\alpha_k}(x_1^j) + r - 2, & 1 \leq j \leq m-1. \end{aligned}$$

For all cases, it can be checked that the edge labeling $\beta_k$ is the $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$-labeling for $k = 2, 3$. Then, under the labeling $\beta_k$ the set weights of the vertices $\{wt_{\beta_k}(x_1^j) : 1 \leq j \leq m,\ 1 \leq i \leq n,\ k = 2, 3\}$ attain the values $\{r, r+1, ..., mn+r-2, mn+r-1\}$. Thus, the set of modular weights $\{wt_{\beta_k}(x_1^j) \pmod{mn} : 1 \leq j \leq m,\ 1 \leq i \leq n,\ k = 2, 3\}$ are $\{0, 1, 2, ..., mn-1\}$, which implies that $ms(P_m \triangleright r - C_{n,k}) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil$. □

Figure 3 shows an example of modular irregular labeling of $P_4 \triangleright 3 - C_{8,4}$.

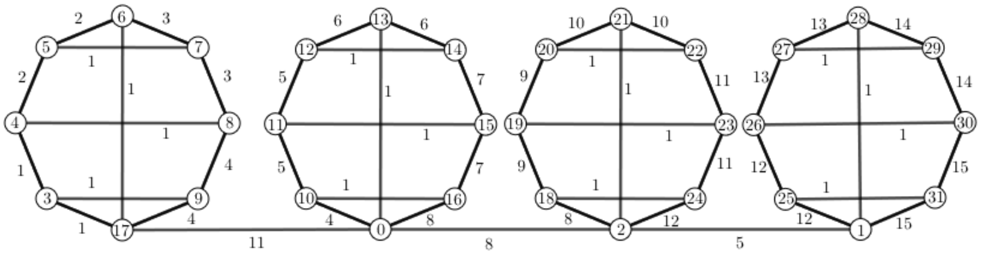

Figure 3. Example of Modular Irregular Labeling $P_4 \triangleright 3 - C_{8,4}$ with ms$(P_4 \triangleright 3 - C_{8,4}) = 15$

Let $n, q$, and $a_1, a_2, ..., a_q$ be positive integers with $1 \leq a_1 < a_2 < ... < a_p < ... < a_q \leq \left\lfloor \frac{n}{2} \right\rfloor$. An undirected graph with the set of vertices $V = \{x_i : 1 \leq i \leq n\}$ and the set of edges $E = x_i x_{i+a_p} : 1 \leq i \leq n, 1 \leq p \leq q$, with the indices $a_p$ being taken modulo $n$, is called a circulant graph and is denoted by $C_n(a_1, a_2, ..., a_p, ..., a_q)$.

**Corollary 2.1.** *Let $P_m$ be a path graph of order $m$ and $C_n(1, a_2, ..., a_p, ..., a_q)$ be a circulant graph of order $n$. Then,*

$$ms(P_m \triangleright C_n(1, a_2, ..., a_q)) = \begin{cases} \infty, & m \equiv 1,3 \pmod 4,\ n \equiv 2 \pmod 4, \\ \infty, & m \equiv 2 \pmod 4,\ n \equiv 1,3 \pmod 4, \\ \left\lceil \dfrac{(n-1)m+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

*Proof.* $P_m \triangleright C_n(1, a_2, \ldots, a_p, \ldots, a_q)$ have $mn$ vertices with a vertex set $V(P_m \triangleright C_n(1, a_2, \ldots, a_p, \ldots, a_q)) = \{x_i^j : 1 \le i \le n, 1 \le j \le m\}$ and an edge set $E(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \{x_1^j x_1^{j+1} : 1 \le j \le m-1\} \cup \{x_i^j x_{i+1}^j : 1 \le i \le n, 1 \le j \le m\} \cup \{x_i^j x_{i+a_p \pmod{mn}}^j : 1 \le i \le n, 1 \le j \le m, 2 \le p \le q\}$. To simplify labeling, we change the edge notation $x_n^j x_{n+1}^j = x_n^j x_1^j$.

For $m \equiv 1,3 \pmod 4$ and $n \equiv 2 \pmod 4$ or $m \equiv 2 \pmod 4$ and $n \equiv 1,3 \pmod 4$, $P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)$ have order $mn \equiv 2 \pmod 4$. According to Theorem 1.2, the $ms(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \infty$. Based on the pattern of the edge labels, we divided the proof of $mn \not\equiv 2 \pmod 4$ into five cases, each case we label its edge by $\beta_k$, for $k = 1, 2, ..., 5$.

For $k = 1, 2, ..., 5$, define the edge labeling $\beta_k : E(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) \to \{1, 2, \ldots, \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\begin{aligned} \beta_k(x_i^j x_{i+1}^j) &= \alpha_k(x_i^j x_{i+1}^j), \quad 1 \le i \le n,\ 1 \le j \le m, \\ \beta_k\left(x_i^j x_{i+a_p \pmod{mn}}^j\right) &= 1, \qquad\qquad\quad 1 \le i \le n,\ 1 \le j \le m, \\ \beta_k(x_1^j x_1^{j+1}) &= \alpha_k(x_1^j x_1^{j+1}), \qquad\qquad 1 \le j \le m-1. \end{aligned}$$

If $|a_p| = s$, we have the vertex weight

$$\begin{aligned} wt_{\beta_k}(x_i^j) &= wt_{\alpha_k}(x_i^j) + s, \quad 2 \le i \le n,\ 1 \le j \le m, \\ wt_{\beta_k}(x_1^j) &= wt_{\alpha_k}(x_1^j) + s, \qquad\quad 1 \le j \le m-1. \end{aligned}$$

For all cases, it can be checked that the edge labeling $\beta_k$ is the $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$-labeling for $k = 1, 2, ..., 5$. Then, under labeling $\beta_k$ the set weights of the vertices $\{wt_{\beta_k}(x_1^j) : 1 \le j \le m,\ 1 \le i \le n,\ 1 \le k \le 4\}$ reach the values $\{2+s, 3+s, ..., mn+s, mn+1+s\}$ and for $\{wt_{\beta_5}(x_1^j) : 1 \le j \le m,\ 1 \le i \le n\}$ attain the values $\{3+s, 4+s, ..., mn+s, mn+1+s, mn+2+s\}$. Thus, the set of modular weights $\{wt_{\beta_k}(x_1^j) \pmod{mn} : 1 \le j \le m,\ 1 \le i \le n,\ 1 \le k \le 5\}$ is equal to $\{0, 1, 2, ..., mn-1\}$, which implies that $ms(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil$. $\square$

An example of modular irregular labeling of $P_4 \triangleright C_8(1, 2)$ can be seen in Figure 4 as follows.

If we add an edge from vertex $x_1^m$ to vertex $x_1^1$ in the path, it becomes a cycle graph $C_m$ that has $m$ vertices with a set of vertices $V = \{x_1^j : 1 \le j \le m\}$ and a set of edges $E = \{x_1^j x_1^{j+1} : 1 \le j \le m\}$.

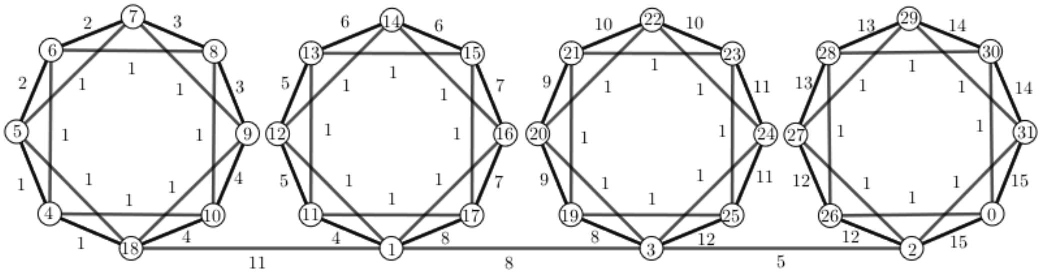

Figure 4. Example of Modular Irregular Labeling $P_4 \triangleright C_8(1,2)$ with ms$(P_4 \triangleright C_8(1,2)) = 15$

**Theorem 2.4.** *Let $C_m$ and $C_n$ be cycle graphs of order $m$ and $n$ respectively. Then,*

$$ms(C_m \triangleright C_n) = \begin{cases} \infty, & m \equiv 1,3 \pmod 4,\ n \equiv 2 \pmod 4, \\ \infty, & m \equiv 2 \pmod 4,\ n \equiv 1,3 \pmod 4, \\ \left\lceil \dfrac{(n-1)m+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

*Proof.* $C_m \triangleright C_n$ have $mn$ vertices with a vertex set $V(C_m \triangleright C_n = \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and an edge set $E(C_m \triangleright C_n) = \{x_1^j x_1^{j+1} : 1 \leq j \leq m\} \cup \{x_i^j x_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq m\}$. To simplify labeling, we change the edge notation $x_1^m x_1^{m+1} = x_1^1 x_1^m$ and $x_n^j x_{n+1}^j = x_n^j x_1^j$.

For $m \equiv 1,3 \pmod 4$ and $n \equiv 2 \pmod 4$ or $m \equiv 2 \pmod 4$ and $n \equiv 1,3 \pmod 4$, $C_m \triangleright C_n$ have order $mn \equiv 2 \pmod 4$. According to Theorem 1.2, the $ms(C_m \triangleright C_n) = \infty$. Based on the pattern of the edge labels, we divided the proof of $mn \not\equiv 2 \pmod 4$ into 5 cases, each case labeling its edge by $\gamma_k$, for $k = 1, 2, .., 6$.

**Case 1.** For $m$ even, define edge labeling $\gamma_k : E(C_m \triangleright C_n) \rightarrow \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ for $k = 1, 2$ as follows.

$$\gamma_k(x_i^j x_{i+1}^j) = \alpha_k(x_i^j x_{i+1}^j), \quad 1 \leq i \leq n,\ 1 \leq j \leq m.$$

$$\gamma_k(x_1^j x_1^{j+1}) = \begin{cases} \alpha_k(x_1^j x_1^{j+1}) - 1, & 1 \leq j \leq \dfrac{m}{2},\ j \text{ is odd}, \\ \alpha_k(x_1^j x_1^{j+1}) + 1, & 1 \leq j \leq \dfrac{m}{2},\ j \text{ is even}, \\ \alpha_k(x_1^j x_1^{j+1}) - 1, & \dfrac{m+2}{2} \leq j \leq m-1,\ j \text{ is odd}, \\ \alpha_k(x_1^j x_1^{j+1}) + 1, & \dfrac{m+2}{2} \leq j \leq m-1,\ j \text{ is even}, \\ 1, & j = m. \end{cases}$$

We have the vertex weight

$$\begin{aligned} wt_{\gamma_k}(x_i^j) &= wt_{\alpha_k}(x_i^j), \quad 2 \leq i \leq n,\ 1 \leq j \leq m, \\ wt_{\gamma_k}(x_1^j) &= wt_{\alpha_k}(x_i^j), \quad\quad\quad\quad\quad 1 \leq j \leq m. \end{aligned}$$

**Case 2.** For $m$ odd, $n$ even, define an edge labeling $\gamma_3 : E(C_m \triangleright C_n) \rightarrow \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\gamma_3(x_i^j x_{i+1}^j) = \begin{cases} \alpha_3(x_i^j x_{i+1}^j), & 1 \le i \le n,\ 1 \le j \le m-1, \\ \dfrac{(n-1)(m-1)}{2} + 2\left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le \dfrac{n}{2},\ j = m, \\ \dfrac{(n-1)m+n+3}{2} - i, & \dfrac{n+2}{2} \le i \le n,\ j = m. \end{cases}$$

$$\gamma_3(x_1^j x_1^{j+1}) = \begin{cases} \alpha_3(x_1^j x_1^{j+1}) - \dfrac{n-1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is odd}, \\ \alpha_3(x_1^j x_1^{j+1}) + \dfrac{n-1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is even}, \\ \alpha_3(x_1^j x_1^{j+1}) - \dfrac{n-1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is odd}, \\ \alpha_3(x_1^j x_1^{j+1}) + \dfrac{n-1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is even}, \\ \dfrac{n-1}{4}, & j = m. \end{cases}$$

We have the vertex weight

$$wt_{\gamma_3}(x_i^j) = \big\{ wt_{\alpha_3}(x_i^j), \quad 2 \le i \le n,\ 1 \le j \le m-1.$$

$$wt_{\gamma_3}(x_i^m) = \begin{cases} (n-1)(m-1) + 2(i-1), & 1 \le i \le \dfrac{n}{2} \\ m(n-1)+1, & i = \dfrac{n+2}{2} \\ (n-1)m + n - 2(i-2), & \dfrac{n+4}{2} \le i \le n, \end{cases}$$

$$wt_{\gamma_3}(x_1^j) = wt_{\alpha_3}(x_i^j), \quad 1 \le j \le m.$$

**Case 3.** For $m, n$ odd, $m = n$, define edge labeling $\gamma_4 : E(C_m \triangleright C_n) \rightarrow \{1, 2, ..., \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\gamma_4(x_i^j x_{i+1}^j) = \alpha_1(x_i^j x_{i+1}^j), \quad 1 \le i \le n,\ 1 \le j \le \frac{m+1}{2} \text{ and } \frac{m+3}{2} \le j \le m.$$

$$\gamma_4(x_1^j x_1^{j+1}) = \begin{cases} \alpha_4(x_1^j x_1^{j+1}) + \dfrac{n-1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is odd}, \\ \alpha_4(x_1^j x_1^{j+1}) - \dfrac{n-1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is even}, \\ \alpha_4(x_1^j x_1^{j+1}) + \dfrac{n-1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is odd}, \\ \alpha_4(x_1^j x_1^{j+1}) - \dfrac{n-1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is even}, \\ \dfrac{n-1}{4}, & j = m. \end{cases}$$

We have the vertex weight

$$\begin{aligned} wt_{\gamma_4}(x_i^1) &= i, & 2 \le i \le n \\ wt_{\gamma_4}(x_i^j) &= wt_{\alpha_4}(x_i^j), & 2 \le i \le n,\ 2 \le j \le m, \\ wt_{\gamma_4}(x_1^j) &= wt_{\alpha_4}(x_i^j), & 1 \le j \le m. \end{aligned}$$

**Case 4.** For $m \equiv 1 \pmod 4$, $n \equiv 3 \pmod 4$, define the edge labeling $\gamma_5 : E(C_m \triangleright C_n) \to \{1, 2, ..., \lceil \frac{(n-1)m+1}{2} \rceil\}$ as follows.

$$\gamma_5(x_i^j x_{i+1}^j) = \begin{cases} \alpha_5(x_i^j x_{i+1}^j), & 1 \le i \le n,\ 1 \le j \le m,\ j \ne \dfrac{m+1}{2} \\ \dfrac{(n-1)(m+1)-4n+8}{4} + \left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le n,\ j = \dfrac{m+1}{2}. \end{cases}$$

$$\gamma_5(x_1^j x_1^{j+1}) = \begin{cases} \alpha_5(x_1^j x_1^{j+1}) - \dfrac{n+1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is odd}, \\ \alpha_5(x_1^j x_1^{j+1}) + \dfrac{n+1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is even}, \\ \alpha_5(x_1^j x_1^{j+1}) + \dfrac{n+1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is odd}, \\ \alpha_5(x_1^j x_1^{j+1}) - \dfrac{n+1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is even}, \\ \dfrac{n+1}{4}, & j = m. \end{cases}$$

We have the vertex weight

$$wt_{\gamma_5}(x_i^j) = \begin{cases} wt_{\alpha_5}(x_i^j), & 2 \le i \le n,\ 1 \le j \le m-1,\ j \ne \dfrac{m+1}{2}, \\ \dfrac{(n-1)(m+1)}{2} - 2n + i + 3, & 2 \le i \le n,\ j = \dfrac{m+1}{2}. \end{cases}$$

$$wt_{\gamma_5}(x_1^j) = wt_{\alpha_5}(x_i^j), \quad 1 \le j \le m.$$

**Case 5.** For $m \equiv 3 \pmod 4$, $n \equiv 1 \pmod 4$, define edge labeling $\gamma_6 : E(C_m \triangleright C_n) \to \{1, 2, ..., \lceil \frac{(n-1)m+1}{2} \rceil\}$ as follows.

$$\gamma_6(x_i^j x_{i+1}^j) = \begin{cases} \alpha_5(x_i^j x_{i+1}^j), & 1 \le i \le n,\ 1 \le j \le m-1, \\ \dfrac{(n-1)(m-1)}{2} + 1 + 2\left\lceil \dfrac{i-1}{2} \right\rceil, & 1 \le i \le \dfrac{n-1}{2},\ j = m, \\ \dfrac{(n-1)m+n+3}{2} - i, & \dfrac{n+1}{2} \le i \le n,\ j = m. \end{cases}$$

$$\gamma_6(x_1^j x_1^{j+1}) = \begin{cases} \alpha_5(x_1^j x_1^{j+1}) - \dfrac{n+1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is odd},\\ \alpha_5(x_1^j x_1^{j+1}) + \dfrac{n+1}{4}, & 1 \le j \le \dfrac{m-1}{2},\ j \text{ is even},\\ \alpha_5(x_1^j x_1^{j+1}) - \dfrac{n+1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is odd},\\ \alpha_5(x_1^j x_1^{j+1}) + \dfrac{n+1}{4}, & \dfrac{m+1}{2} \le j \le m-1,\ j \text{ is even},\\ \dfrac{n+1}{4}, & j = m. \end{cases}$$

We have the vertex weight

$$wt_{\gamma_6}(x_i^j) = \begin{cases} wt_{\alpha_5}(x_i^j), & 2 \le i \le n,\ 1 \le j \le m-1,\ j \ne \dfrac{m+1}{2},\\ \dfrac{(n-1)(m+1)}{2} - 2n + i + 2, & 2 \le i \le n,\ j = \dfrac{m+1}{2}. \end{cases}$$

$$wt_{\gamma_6}(x_i^m) = \begin{cases} (n-1)(m-1) + 2i, & 2 \le i \le \dfrac{n-1}{2}\\ m(n-1) + 2, & i = \dfrac{n+1}{2}\\ (n-1)m + n - 2(i-2), & \dfrac{n+3}{2} \le i \le n, \end{cases}$$

$$wt_{\gamma_6}(x_1^j) = wt_{\alpha_5}(x_i^j), \quad 1 \le j \le m.$$

For all cases, it can be checked that the edge labeling $\gamma_k$ is the $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$-labeling for $k = 1, 2, ..., 6$. Then, under the labeling $\gamma_k$, the set weights of the vertices $\{wt_{\gamma_k}(x_i^j) : 1 \le j \le m,\ 1 \le i \le n,\ 1 \le k \le 4\}$ attain the values $\{2, 3, ..., mn, mn+1\}$ and for $\{wt_{\gamma_k}(x_i^j) : 1 \le j \le m,\ 1 \le i \le n,\ 5 \le k \le 6\}$ attain the values $\{3, 4, ..., mn, mn+1, mn+2\}$. Thus, the set of modular weights $\{wt_{\gamma_k}(x_i^j) : 1 \le j \le m,\ 1 \le i \le n,\ 1 \le k \le 6\}$ are $\{0, 1, 2, ..., mn-1\}$, which implies that $ms(C_m \triangleright C_n) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil$. $\square$

Figure 5 gives an example of modular irregular labeling of $C_4 \triangleright C_8$,

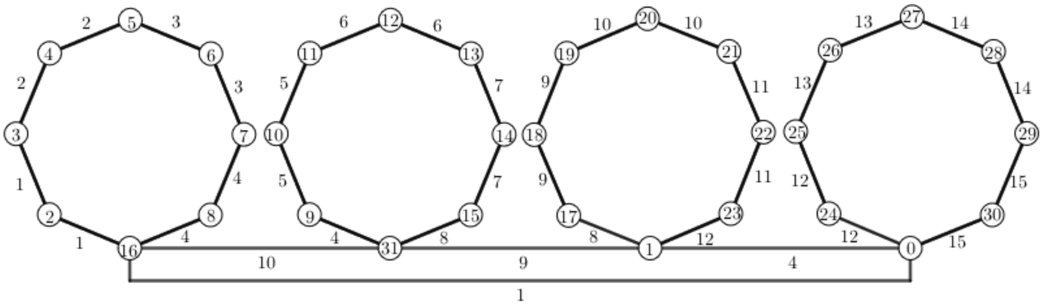

Figure 5. Example of Modular Irregular Labeling $C_4 \triangleright C_8$ with $\text{ms}(C_4 \triangleright C_8) = 15$

If we add $k$ chords in $C_n$, then we denote the graph as $C_{n,k}$. In the following theorem, we consider the $r$-regular $C_{n,k}$ graphs, denoted as $r - C_{n,k}$ In these graphs the chords are all 1-factor of $C_{n,k}$

**Theorem 2.5.** *Let $C_m$ be a cycle graph of order $m$ and $r - C_{n,k}$ be a $r$ regular cycle graph of order $n$ containing $k$ chords. Then,*

$$ms(C_m \triangleright r - C_{n,k}) = \begin{cases} \infty, & m \equiv 1,3 \pmod 4,\ n \equiv 2 \pmod 4, \\ \infty, & m \equiv 2 \pmod 4,\ n \equiv 1,3 \pmod 4, \\ \left\lceil \dfrac{(n-1)m+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

*Proof.* Let $C_m$ be a cycle graph of order $m$ and $r - C_{n,k}$ be a $r$-regular graph with cycle of order $n$ containing $k$ chords that have 1-factor $M(r - C_{n,k})$.. Thus, $n$ is even. $C_m \triangleright r - C_{n,k}$ have $mn$ vertices with a set of vertex $V(C_m \triangleright r - C_{n,k}) = \{x_i^j : 1 \le i \le n, 1 \le j \le m\}$ and an set of edges $E(C_m \triangleright r - C_{n,k}) = \{x_1^j x_1^{j+1} : 1 \le j \le m\} \cup \{x_i^j x_{i+1}^j : 1 \le i \le n, 1 \le j \le m\} \cup \{e : e \in M(r - C_{n,k}), e \notin x_i^j x_{i+1}^j\}$. To simplify labeling, we change the edge notation $x_m^j x_{m+1}^j = x_m^j x_1^j$ and $x_n^j x_{n+1}^j = x_n^j x_1^j$.

For $m \equiv 1,3 \pmod 4$ and $n \equiv 2 \pmod 4$ or $m \equiv 2 \pmod 4$ and $n \equiv 1,3 \pmod 4$, $C_m \triangleright r - C_{n,k}$ have order $mn \equiv 2 \pmod 4$. According to Theorem 1.2, the $ms(C_m \triangleright r - C_{n,k}) = \infty$. Based on the pattern of the edge labels, we divided the proof of $mn \not\equiv 2 \pmod 4$ into 2 cases, each case labeling its edge by $\delta_k$, for $k = 2,3$.

For $k = 2,3$, define the edge labeling $\beta_k : E(P_m \triangleright r - C_{n,k}) \to \{1,2,...,\left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\begin{aligned} \delta_k(x_i^j x_{i+1}^j) &= \gamma_k(x_i^j x_{i+1}^j), & 1 \le i \le n,\ 1 \le j \le m, \\ \delta_k(e) &= 1, & e \in M(r - C_{n,k}), e \notin \delta_k(x_i^j x_{i+1}^j), \\ \delta_k(x_1^j x_1^{j+1}) &= \gamma_k(x_1^j x_1^{j+1}), & 1 \le j \le m. \end{aligned}$$

We have the vertex weight

$$\begin{aligned} wt_{\delta_k}(x_i^j) &= wt_{\gamma_k}(x_i^j) + r - 2, & 2 \le i \le n,\ 1 \le j \le m, \\ wt_{\delta_k}(x_1^j) &= wt_{\gamma_k}(x_1^j) + r - 2, & 1 \le j \le m. \end{aligned}$$

For all cases, it can be checked that the edge label $\delta_k$ is the $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$ label for $k = 2,3$. Then, under the labeling $\beta_k$ the set weights of the vertices $\{wt_{\delta_k}(x_1^j) : 1 \le j \le m,\ 1 \le i \le n,\ k = 2,3\}$ attain the values $\{r, r+1, ..., mn+r-2, mn+r-1\}$. Thus, the set of modular weights $\{wt_{\delta_k}(x_1^j) \pmod{mn} : 1 \le j \le m,\ 1 \le i \le n,\ k = 2,3\}$ are $\{0,1,2,...,mn-1\}$, which implies that $ms(C_m \triangleright r - C_{n,k}) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil$. □

An example of modular irregular labeling of $C_4 \triangleright 3 - C_{8,4}$ can be seen in Figure 6.

Let $n, q$, and $a_1, a_2, ..., a_p, ..., a_q$ be positive integers, $1 \le a_1 < a_2 < ... < a_p < ... < a_q \le \left\lfloor \frac{n}{2} \right\rfloor$. An undirected graph with the set of vertices $V = \{x_i : 1 \le i \le n\}$ and the set of edges $E = x_i x_{i+a_p} : 1 \le i \le n, 1 \le p \le q$, then the indices $a_p$ being taken modulo $n$, is called a circulant graph and is denoted by $C_n(a_1, a_2, ..., a_p, ..., a_q)$.

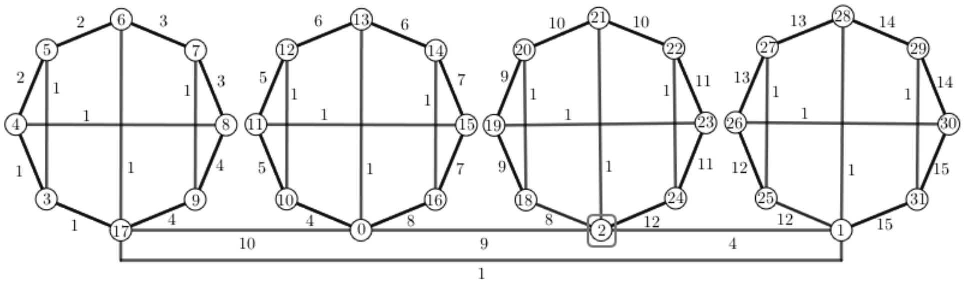

Figure 6. Example of Modular Irregular Labeling $C_4 \triangleright 3 - C_{8,4}$ with $\text{ms}(C_4 \triangleright 3 - C_{8,4}) = 15$

**Corollary 2.2.** *Let $C_m$ be a cycle graph of order $m$ and $C_n(1, a_2, ..., a_p, ..., a_q)$ be a circulant graph of order $n$. Then,*

$$ms(C_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \begin{cases} \infty, & m \equiv 1,3 \pmod 4,\ n \equiv 2 \pmod 4, \\ \infty, & m \equiv 2 \pmod 4,\ n \equiv 1,3 \pmod 4, \\ \left\lceil \dfrac{(n-1)m+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

*Proof.* $C_m \triangleright C_n(1, a_2, \ldots, a_p, \ldots, a_q)$ have $mn$ vertices with a vertex set $V(C_m \triangleright C_n(1, a_2, \ldots, a_p, \ldots, a_q)) = \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and an edge set $E(C_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \{x_1^j x_1^{j+1} : 1 \leq j \leq m\} \cup \{x_i^j x_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{x_i^j x_{i+a_p \pmod n}^j : 1 \leq i \leq n, 1 \leq j \leq m, 2 \leq p \leq q\}$. To simplify labeling, we change the edge notation $x_n^j x_{n+1}^j = x_n^j x_1^j$ and $x_1^m x_1^{m+1} = x_1^m x_1^1$.

For $m \equiv 1,3 \pmod 4$ and $n \equiv 2 \pmod 4$ or $m \equiv 2 \pmod 4$ and $n \equiv 1,3 \pmod 4$, $C_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)$ have order $mn \equiv 2 \pmod 4$. According to Theorem 1.2, the $ms(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \infty$. We divided the proof of $mn \not\equiv 2 \pmod 4$ by 6 cases, each case labeled its edge by $\delta_k$, for $k = 1, 2, .., 6$.

For $k = 1, 2, .., 6$, define the edge labeling $\delta_k : E(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) \rightarrow \{1, 2, \ldots, \left\lceil \frac{(n-1)m+1}{2} \right\rceil\}$ as follows.

$$\begin{aligned} \delta_k(x_i^j x_{i+1}^j) &= \gamma_k(x_i^j x_{i+1}^j), & 1 \leq i \leq n,\ 1 \leq j \leq m, \\ \delta_k(x_i^j x_{i+a_p}^j) &= 1, & 1 \leq i \leq n,\ 1 \leq j \leq m, \\ \delta_k(x_1^j x_1^{j+1}) &= \gamma_k(x_1^j x_1^{j+1}), & 1 \leq j \leq m-1. \end{aligned}$$

If $|a_p| = s$, we have the vertex weight

$$\begin{aligned} wt_{\delta_k}(x_i^j) &= wt_{\gamma_k}(x_i^j) + s, & 2 \leq i \leq n,\ 1 \leq j \leq m, \\ wt_{\delta_k}(x_1^j) &= wt_{\gamma_k}(x_1^j) + s, & 1 \leq j \leq m-1. \end{aligned}$$

For all cases, it can be checked that the edge label $\delta_k$ is the $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$ label for $k = 1, 2, ..., 6$. Then, under the labeling $\delta_k$, the set of vertex weights $\{wt_{\delta_k}(x_i^j) : 1 \leq j \leq m,\ 1 \leq i \leq n,\ 1 \leq$

$k \leq 4\}$ achieves the values $\{2+s, 3+s, ..., mn+s, mn+1+s\}$ and for $\{wt_{\delta_k}(x_i^j) : 1 \leq j \leq m,\ 1 \leq i \leq n,\ 5 \leq k \leq 6\}$ attains the values $\{3+s, 4+s, ..., mn+s, mn+1+s, mn+2+s\}$. Thus, the set of modular weights $\{wt_{\delta_k}(x_i^j) : 1 \leq j \leq m,\ 1 \leq i \leq n,\ 1 \leq k \leq 6\}$ is equal to $\{0, 1, 2, ..., mn-1\}$, which implies that $ms(C_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)) = \left\lceil \frac{(n-1)m+1}{2} \right\rceil$. □

Figure 7 shows an example of modular irregular labeling of $C_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q)$).

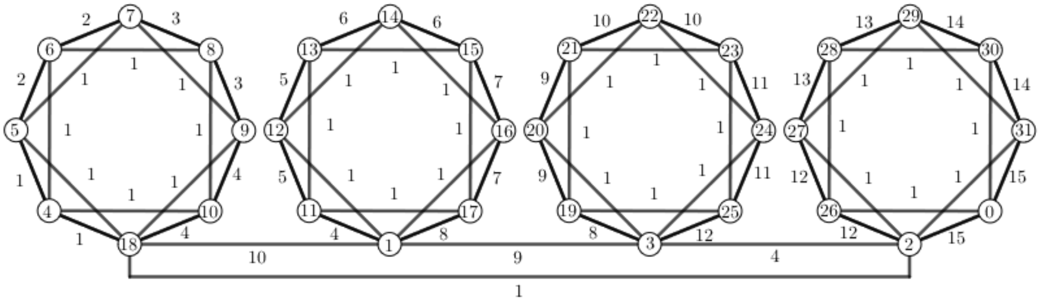

Figure 7. Example of Modular Irregular Labeling $C_4 \triangleright C_8(1, 2)$ with ms$(C_4 \triangleright C_8(1, 2)) = 15$

## 3. Conclusion

In this research, we prove that the modular irregularity strength of Dutch Windmill graph $D_4^m$ is $ms(D_4^m) = \lceil \frac{3m+1}{2} \rceil$ for $m \not\equiv 3 \pmod 4$. Furthermore, we obtained the exact value of modular irregularity strength of several comb product graphs, showing that all of them share a common formula. These graphs include: ms$(P_m \triangleright C_n)$, $ms(P_m \triangleright r - C_{n,k})$, ms$(P_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q))$, ms$(C_m \triangleright C_n)$, $ms(C_m \triangleright r - C_{n,k})$, and ms$(C_m \triangleright C_n(1, a_2, ..., a_p, ..., a_q))$. We found that for these specific graphs, the modular irregularity strength is $\left\lceil \frac{(n-1)m+1}{2} \right\rceil$. For other value of $m$ and $n$ (for the comb product not covered by the above conditions), we have the modular irregularity strength is $ms(G) = \infty$.

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