Electronic Journal of Graph Theory and Applications **14** (2) (2026), 347–361



Electronic Journal of
Graph Theory and Applications

On locating-dominating set of subgraph-amalgamation graphs

Darmaji[a], Suhadi W. Saputro[b]

[a]*Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia*
[b]*Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia*

darmaji@matematika.its.ac.id; suhadi@itb.ac.id

Abstract

The *locating-dominating number* of a graph G is defined as the minimum cardinality of $W \subseteq V(G)$ such that for every two distinct vertices $u, v \in V(G)\backslash W$, $\emptyset \neq N_G(u) \cap W \neq N_G(v) \cap W \neq \emptyset$. For $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a finite collection of simple connected graphs. The *subgraph-amalgamation* of $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ in J, denoted by $Subgraph - Amal\{\mathcal{H}; J\}$, is a graph obtained by identifying all elements of $\mathcal{H}$ in J. The subgraph J is called as a *terminal subgraph* of $\mathcal{H}$. In this paper, we determine general bounds of the locating-dominating number of subgraph-amalgamation graphs for any connected terminal subgraphs. We also determine the locating-dominating number of the vertex-amalgamation graph, which is $Subgraph - Amal\{\mathcal{H}; J\}$ where J is K_1.

Keywords: locating-dominating number, locating-dominating set, subgraph-amalgamation graph, vertex-amalgamation graph
Mathematics Subject Classification : 05C69; 05C76
DOI: 10.5614/ejgta.2026.14.2.7

1. Introduction

Fault diagnosis in multiprocessor systems is an application of locating-dominating set. Let G be a multiprocessor system where vertices are processors, and edges are links between processors.

Received: 1 November 2024, Revised: 11 February 2026, Accepted: 3 August 2026.

www.ejgta.org

Assume that at most one processor is malfunctioning. We want to test the system and find the faulty processor. Some processors can be chosen and assigned to check their neighbours. If a chosen processor detect a fault, it sends an alarm signal. In this case, we need an exact location of a faulty processor. So, we must choose some processors such that the chosen processors can uniquely tell the location of the malfunctioning processor. A locating-dominating set of G can be used to choose those processors.

Let G be a graph. For a vertex v of G, we define $N_G(v)$ as the set of vertices which are adjacent to v in G. A set of vertices $W \subseteq V(G)$ is called a *locating-dominating* set of a graph G if $\emptyset \neq N_G(u) \cap W \neq N_G(v) \cap W \neq \emptyset$ for any two distinct vertices $u, v \in V(G) \setminus W$. The *locating-dominating number* of G, denoted by $\lambda(G)$, is the minimum cardinality of all locating-dominating sets of G. This concept was introduced by Slater [26, 27].

It has been proven that the locating-dominating problem is NP-complete [10, 11, 16]. Furthermore, Cappelle *et al.* [9] have shown that the locating-dominating problem is also W[1]-hard when parameterized by the size of a minimum clique cover. However, an efficient algorithm to determine a locating-dominating set in certain classes of graph has been constructed. Slater [26] formulated a linear-time algorithm to solve the locating-dominating problem in a tree. Meanwhile Rajasekar and Nagarajan [24] provided the precise algorithm for finding the locating-dominating number of graphs which are connected by bridge.

All graphs with certain locating-dominating number have been characterized. Henning and Oellermann [19] have proved that for a connected graph G of order $n \geq 2$, $\lambda(G) = n - 1$ if and only if G is either complete graphs K_n or star graphs $K_{1,n-1}$. They also characterized all graphs G of order $n \geq 4$ with locating-dominating number $n - 2$. Meanwhile, Caceres *et al.* [7] have proved that there are 16 non-isomorphic graphs G satisfying $\lambda(G) = 2$.

Some authors also obtained the locating-dominating number of certain class of graphs. Some of them are paths [27], cycles [4], stars [1], complete graphs [19], bipartite graphs [7, 20], wheels [7], twin-free graphs [15, 18], and hypergraphs [13]. Blidia *et al.* [5] have been investigated the locating-dominating set of trees. The locating-dominating number of some regular graphs can be seen in [4, 7, 17]. Esther and Vivik [12] studied the locating-dominating number of prism, antiprism, crossed prism, and circular ladder prism graph. Meanwhile a locating-dominating set of graphs with girth at least 5 has been investigated in [3].

Some authors also have determined the locating-dominating number of graphs obtained from a product graphs. The locating-dominating number of corona and composition product graphs have been determined in [8]. Pribadi and Saputro [23] investigated the locating-dominating number of comb product of any two connected graphs. The locating-dominating number of a Cartesian product graph between certain graphs have been determined in [25]. The locating-dominating number of functigraphs of complete graphs has been studied by Murtaza *et al.* [22]. Moreover, Argiroffo *et al.* [2] investigated a locating-dominating set of graph by adding a universal vertex.

In this paper, we consider the subgraph-amalgamation graphs, which is a graph operation where multiple graphs are joined by identifying them at a terminal subgraph. One can obtain a larger graph synthesized from small graphs. Its application primarily focus on simplifying the study of complex networks by breaking them into smaller manageable structures while preserving specific properties [29].

We are using the definition of the subgraph-amalgamation graph which is stated in [14]. For

www.ejgta.org

$n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a finite collection of simple connected graphs. The *subgraph-amalgamation* of $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ in J, denoted by $Subgraph - Amal\{\mathcal{H}; J\}$, is a graph obtained by identifying all elements of $\mathcal{H}$ in J. The subgraph J is called as a *terminal subgraph* of $\mathcal{H}$.

This graphs have been widely investigated in domination parameters, including the domination number [6, 28] and the restrained domination number [21]. Note that for every vertex $v \in V(H)$ where $H \in \mathcal{H}$, if $v \notin J$, then we have $N_{Subgraph-Amal\{\mathcal{H},J\}}(v) = N_H(v)$, otherwise, we have $N_H(v) \subseteq N_{Subgraph-Amal\{\mathcal{H},J\}}(v)$. In domination parameters, by considering these properties, a dominating set of subgraph-amalgamation graph may be constructed by combining a dominating set from each element of $\mathcal{H}$. However, in case of locating-dominating parameter, this set may be not optimized. We need to explore some other properties of subgraph-amalgamation graph to obtain a locating-dominating set with minimum cardinality.

The purpose of this paper is to further investigate the locating-dominating number of subgraph-amalgamation graphs. We obtain two main results, one of them is the following result related with the general bounds of the locating-dominating number of subgraph-amalgamation graphs for any connected terminal subgraphs.

Theorem 1.1. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph containing a connected subgraph J, and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Then*

$$\sum_{i=1}^{n}(\lambda(H_i) - |V(J)|) \leq \lambda(Subgraph - Amal\{\mathcal{H}; J\}) \leq \sum_{i=1}^{n}\lambda(H_i).$$

The bounds are sharp.

Our second result is related with a subgraph-amalgamation graph $Subgraph - Amal\{\mathcal{H}; J\}$ where J is a vertex. We call this case of subgraph-amalgamation graph as the vertex-amalgamation graph. For $H \in \mathcal{H}$, let x be a vertex of H which is also belong to the terminal subgraph J. By the definition of subgraph-amalgamation graph, x is the only vertex of H which is adjacent to some other vertices in $K \in \mathcal{H} \setminus \{H\}$. It follows that we may choose sets $S_H \subseteq V(H)$ and $S_K \subseteq V(K)$ such that $S_H \cup S_K$ is a subset of a locating-dominating set of a vertex-amalgamation graph where $N_H(x) \cap S_H \neq \emptyset$ or $N_K(x) \cap S_K \neq \emptyset$. So, we need to classify the graph $H \in \mathcal{H}$ based on the vertex x and its locating-dominating set. We say that:

- H is of type $\mathcal{A}_x$, if every locating-dominating set of H with $\lambda(H)$ vertices does not contain x.
- Otherwise, H is of type $\mathcal{B}_x$, if there exists a locating-dominating set W of H with $\lambda(H)$ vertices containing x, such that $D = W \setminus \{x\}$ is a locating-dominating set of $H \setminus \{x\}$ and $N_H(x) \cap D \neq \emptyset$.
- Otherwise, H is of type $\mathcal{C}_x$, if there exists a locating-dominating set W of H with $\lambda(H)$ vertices containing x, such that $D = W \setminus \{x\}$ is a locating-dominating set of $H \setminus \{x\}$ and $N_H(x) \cap D = \emptyset$.

www.ejgta.org

- Otherwise, H is of type $\mathcal{D}_x$, if there exists a locating-dominating set W of H with $\lambda(H)$ vertices containing x, such that $D = W \setminus \{x\}$ is not a locating-dominating set of $H \setminus \{x\}$ and for every $y \in N_H(x)$, $N_H(y) \cap D \neq \emptyset$.
- Otherwise, H is of type $\mathcal{E}_x$, that is every locating-dominating set W of H with $\lambda(H)$ vertices containing x, satistifes $D = W \setminus \{x\}$ is not a locating-dominating set of $H \setminus \{x\}$ and there exists $y \in N_H(x)$ such that $N_H(y) \cap D = \emptyset$.

Note that, the type of H is based on the vertex x whose belong to J. Let us consider three examples as follows. First, let H_1 be a path P_5 of order 5. If $x \in V(H_1)$ which is adjacent to a vertex of degree one, then H_1 is of type $\mathcal{E}_x$, otherwise H_1 is of type $\mathcal{A}_x$. Second, let H_2 be a path P_6 of order 6. If $x \in V(H_2)$ which is adjacent to a vertex of degree one, then H_2 is of type $\mathcal{E}_x$. Otherwise, if x is a vertex of degree one, then H_2 is of type $\mathcal{C}_x$, otherwise H_2 is of type $\mathcal{D}_x$. Third, let H_3 be a star graph $K_{1,n}$ of order $n+1$. If $x \in V(H_3)$ is a vertex of degree one, then H_3 is of type $\mathcal{B}_x$, otherwise H_3 is of type $\mathcal{E}_x$.

In Theorem 1.2 below, we give an exact value of the locating-dominating number of any vertex-amalgamation graphs.

Theorem 1.2. *For $n \geq 2$ and $i \in \{1, 2, \ldots, n\}$, let H_i be a simple connected graph where $x \in V(H_i)$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Let $J \subseteq H_i$, where $V(J) = \{x\}$, and G be a subgraph-amalgamation of $\mathcal{H}$ in J. For $Z \in \{\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{E}\}$, let t_Z be the number of type Z_x in $\mathcal{H}$. Then*

$$\lambda(G) = \begin{cases} t_{\mathcal{A}} + \sum_{i=1}^{n}(\lambda(H_i) - 1), & \text{if } t_{\mathcal{D}} = t_{\mathcal{E}} = 0 \text{ and } (t_{\mathcal{A}} \geq 1 \text{ or } t_{\mathcal{B}} \geq 2), \\ t_{\mathcal{A}} + t_{\mathcal{E}} + \sum_{i=1}^{n}(\lambda(H_i) - 1), & \text{if } t_{\mathcal{E}} \geq 1, \\ t_{\mathcal{A}} + 1 + \sum_{i=1}^{n}(\lambda(H_i) - 1), & \text{otherwise}. \end{cases}$$

2. General Bounds and Proof of Theorem 1.1

Theorem 1.1 is a direct consequence of Lemma 2.1, Lemma 2.2, and Theorem 2.1 in this section.

Let us consider $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ where H_i $(1 \leq i \leq n)$ is a connected graph containing a connected subgraph J. Let G be a subgraph-amalgamation graph of $\mathcal{H}$ in J. We define $V(G) = V(J) \cup \{(i, a) | 1 \leq i \leq n;\ a \in V(H_i) \setminus V(J)\}$. We also define $H_i^* = \{(i, a) \mid a \in V(H_i) \setminus V(J)\}$. The lower bound in Theorem 1.1 is a direct consequence of Lemma 2.1 below.

Lemma 2.1. *Let W be a locating-dominating set of a subgraph-amalgamation graph G of $\mathcal{H}$ in J. Then every graph $H \in \mathcal{H}$ contributes at least $\lambda(H) - |V(J)|$ vertices in W.*

Proof. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. Suppose that there exists $i \in \{1, 2, \ldots, n\}$ such that H_i contributes at most $\lambda(H_i) - |V(J)| - 1$ vertices in W. Let $W_i = W \cap H_i^*$ and $B_i = \{b \in V(H_i) \mid (i, b) \in W_i\}$. Now, we define $S_i = B_i \cup V(J)$. Note that $|S_i| \leq \lambda(H_i) - 1$.

If there exists a vertex $x \in V(H_i) \setminus S_i$ such that $N_{H_i}(x) \cap S_i = \emptyset$, then it is clear that $N_G((i, x)) \cap W_i = \emptyset$. It follows that W is not a locating-dominating set of G, a contradiction.

Now, we assume that every vertex $x \in V(H_i) \setminus S_i$ satisfies $N_{H_i}(x) \cap S_i \neq \emptyset$. Since $|S_i| \leq \lambda(H_i) - 1$, there exist two distinct vertices $x, y \in V(H_i) \setminus S_i$ such that $N_{H_i}(x) \cap S_i = N_{H_i}(y) \cap S_i$. It follows that $N_G((i, x)) \cap W_i = N_G((i, y)) \cap W_i$, which implies $N_G((i, x)) \cap W = N_G((i, y)) \cap W$, a contradiction with W is a locating-dominating set of G. □

www.ejgta.org

In order to show the upper bound in Theorem 1.1, first we need to define some notations. Let S_i be a locating-dominating set of H_i. We define $S_i^* = S_i \setminus V(J)$ and $J_i = S_i \cap V(J)$. Now, we consider some properties for two different vertices in G which correspond to $u_i \in V(H_i) \setminus S_i$ and $u_j \in V(H_j) \setminus S_j$.

Proposition 2.1. *Let u_i and u_j be vertices in $V(J) \setminus J_i$ and $V(J) \setminus J_j$, respectively. Let $W = \{(a,b)|a \in \{i,j\};\ b \in S_i^* \cup S_j^*\} \cup J_i \cup J_j$. If u_i and u_j are different vertices in J of G which also are not elements of W, then $N_G(u_i) \cap W \neq N_G(u_j) \cap W$.*

Proof. By the definition of the subgraph-amalgamation of $\mathcal{H}$ in J, all vertices of J are also elements of $V(H_i)$. Thus, it is clear that $u_i, u_j \in V(H_i)$. Since $u_i, u_j \notin W$, it follows that $u_i, u_j \notin S_i$. Then there exists a vertex $z \in S_i$ such that $z \in N_{H_i}(u_i)$ but $z \notin N_{H_i}(u_j)$. If $z \in S_i^*$, then we have $(i,z) \in N_G(u_i)$ but $(i,z) \notin N_G(u_j)$. In the other hand, if $z \in J_i$, then we have $z \in N_G(u_i)$ but $z \notin N_G(u_j)$. All cases follow $N_G(u_i) \cap W \neq N_G(u_j) \cap W$. □

Proposition 2.2. *Let u_i and u_j be vertices in $V(J) \setminus J_i$ and $V(H_j) \setminus (S_j \cup J)$, respectively. Let $W = \{(a,b)|a \in \{i,j\};\ b \in S_i^* \cup S_j^*\} \cup J_i \cup J_j$. If u_i in J of G is not an element of W, then $N_G(u_i) \cap W \neq N_G((j,u_j)) \cap W$.*

Proof. Proposition is completed by using the similar argument of Proposition 2.1. □

Proposition 2.3. *Let u_i and u_j be vertices in $V(H_i) \setminus (S_i \cup J)$ and $V(H_j) \setminus (S_j \cup J)$, respectively. Let $W = \{(a,b)|a \in \{i,j\};\ b \in S_i^* \cup S_j^*\} \cup J_i \cup J_j$. If $N_{H_i}(u_i) \cap S_i^* \neq \emptyset$ or $N_{H_j}(u_j) \cap S_j^* \neq \emptyset$, then $N_G((i,u_i)) \cap W \neq N_G((j,u_j)) \cap W$.*

Proof. We assume that $N_{H_i}(u_i) \cap S_i^* \neq \emptyset$. Let $v_i \in N_{H_i}(u_i) \cap S_i^*$. Thus, it is clear that $(i,v_i) \in W$. By the definition of the subgraph-amalgamation of $\mathcal{H}$ in J, it is easy to verify that (i,v_i) is not adjacent to (j,u_j). Therefore, we obtain $N_G((i,u_i)) \cap W \neq N_G((j,u_j)) \cap W$. □

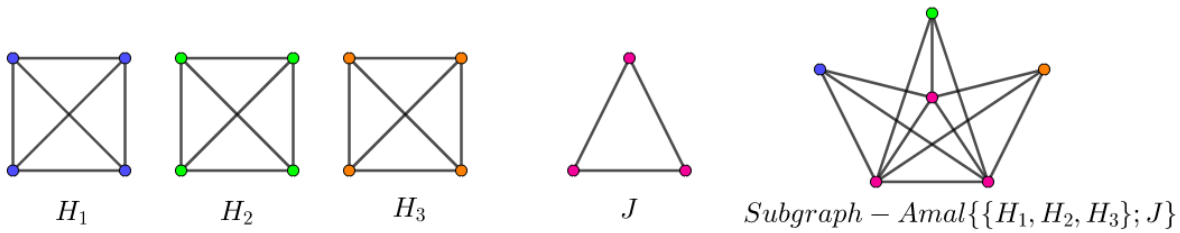


Figure 1. An example of $Subgraph - Amal\{\{H_1, H_2, H_3\}; J\}$.

Note that according to Propositions 2.1-2.3, we have a possibility for $u_i \in V(H_i) \setminus S_i$ and $u_j \in V(H_j) \setminus S_j$ such that $N_G((i,u_i)) \cap W = N_G((j,u_j)) \cap W$ where $W = \{(a,b)|a \in \{i,j\};\ b \in S_i^* \cup S_j^*\} \cup J_i \cup J_j$. It means that the union set $\bigcup_{i=1}^{n} (\{(i,u) \mid u \in S_i^*\} \cup J_i)$ may be not a locating-dominating set of G.

For an example, let us consider a subgraph-amalgamation graph G as illustrated in Figure 1. For $1 \leq i \leq 3$, let $V(H_i) = \{x_j^i \mid 1 \leq j \leq 4\}$. Note that H_i is isomorphic to K_4. Henning

www.ejgta.org

and Oellermann [19] have been determined that $\lambda(K_4) = 3$. Thus, any sets $S_i \subseteq V(H_i)$ with cardinality 3 is a locating-dominating set of H_i.

Now, assume that $V(J) = \{y_1, y_2, y_3\}$ where $y_j = x_j^i$. Let $W = \bigcup_{i=1}^{3} (\{(i, u) \mid u \in S_i^*\} \cup J_i)$. If $S_1 = \{x_1^1, x_2^1, x_4^1\}$, $S_2 = \{x_2^2, x_3^2, x_4^2\}$, and $S_3 = \{x_2^3, x_3^3, x_4^3\}$, then $W = V(G)$. It follows that W is a locating-dominating set of the subgraph-amalgamation graph G. However, if for $1 \leq i \leq 3$, $S_i = \{x_j^i \mid 1 \leq j \leq 3\}$, then $W = V(J)$. Since $N_G((1, x_4^1)) \cap W = V(J) = N_G((2, x_4^2)) \cap W$, it is clear that W is not a locating-dominating set of G. Therefore, we need to construct a new set in G such that this set is always a locating-dominating set of G.

The general construction of a locating-dominating set for any subgraph-amalgamation graphs, is given in Algorithm 1 below.

Algorithm 1 Locating-dominating set of $Subgraph - Amal\{\mathcal{H}; J\}$

Input: $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, J, and a locating-dominating set S_i of H_i with $\lambda(H_i)$ elements.
Output: W

1. **for** $i = 1 \to n$ **do** Define $S_i^* = S_i \setminus V(J)$ and $J_i = S_i \cap V(J)$;
2. Let $K_1 = \emptyset$, $W_J = J_1$, and $W^* = \{(1, a) | a \in S_1^*\}$;
3. Let $i := 2$ and $K_i = \emptyset$;
4. **if** $J_i \cap W_J \neq \emptyset$
   then For every element $u \in J_i \cap W_J$, if there exists a neighbor vertex of u in $V(H_i) \setminus (S_i \cup J)$ which is not adjacent to all vertices in S_i^*, then put it into a set K_i;
5. Define $W_J := W_J \cup J_i$ and $W^* := W^* \cup \{(i, a) | a \in S_i^* \cup K_i\}$;
6. Define $i := i + 1$;
7. **if** $i \leq n$
   then Repeat Step 4
   else Define $W = W_J \cup W^*$.

Let us reconsider a subgraph-amalgamation graph G as illustrated in Figure 1 where for $1 \leq i \leq 3$, $V(H_i) = \{x_j^i \mid 1 \leq j \leq 4\}$ and $V(J) = \{y_1, y_2, y_3\}$ with $y_j = x_j^i$. If $S_1 = \{x_1^1, x_2^1, x_4^1\}$, $S_2 = \{x_2^2, x_3^2, x_4^2\}$, and $S_3 = \{x_2^3, x_3^3, x_4^3\}$, then according to Algorithm 1, we obtain a set $W = V(G)$, which is implies W is a locating-dominating set of G. Now, for $1 \leq i \leq 3$, let $S_i = \{x_j^i \mid 1 \leq j \leq 3\}$. So, $S_i^* = S_i \setminus V(J) = \emptyset$ and $J_i = S_i \cap V(J) = V(J)$. By using Algorithm 1, for every iteration, the sets W_J, K_i, and W^* can be seen in the following table.

i	W_J	K_i	W^*
1	$V(J)$	$\emptyset$	$\emptyset$
2	$V(J)$	$\{x_4^2\}$	$\{(2, x_4^2)\}$
3	$V(J)$	$\{x_4^3\}$	$\{(2, x_4^2), (3, x_4^*)\}$

Thus, we obtain $W = V(J) \cup \{(2, x_4^2), (3, x_4^*)\}$. Since $|W| = |V(G)| - 1$, it is clear that W is a locating-dominating set of G. Note that, the set K_i is needed in case there is a vertex

www.ejgta.org

$z_i \in V(H_i)\backslash S_i$ such that $N_{H_i}(z_i) \cap S_i^* = \emptyset$. So that, if there are $z_i \in V(H_i)\backslash S_i$ and $z_j \in V(H_j)\backslash S_j$ with $i \neq j$, satisfying this condition, then we will obtain $(i, z_i) \in W$ or $(j, z_j) \in W$.

In Lemma 2.2, we show that the set obtained from Algorithm 1 is a locating-dominating set of G with at most $\sum_{i=1}^{n} \lambda(H_i)$ elements.

Lemma 2.2. *Let W be a set obtained from Algorithm 1. Then W is a locating-dominating set of G with at most $\sum_{i=1}^{n} \lambda(H_i)$ elements.*

Proof. Let S_i be a locating-dominating set of H_i with $\lambda(H_i)$ elements. According to Algorithm 1 above, since $W = W_J \cup W^* = \bigcup_{i=1}^{n}(J_i \cup \{(i, a)|a \in S_i^* \cup K_i\})$ and

$$|W_J \cup W^*| = \sum_{i=1}^{n} (|J_i| + |S_i^*| - |K_i|) \leq \sum_{i=1}^{n} (|J_i| + |S_i^*|) = \sum_{i=1}^{n} \lambda(H_i)$$

we obtain that W contains at most $\sum_{i=1}^{n} \lambda(H_i)$ elements. Now, we will show that W is a locating-dominating set of $Subgraph - Amal\{\mathcal{H}; J\}$.

Let x and y be two distinct vertices in $V(G) \setminus W$. If there exists $i \in \{1, 2 \ldots, n\}$ such that the corresponding vertex x' of x, and y' of y are in H_i, then $N_{H_i}(x') \cap S_i \neq N_{H_i}(y') \cap S_i$ which implies $N_G(x) \cap W \neq N_G(y) \cap W$ since $J_i \cup \{(i, a)|a \in S_i^*\} \subseteq W$. Now, we assume that the corresponding vertex x' of x, and y' of y is in H_i and H_j, respectively. So, x' and y' must satisfy the condition in Proposition 2.1, Proposition 2.2, or Proposition 2.3, which implies $N_G(x) \cap W \neq N_G(y) \cap W$. Otherwise, we have $N_{H_i}(x') \cap S_i^* = \emptyset$ and $N_{H_j}(y') \cap S_j^* = \emptyset$. According to Steps 4 and 5 of Algorithm 1, it follows $x \in W$ or $y \in W$, a contradiction. □

In order to show that the lower and upper bounds in Theorem 1.1 are sharp, let us consider a complete bipartite graph $K_{r,s}$ of order $r + s$ $(r, s \geq 3)$, and the graph G of order 10 which can be seen in Figure 2 below. Henning and Oellermann [19] have been proven that $\lambda(K_{r,s}) = r + s - 2$. It is easy to verify that two vertices which are not in the locating-dominating set with cardinality $r + s - 2$, are adjacent. Meanwhile, in lemma below, we will show that the locating-dominating number of the graph G in Figure 2 is 4 and the set $\{z_2, z_4, z_7, z_9\}$ is the only locating-dominating set of G with 4 elements.

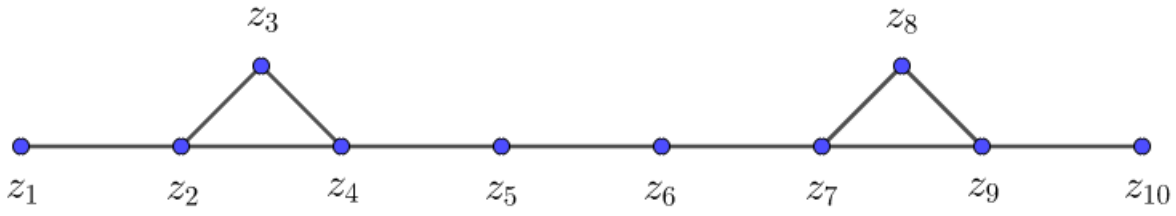


Figure 2. A graph G of order 10.

Proposition 2.4. *Let G be a connected graph as illustrated in Figure 2. If S is a locating-dominating set of G, then at least two vertices in $\{z_1, z_2, z_3, z_4\}$ are elements of S.*

www.ejgta.org

Proof. Suppose that $|\{z_1, z_2, z_3, z_4\} \cap S| \leq 1$. If $z_2 \in S$, then we obtain $N_G(z_1) \cap S = N_G(z_3) \cap S$, a contradiction. Thus, we assume that $z_2 \notin S$. However, there exists $v \in \{z_1, z_3\}$ such that $N_G(v) \cap S = \emptyset$, a contradiction. □

Lemma 2.3. *Let G be a connected graph as illustrated in Figure 2. Then $\lambda(G) = 4$. Moreover, $\{z_2, z_4, z_7, z_9\}$ is the only locating-dominating set of G with 4 vertices.*

Proof. The lower bound $\lambda(G) \geq 4$ is a direct consequences of Proposition 2.4.

Now, define $W = \{z_2, z_4, z_7, z_9\}$. Note that

$$\begin{array}{ll} N_G(z_1) \cap W = \{z_2\}, & N_G(z_6) \cap W = \{z_7\}, \\ N_G(z_3) \cap W = \{z_2, z_4\}, & N_G(z_8) \cap W = \{z_7, z_9\}, \\ N_G(z_5) \cap W = \{z_4\}, & N_G(z_{10}) \cap W = \{z_9\}. \end{array}$$

Since every distinct vertices $u, v \in V(G) \setminus W$ satisfies $\emptyset \neq N_G(u) \cap W \neq N_G(v) \cap W \neq \emptyset$, the set W is a locating-dominating set of G.

Let S be a locating-dominating set of G with 4 elements. First, let $z_2 \notin S$. It follows that $z_1 \in S$. If $|\{z_3, z_4, z_5, z_6\} \cap S| = 1$, then we have a contradiction since $N_G(z_3) \cap S = N_G(z_5) \cap S$ or there exists $v \in \{z_3, z_4, z_5\}$ such that $N_G(v) \cap S = \emptyset$. Thus, assume that $|\{z_3, z_4, z_5, z_6\} \cap S| \geq 2$. If there exists $v \in \{z_8, z_9, z_{10}\}$ such that $N_G(v) \cap S = \emptyset$, then we have a contradiction. Otherwise, $z_9 \in S$ which implies $N_G(z_8) \cap S = N_G(z_{10}) \cap S$, a contradiction.

Now, assume that $z_2 \in S$ but $z_4 \notin S$. It follows that $z_1 \in S$ or $z_3 \in S$. If $|\{z_6, z_7, \ldots, z_{10}\} \cap S| = 1$, then we have a contradiction since $N_G(z_8) \cap S = N_G(z_{10}) \cap S$ or there exists $v \in \{z_7, z_8, z_9, z_{10}\}$ such that $N_G(v) \cap S = \emptyset$. So, it must be $|\{z_6, z_7, \ldots, z_{10}\} \cap S| = 2$. If $z_6 \notin S$, then we have $N_G(z_5) \cap S = \emptyset$, a contradiction. Thus, it must be $z_6 \in S$. If there exists $v \in \{z_8, z_9, z_{10}\}$ such that $N_G(v) \cap S = \emptyset$, then we have a contradiction. Otherwise, $z_9 \in S$ which implies that $N_G(z_8) \cap S = N_G(z_{10}) \cap S$, a contradiction. □

Now, we are ready to determine the locating-dominating number of a subgraph-amalgamation graph which obtained from a collection $\mathcal{H}$ of n graphs, that consists of r graphs G as illustrated in Figure 2 and $n-r$ complete bipartite graphs $K_{3,3}$, and the terminal subgraph is an edge. For every graph G, we assume that the terminal subgraph J is an edge z_5z_6. In case $r = 0$, the locating-dominating number of this subgraph-amalgamation graph will attain the lower bound of Theorem 1.1. Meanwhile, the locating-dominating number of this graph will reach the upper bound of Theorem 1.1 when there is no $K_{3,3}$ in $\mathcal{H}$.

Theorem 2.1. *For $n \geq 2$ and $r \in \{0, 1, \ldots, n\}$, let $\mathcal{H}$ consists of r connected graphs G as illustrated in Figure 2, and $n - r$ complete bipartite graphs $K_{3,3}$. Let J be a path of order two. If every graph G is identified on an edge z_5z_6, then*

$$\lambda(Subgraph - Amal\{\mathcal{H}; J\}) = 2n + 2r.$$

Proof. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ where the first r graphs are isomorphic to G. Let $V(K_{3,3}) = \{a_i, b_i \mid 1 \leq i \leq 3\}$ where $E(K_{3,3}) = \{a_ib_j \mid i, j \in \{1, 2, 3\}\}$. WLOG, let an edge a_1b_1 be a terminal subgraph of $K_{3,3}$. Let Γ be a subgraph-amalgamation graph of $\mathcal{H}$ in J.

www.ejgta.org

For $i \in \{1, 2, \ldots, n\}$, let $H_i \in \mathcal{H}$. Let B be a locating-dominating set of Γ. By considering Lemma 2.1, H_i contributes at least $\lambda(H_i) - 2$ vertices in B. Since $\lambda(K_{3,3}) = 4$ [19], if $H_i \cong K_{3,3}$, then H_i contributes at least 2 vertices in B.

Now, let $H_i \cong G$. In Lemma 2.3, we prove that $\lambda(G) = 4$. Therefore, according to Lemma 2.1, the graph G in $\mathcal{H}$ contributes at least 2 vertices in B. Moreover, we will show that G contributes at least 4 vertices in B. By Lemma 2.3, all locating-dominating sets of G with $\lambda(G)$ vertices do not contain z_5 and z_6. Note that for $k \in \{1, 2, 3, 4\}$ (or $k \in \{7, 8, 9, 10\}$), we have $N_\Gamma((i, z_k)) = \{(i, z_t) \mid z_t \in N_G(z_k)\}$. By using the similar proof of Proposition 2.4, at least two vertices in $\{(i, z_1), (i, z_2), (i, z_3), (i, z_4)\}$ (or $\{(i, z_7), (i, z_8), (i, z_9), (i, z_{10})\}$) are elements of B. Thus, we obtain that G contributes at least 4 vertices. Therefore, we have $\lambda(\Gamma) \geq 4r + 2(n - r) = 2n + 2r$.

Now, define

$$W = \begin{cases} \{(i, z_j) \mid 1 \leq i \leq n;\ j \in \{2, 4, 7, 9\}\}, & \text{if } r = n, \\ \{(i, z_j) \mid 1 \leq i \leq r;\ j \in \{2, 4, 7, 9\}\} \cup \{(i, a_2), (i, b_2) \mid r + 1 \leq i \leq n\}, & \text{if } 0 < r < n, \\ \{(i, a_2), (i, b_2) \mid 1 \leq i \leq n\}, & \text{if } r = 0. \end{cases}$$

Note that $|W| = 2n + 2r$. We will show that W is a locating-dominating set of Γ.

For the vertices which are not in J, we obtain

$$\begin{array}{cc} N_\Gamma((i, a_3)) \cap W = \{(i, b_2)\}, & N_\Gamma((i, b_3)) \cap W = \{(i, a_2)\}, \\ N_\Gamma((i, z_1)) \cap W = \{(i, z_2)\}, & N_\Gamma((i, z_3)) \cap W = \{(i, z_2), (i, z_4)\}, \\ N_\Gamma((i, z_8)) \cap W = \{(i, z_7), (i, z_9)\}, & N_\Gamma((i, z_{10})) \cap W = \{(i, z_9)\}. \end{array}$$

Now, let us consider two distinct vertices x and y of J. For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, let $H_i, H_j \in \mathcal{H}$.

- $H_i, H_j \cong G$

  If the vertex z_5 of H_i and the vertex z_5 of H_j are identified to x, then $(i, z_4), (j, z_4) \in N_\Gamma(x) \cap W$ and $(i, z_7), (j, z_7) \in N_\Gamma(y) \cap W$. If the vertex z_5 of H_i and the vertex z_6 of H_j are identified to x, then $(i, z_4), (j, z_7) \in N_\Gamma(x) \cap W$ and $(i, z_7), (j, z_4) \in N_\Gamma(y) \cap W$.

- $H_i \cong G$ and $H_j \cong K_{3,3}$

  If the vertex z_5 of H_i and the vertex a_1 of H_j are identified to x, then $(i, z_4), (j, b_2) \in N_\Gamma(x) \cap W$ and $(i, z_7), (j, a_2) \in N_\Gamma(y) \cap W$. If the vertex z_5 of H_i and the vertex b_1 of H_j are identified to x, then $(i, z_4), (j, a_2) \in N_\Gamma(x) \cap W$ and $(i, z_7), (j, b_2) \in N_\Gamma(y) \cap W$.

- $H_i, H_j \cong K_{3,3}$

  If the vertex a_1 of H_i and the vertex a_1 of H_j are identified to x, then $(i, b_2), (j, b_2) \in N_\Gamma(x) \cap W$ and $(i, a_2), (j, a_2) \in N_\Gamma(y) \cap W$. If the vertex a_1 of H_i and the vertex b_1 of H_j are identified to x, then $(i, b_2), (j, a_2) \in N_\Gamma(x) \cap W$ and $(i, a_2), (j, b_2) \in N_\Gamma(y) \cap W$.

Since every distinct vertices $u, v \in V(\Gamma) \setminus W$ satisfies $\emptyset \neq N_\Gamma(u) \cap W \neq N_\Gamma(v) \cap W \neq \emptyset$, the set W is a locating-dominating set of Γ. □

www.ejgta.org

3. Vertex-Amalgamation Graph and Proof of Theorem 1.2

In this section, we assume that every $H \in \mathcal{H}$ satisfies $x \in V(H)$ where for the terminal subgraph J, $V(J) = \{x\}$. So, $|J| = 1$. Let G be a subgraph-amalgamation of $\mathcal{H}$ in J. Note that G is a vertex-amalgamation graph. We define G_i as an induced subgraph of G by $\{(i,a)|a \in V(H_i)\} \cup \{x\}$. It is easy to see that G_i is isomorphic to H_i. According to Lemma 2.1 and Algorithm 1, we obtain a direct consequence which can be seen in proposition below.

Proposition 3.1. *Let W be a locating-dominating set of a vertex-amalgamation G of $\mathcal{H}$ in J. Then every graph $H \in \mathcal{H}$ contributes $\lambda(H) - 1$ or $\lambda(H)$ vertices in W.*

Thus, by considering Proposition 3.1 above, we characterize $H \in \mathcal{H}$ based on the only vertex x of H in J. We recall that:

- H is of type $\mathcal{A}_x$, if every locating-dominating set of H with $\lambda(H)$ vertices does not contain x.
- Otherwise, H is of type $\mathcal{B}_x$, if there exists a locating-dominating set W of H with $\lambda(H)$ vertices containing x, such that $D = W \setminus \{x\}$ is a locating-dominating set of $H \setminus \{x\}$ and $N_H(x) \cap D \neq \emptyset$.
- Otherwise, H is of type $\mathcal{C}_x$, if there exists a locating-dominating set W of H with $\lambda(H)$ vertices containing x, such that $D = W \setminus \{x\}$ is a locating-dominating set of $H \setminus \{x\}$ and $N_H(x) \cap D = \emptyset$.
- Otherwise, H is of type $\mathcal{D}_x$, if there exists a locating-dominating set W of H with $\lambda(H)$ vertices containing x, such that $D = W \setminus \{x\}$ is not a locating-dominating set of $H \setminus \{x\}$ and for every $y \in N_H(x)$, $N_H(y) \cap D \neq \emptyset$.
- Otherwise, H is of type $\mathcal{E}_x$, that is every locating-dominating set W of H with $\lambda(H)$ vertices containing x, satisfies $D = W \setminus \{x\}$ is not a locating-dominating set of $H \setminus \{x\}$ and there exists $y \in N_H(x)$ such that $N_H(y) \cap D = \emptyset$.

In order to prove Theorem 1.2, first we need to define or recall some notations. Let S_i be a locating-dominating set of H_i. We recall $S_i^* = S_i \setminus V(J)$ and $J_i = S_i \cap V(J)$. For a locating-dominating set W of a vertex-amalgamation graph G, we recall $W_i = \{(i,b) \in V(G) \mid (i,b) \in W\}$ and $B_i = \{b \in V(H_i) \mid (i,b) \in W_i\}$. For $w \in W$, we define $W_w = W \setminus \{w\}$.

In Proposition 3.2 below, we give a property of $H \in \mathcal{H}$ which is of type $\mathcal{A}_x$. This proposition is needed to show the vertices property of H of type $\mathcal{A}_x$ that will be contributed to a locating-dominating set of a vertex-amalgamation graph G, which can be seen in Lemma 3.1. Furthermore, in Lemma 3.1 we also provide the vertices property for the others type of H.

Proposition 3.2. *Let $x \in V(H)$ and H be a graph of type $\mathcal{A}_x$. Let W be a locating-dominating set of H with $\lambda(H)$ vertices and $w \in W$. Then there exists a vertex $p \in V(H) \setminus (W_w \cup \{x\})$ such that $N_H(p) \cap W_w = \emptyset$ or there exist two distinct vertices $p, q \in V(H) \setminus (W_w \cup \{x\})$ such that $N_H(p) \cap W_w = N_H(q) \cap W_w$.*

www.ejgta.org

Proof. Suppose that every vertex $p \in V(H) \setminus (W_w \cup \{x\})$ satisfies $N_H(p) \cap W_w \neq \emptyset$ and every distinct vertices $p, q \in V(H) \setminus (W_w \cup \{x\})$ satisfies $N_H(p) \cap W_w \neq N_H(q) \cap W_w$. Define $W^* = W_w \cup \{x\}$. We distinguish three cases.

1. x is adjacent to both p and q
   Then $N_H(p) \cap W^* = (N_H(p) \cap W_w) \cup \{x\} \neq (N_H(q) \cap W_w) \cup \{x\} = N_H(q) \cap W^*$.
2. x is adjacent to p but x is not adjacent to q
   Then $N_H(p) \cap W^* = (N_H(p) \cap W_w) \cup \{x\} \neq N_H(q) \cap W_w = N_H(q) \cap W^*$.
3. x is not adjacent to both p and q
   Then $N_H(p) \cap W^* = N_H(p) \cap W_w \neq N_H(q) \cap W_w = N_H(q) \cap W^*$.

According to all cases above, we obtain $N_H(p) \cap W^* \neq N_H(q) \cap W^*$. It follows that W^* is a locating-dominating set of H with $\lambda(H)$ vertices containing x, a contradiction with H is of type $\mathcal{A}_x$. □

Lemma 3.1. *Let W be a locating-dominating set of a vertex-amalgamation G of $\mathcal{H}$ in J where $V(J) = \{x\}$.*

1. *If $H_i \in \mathcal{H}$ is of type $\mathcal{A}_x$, then $H_i \setminus \{x\}$ contributes at least $\lambda(H_i)$ vertices in W.*
2. *If $H_i \in \mathcal{H}$ is of type $\mathcal{B}_x$ or $\mathcal{C}_x$, then $H_i \setminus \{x\}$ contributes at least $\lambda(H_i) - 1$ vertices in W.*
3. *If $H_i \in \mathcal{H}$ is of $\mathcal{D}_x$ or $\mathcal{E}_x$, then H_i contributes at least $\lambda(H_i)$ vertices in W. Moreover, there exists W such that H_i contributes $\lambda(H_i)$ vertices, including x.*

Proof. We distinguish three cases.

1. H_i is of type $\mathcal{A}_x$
   Suppose that $H_i \setminus \{x\}$ contributes at most $\lambda(H_i) - 1$ in W. Thus, $|B_i| \leq \lambda(H_i) - 1$. Considering Proposition 3.2, we have two possibilities.
   (a) There exists a vertex $p \in V(H_i) \setminus (B_i \cup \{x\})$ such that $N_{H_i}(p) \cap B_i = \emptyset$.
   It follows that $N_G((i,p)) \cap W_i = \emptyset$, which implies $N_G((i,p)) \cap W = \emptyset$.
   (b) There exist two distinct vertices $p, q \in V(H_i) \setminus (B_i \cup \{x\})$ such that $N_{H_i}(p) \cap B_i = N_{H_i}(q) \cap B_i$
   It follows that $N_G((i,p)) \cap W_i = N_G((i,q)) \cap W_i$, which implies $N_G((i,p)) \cap W = N_G((i,q)) \cap W$.
   Both possibilities above imply a contradiction.
2. H_i is of type $\mathcal{B}_x$ or $\mathcal{C}_x$
   According to Lemma 2.1, it is clear that H_i contributes at least $\lambda(H_i) - 1$ vertices in W. By the definition of type $\mathcal{B}_x$ or $\mathcal{C}_x$, there exists a locating-dominating set S_i of H_i with $\lambda(H_i)$ vertices containing x such that S_i^* is a locating-dominating set of $H_i \setminus \{x\}$. It follows that $T_i = \{(i,b) | b \in S_i^*\}$ is also a locating-dominating set of $G_i \setminus \{x\}$. Therefore, it will be $T_i \subseteq W_i$.
3. H_i is of type $\mathcal{D}_x$ or $\mathcal{E}_x$
   According to Lemma 2.1, it is clear that H_i contributes at least $\lambda(H_i) - 1$ vertices in W. By the definition of type $\mathcal{D}_x$ or $\mathcal{E}_x$, a locating-dominating set S_i of H_i with $\lambda(H_i)$ vertices containing x will imply S_i^* is not a locating-dominating set of $H_i \setminus \{x\}$. It follows that

www.ejgta.org

$T_i = \{(i,b)|b \in S_i^*\}$ is also not a locating-dominating set of G_i. Therefore, we need to add at least one more vertex on T_i such that a new set is a locating-dominating set of G_i. Note that, $T_i \cup \{x\}$ is a locating-dominating set of G_i. Thus, the lemma is completed.

□

Let W be a locating-dominating set of a vertex-amalgamation graph G of $\mathcal{H}$ in J where $V(J) = \{x\}$. According to Lemma 3.1 above, if $H_i, H_j \in \mathcal{H}$ are both of type $\mathcal{E}_x$, then H_i contributes at least $\lambda(H_i)$ vertices which may be include x, in W, and so does H_j. Thus, H_i and H_j contributes at least $\lambda(H_i) + \lambda(H_j) - 1$ vertices in W since x in G is correspond to x in both H_i and H_j. However, if H_i and H_j contribute exactly $\lambda(H_i) + \lambda(H_j) - 1$ vertices in W, then it can be verified that these vertices is not a locating-dominating set of an induced subgraph of G by all vertices correspond to vertices in H_i or H_j. In Lemma 3.2 below, we provide the minimum number of vertices contributed in W from $t \geq 2$ graphs of type $\mathcal{E}_x$.

Lemma 3.2. *Let W be a locating-dominating set of a vertex-amalgamation G of $\mathcal{H}$ in J where $V(J) = \{x\}$. For $t \geq 2$, let $H_1, H_2, \ldots, H_t$ of $\mathcal{H}$ are of type $\mathcal{E}_x$. Then they are contributes at least $\sum_{i=1}^{t} \lambda(H_i)$ vertices in W.*

Proof. For $1 \leq i \leq t$, let $|S_i| = \lambda(H_i)$. Then S_i contains x but S_i^* is not a locating-dominating set of $H_i \setminus \{x\}$. Also, there exists $y_i \in N_{H_i}(x)$ such that $N_{H_i}(y_i) \cap S_i^* = \emptyset$. Note that y_i is the only vertex in H_i having this property. If there are two distinct vertices $y_i, z_i \in V(H_i)$ which are adjacent to x such that $N_{H_i}(y_i) \cap S_i^* = \emptyset = N_{H_i}(z_i) \cap S_i^*$, then $N_{H_i}(y_i) \cap S_i = \{x\} = N_{H_i}(z_i) \cap S_i$, a contradiction with S_i is a locating-dominating set of H_i.

By Lemma 3.1, $H_1, H_2, \ldots, H_t$ of $\mathcal{H}$ contributes at least $1 + \sum_{i=1}^{t}(\lambda(H_i) - 1)$ vertices in W. This minimum contribution is included x. Suppose that $H_1, H_2, \ldots, H_t$ of $\mathcal{H}$ contributes at most $(\sum_{i=1}^{t} \lambda(H_i)) - 1$ vertices in W. So there exist different $i, j \in \{1, 2, \ldots, t\}$ such that the number of vertices of $W_r \subseteq W$ is at most $\lambda(H_r) - 1$ ($r \in \{i, j\}$). Since W is a locating-dominating set of G, then $W_r \cup \{x\}$ must be a locating-dominating set of G_r. It follows that $\{a \in V(H_r)|(r,a) \in W_r\} \cup \{x\}$ is also a locating-dominating set of H_r. This set contains $\lambda(H_r)$ vertices. By the definition of type $\mathcal{E}_x$, there is a vertex $y_r \in N_{H_r}(x)$ such that $N_{H_r}(y_r) \cap \{a \in V(H_r)|(r,a) \in W_r\} = \emptyset$. It follows that $N_G((r, y_r)) \cap (W_r \cup \{x\}) = \{x\}$. Since $r \in \{i, j\}$, we have that $N_G((i, y_i)) \cap (W_i \cup \{x\}) = \{x\} = N_G((j, y_j)) \cap (W_j \cup \{x\})$ which implies $N_G((i, y_i)) \cap W = \{x\} = N_G((j, y_j)) \cap W$, a contradiction. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2
By considering Lemmas 3.1 and 3.2, it is enough to show the upper bound of the locating-dominating number of the vertex-amalgamation graph G. Assume that the first $t_{\mathcal{E}_x}$ elements of $\mathcal{H}$ are of type $\mathcal{E}_x$. For $1 \leq i \leq n$, let S_i be a locating-dominating set of H_i with $\lambda(H_i)$ vertices satisfying the definition of the type of H_i. Define $L_i = \{(i,a)|a \in S_i^*\}$. Note that, if H_i is of type $\mathcal{A}_x$, then $|L_i| = \lambda(H_i)$, otherwise $|L_i| = \lambda(H_i) - 1$. For H_i is of type $\mathcal{E}_x$, let $y_i \in V(H_i)$ be

www.ejgta.org

a vertex satisfying $y_i \in N_{H_i}(x)$ and $N_{H_i}(y_i) \cap S_i^* = \emptyset$. Define

$$L = \begin{cases} \cup_{i=1}^{n} L_i, & \text{if } t_{\mathcal{D}} = t_{\mathcal{E}} = 0 \text{ and } (t_{\mathcal{A}} \geq 1 \text{ or } t_{\mathcal{B}} \geq 2), \\ (\cup_{i=1}^{n} L_i) \cup \{(i, y_i) | 2 \leq i \leq t_{\mathcal{E}} - 1\} \cup \{x\}, & \text{if } t_{\mathcal{E}} \geq 2, \\ (\cup_{i=1}^{n} L_i) \cup \{x\}, & \text{otherwise.} \end{cases}$$

Let us consider two distinct vertices $a, b \in V(G) \setminus L$. Let a' be a correspond vertex of a in H_i and b' be a correspond vertex of b in H_j. If $i = j$, then it is clear that $\emptyset \neq N_G(a) \cap L \neq N_G(b) \cap L \neq \emptyset$ since all correspond vertices of S_i in G are in L. Now, we assume that $i \neq j$. Then there exists a vertex $z_i \in L_i$ such that $a \in N_G(z_i)$ or there exists a vertex $z_j \in L_j$ such that $b \in N_G(z_j)$. Since $b \notin N_G(z_i)$ and $a \notin N_G(z_j)$, we obtain that $\emptyset \neq N_G(a) \cap L \neq N_G(b) \cap L \neq \emptyset$. Therefore, L is a locating-dominating set of G. ■

Acknowledgement

The authors are thankful to the anonymous referee for some comments that helped to improve the presentation of the manuscript.

References

[1] G.R. Argiroffo, S.M. Bianchi, and A.K. Wagler, A polyhedral approach to locating-dominating sets in graphs, *Electron. Notes Discrete Math.*, **50** (2015), 89–94.

[2] G. Argiroffo, S. Bianchi, Y. Lucarini, and A. Wagler, The identifying code, the locating-dominating, the open locating-dominating and the locating total-dominating problems under some graph operations, *Electron. Notes Theoret. Comput. Sci.*, **346** (2019), 135–145.

[3] C. Balbuena, F. Foucaud, and A. Hansberg, Locating-dominating sets and identifying codes in graphs of girth at least 5, *Electron. J. Combin.*, **22**(2) (2015), P2.15.

[4] N. Bertrand, I. Charon, O. Hudry, and A. Lobstein, Identifying and locating-dominating codes on chains and cycles, *European J. Combin.*, **25**(7) (2004), 969–987.

[5] M. Blidia, M. Chellali, F. Maffray, J. Moncel, and A. Semri, Locating-domination and identifying codes in trees, *Australasian J. Combin.*, **39** (2007), 219–232.

[6] P. Boonmee, J. Ma-In, and S. Panma, Domination numbers of amalgamations of cycles at connected subgraphs, *J. Math.*, **2022** (2022), Article ID 7336728.

[7] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, and M.L. Puertas, Locating-dominating codes: Bounds and extremal cardinalities, *Appl. Math. Comput.*, **220** (2013), 38–45.

[8] S.R. Canoy, Jr., G.A. Malacas, and D. Tarepe, Locating-dominating sets in graphs, *Appl. Math. Sci.*, **8** No.8 (2014), 4381–4388.

www.ejgta.org

[9] M.R. Cappelle, G.C.M. Gomes, and V.F. dos Santos, Parameterized algorithms for locating-dominating sets, *Procedia Comput. Sci.*, **195** (2021), 68–76.

[10] I. Charon, O. Hudry, and A. Lobstein, Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard, *Theoret. Comput. Sci.*, **290**(3) (2003), 2109–2120.

[11] C.J. Colbourn, P.J. Slater, and L.K. Stewart, Locating-dominating sets in series parallel networks, *Congr. Numer.*, **56** (1987), 135–162.

[12] S.J. Esther and J.V. Vivik, The locating and locating domination of prism family graphs, *Math. Stat.*, **12**(3) (2024), 292–302.

[13] M. Fazil, I. Javaid, M. Salman, and U. Ali, Locating-dominating sets in hypergraphs, *Periodica Math. Hungar.*, **72**(2) (2016), 224–234.

[14] D. Fitriani, and S.W. Saputro, The local metric dimension of amalgamation of graphs, *Electron. J. Graph Theory Appl.*, **12**(1) (2024), 125–146.

[15] F. Foucaud, M.A. Henning, and C. Löwenstein, Locating-dominating sets in twin-free graphs, *Discrete Appl. Math.*, **200** (2015), 52–58.

[16] F. Foucaud, G.B. Mertzios, R. Naserasr, A. Parreau, and P. Valicov, Identification, location-domination and metric dimension on interval and permutation graphs. i. bounds, *Theoret. Comput. Sci.*, **668** (2017), 43–58.

[17] A.K.A. Gafur, and S.W. Saputro, On locating-dominating set of regular graphs, *J. Math.*, **2021** (2021), Article ID 8147514.

[18] D. Garijo, A. González, and A. Márquez, The difference between the metric dimension and the determining number of a graph, *Appl. Math. Comput.*, **249** (2014), 487–501.

[19] M.A. Henning, and O.R. Oellermann, Metric-locating-dominating sets in graphs, *Ars Comb.*, **73** (2004), 129–141.

[20] C. Hernando, M. Mora, and I.M. Pelayo, Locating domination in bipartite graphs and their complements, *Discrete Appl. Math.*, **263** (2019), 195–203.

[21] L. Maro, A. Sanga, and M.E. Tuaty, Determination of the restrained domination number on vertex amalgamation and edge amalgamation of the path graph with the same order, *Barekeng: Jurnal Ilmu Matematika dan Terapan*, **16**(2) (2022), 421–426.

[22] M. Murtaza, M. Fazil, and I. Javaid, Locating-dominating sets of functigraphs, *Theoret. Comput. Sci.*, **799** (2019), 115–123.

[23] A.A. Pribadi, and S.W. Saputro, On locating-dominating number of comb product graphs, *Indones. J. Combin.*, **4**(1) (2020), 27–33.

www.ejgta.org

[24] G. Rajasekar, and K. Nagarajan, Algorithm for finding location domination number of a graph connected by a bridge, *Int. J. Pure Appl. Math.*, **118**(6) (2018), 313–321.

[25] S.W. Saputro, and I.R.A. As-Shidiq, On locating-dominating set of the Cartesian product of complete bipartite graphs and a path of order two, *Jurnal Matematika UNAND*, **14**(3) (2025), 208–215.

[26] P.J. Slater, Domination and location in acyclic graphs, *Networks*, **17** (1987), 55–64.

[27] P.J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.*, **22** (1988), 445–455.

[28] Y. Wahyuni, M.I. Utoyo, and Slamin, Domination number of vertex amalgamation of graphs, *IOP Conf. Series: J. Phys: Conf. Series*, **855** (2017), 012059.

[29] Y. Yang, and X. Kong, The thickness of amalgamations of graphs, (2012), arXiv: 1201.6483v1.

www.ejgta.org