# Electronic Journal of
# Graph Theory and Applications

# On the nullity number of graphs

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## Abstract

The paper discusses bounds on the nullity number of graphs. It is proved in [B. Cheng and B. Liu, *On the nullity of graphs.* Electron. J. Linear Algebra 16 (2007) 60–67] that $\eta \leq n - D$, where $\eta$, $n$ and $D$ denote the nullity number, the order and the diameter of a connected graph, respectively. We first give a necessary condition on the extremal graphs corresponding to that bound, and then we strengthen the bound itself using the maximum clique number. In addition, we prove bounds on the nullity using the number of pendant neighbors in a graph. One of those bounds is an improvement of a known bound involving the domination number.

## 1. Introduction

In the present paper, we consider only finite, undirected and simple graphs, *i.e.*, graphs without loops or multiple edges. A graph is denoted by $G = (V, E)$, where $V$ is its vertex set and $E$ its edge set. The order $n$ of $G$ is the number of its vertices, *i.e.*, $n = |V|$. The size $m$ of $G$ is the number of its edges, *i.e.*, $m = |E|$. If $W$ is a subset of the vertex set of $G$, we denote by $G[W]$ the graph with vertex set $W$ and edge set composed of the edges of $G$ whose end-vertices belong to $W$. It is called *the subgraph of $G$ induced by $W$* and we say that $W$ induces $G[W]$. A graph $H$ is said to be an *induced subgraph* of $G$ if there exists a subset of vertices $W$ of $G$ such that $H = G[W]$.

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As usual, $S_n$, $P_n$, $C_n$ and $K_n$ denote the star, the path, the cycle and complete graph on $n$ vertices, respectively. $K_{p,n-p}$ denotes the complete bipartite graph on $n$ vertices whose partition classes contain $p$ and $n - p$ vertices respectively.

Assume that $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A = (a_{i,j})_{1 \le i,j \le n}$ of $G$ is the 0-1 matrix defined by $a_{i,j} = 1$ if and only if $v_i v_j \in E$. The eigenvalues of $G$ are those of its adjacency matrix $A$. The *nullity* of a graph $G$, denoted by $\eta = \eta(G)$, is the multiplicity of 0 as an eigenvalue of $G$ with $\eta = 0$ if $A$ is not a singular matrix. The *rank* of a graph, denoted by $rk = rk(G)$, is the rank of its adjacency matrix, thus for any graph $G$ on $n$ vertices, we have $\eta(G) = n - rk(G)$. Therefore, any result about the nullity can be stated in terms of rank, and vice versa.

The problem of characterizing nonsingular graphs, *i.e.*, graphs with $\eta > 0$, was first posed by Collatz and Sinogowitz [12] in 1957 (see also [33]). This question is of great interest in chemical graph theory because of the relationship between the nullity of a graph representing an alternant hydrocarbon (a bipartite molecular graph) and its molecular stability (see e.g., [14, 28]). The problem also attracted the attention of many mathematicianss. They studied nullity of graphs in general [1, 7, 11, 13, 21, 34, 37, 38, 39, 35, 41]; graphs with zero nullity [5]; nullity of bipartite graphs [15, 18, 31]; nullity of trees [19, 26, 30, 32]; nullity of line graphs of trees [22, 29, 36]; nullity of unicyclic graphs [25, 30, 40]; nullity of bicyclic graphs [23, 27]; nullity of graphs with pendant vertices [24]; and nullity of graphs with pendant trees [20].

In the next section, we prove results about the nullity involving the diameter. We first give a necessary condition on the extremal graphs corresponding to a bound proved in [11]. Then we extend that bound by replacing the diameter by the radius, in two ways. At the end of the section we improve the bound under study by adding the maximum clique number. In the last section, we prove new bounds on the nullity involving the number of pendant neighbors, and then strengthen a well-known bound from [1].

## 2. Results involving the diameter

The distance between two vertices $u$ and $v$ in $G$, denoted by $d(u, v)$, is the length of a shortest path between $u$ and $v$. The eccentricity $e(v)$ of a vertex $v$ in $G$ is the largest distance from $v$ to another vertex of $G$. The minimum eccentricity in $G$, denoted by $r$, is the radius of $G$. The maximum eccentricity of $G$, denoted by $D$, is the diameter of $G$. That is

$$r = \min_{v \in V} e(v) \qquad \text{and} \qquad D = \max_{v \in V} e(v).$$

The following theorem is an elementary and well-known result in matrix theory.

**Theorem 2.1** ([16]). *Let $M$ be a symmetric matrix of rank $k$. Then $M$ contains a nonsingular principal minor of order $k$.*

In graph theory, using the notion of nullity, the above theorem reads: *A graph on $n$ vertices with nullity $\eta$ contains an induced subgraph on $n - \eta$ vertices with nullity 0.*

Exploiting the graph theory version of Theorem 2.1 and the fact that a connected graph of diameter $D$ contains an induced path on $D + 1$ vertices (whose nullity is 1 if $D$ is even and 0 if $D$ is odd; see [6]), Cheng and Liu [11] proved the following result.

**Lemma 2.1** ([11]). *Let $G$ be a connected graph on $n$ vertices with diameter $D$ and nullity number $\eta$. Then*

$$\eta + D \leq \begin{cases} n & \text{if } D \text{ is even} \\ n-1 & \text{if } D \text{ is odd.} \end{cases}$$

We next provide a necessary condition for the sharpness of the above bound, for which we need the following well-known lemma (see, e.g. [14]).

**Lemma 2.2.** *Let $G$ be a graph. If $v$ is a pendant neighbor of $G$ with unique neighbor $u$, then $\eta(G) = \eta(G \setminus \{u, v\})$.*

**Theorem 2.2.** *Let $G$ be a connected graph on $n$ vertices with diameter $D$ and nullity number $\eta$. If $D + \eta = n$ then $G$ is bipartite.*

*Proof.* Let $G$ be a connected graph such that $D + \eta = n$. According to Lemma 2.1, $D$ is necessarily even, and a diametrical path contains an odd number of vertices, namely $D + 1$. Let $P \cong P_{D+1}$ be such a path in $G$.

Now assume that $G$ is not bipartite and let $C \cong C_{2p+1}$ denote an induced odd cycle in $G$.

(i) If $P$ and $C$ are vertex disjoint and there is no edge joining a vertex from $P$ and a vertex from $G$, then the disjoint union of $P \cup C$ is an induced graph of $G$ with rank $D + 2p + 1$. Thus, $n - \eta \geq D + 2p + 1$ and therefore $D + \eta \leq n - 2p - 1 < n$. This is a contradiction.

(ii) If $P$ and $C$ share a single vertex $v$, then using Lemma 2.2 iteratively, the nullity of the graph $H$ induced by $P \cup C$ is 1 if the cardinalities of the components of $P - v$ are odd, and 0 if they are even. Thus, the rank of $G'$ is at least $D + 2p - 1$ and therefore, $n - \eta \geq D + 2p - 1$. It follows that $D + \eta \leq n - 2p + 1 < n$. This is a contradiction.

(iii) If there exists a single edge between a vertex from $P$ and a vertex $w$ from $C$, then the rank of the graph induced by $P \cup C - w$ (which is a disjoint union of two paths $P_{2p}$ and $P_{D+1}$) is $D + 2p$. Thus, $n - \eta \geq D + 2p$ and therefore $D + \eta \leq n - 2p < n$. This is a contradiction.

(iv) If there exist at least two edges between the vertices of $P$ and those of $C$, among the vertices incident to the edges between $P$ and $C$. Let $w_1$ and $w_2$ be two vertices from $C$ such that $v_1 w_1$ and $v_2 w_2$ are edges in $G$ and such that the portion $C'$ of $C$ of odd length is the smallest under these conditions. We consider the following subcases.

- If $v_1 = v_2$, $C'$ contains at least one edge. Consider the cycle $C'' = C' \cup w_1 v_1 w_2$ instead of $C$ and then we are in a case similar to $(ii)$.

- If $v_1 \neq v_2$, $w_1 = w_2$ and $v_1 v_2$ is an edge in $G$. The nullity of the graph induced by $P \cup \{w_1\}$ is 0, thus $n - \eta \geq D + 2$ and therefore $D + \eta \leq n - 2 < n$. This is a contradiction.

- If $v_1 \neq v_2$, $w_1 = w_2$ and $v_1 v_2$ is not an edge in $G$. In this case $d(v_1, v_2) = 2$ and, by the choice of $w_1$ and $w_2$, there is no vertex from $C$ other than $w_1$ adjacent to a vertex from $P$. Denote by $v'$ the common neighbor of $v_1$ and $v_2$ belonging to $P$. Replacing the path $P$ by $(P \setminus \{v'\}) \cup \{w_1\}$ lead to the case $(ii)$.

- If $v_1 \neq v_2$, $w_1 \neq w_2$ and $d(w_1, w_2) \geq d(v_1, v_2) - 1$, the path induced $C'$ and remaining part of $P$ obtained by the deletion of all its vertices lying between $v_1$ and $v_2$. That path contains at least $D+2$ vertices and then its rank is at least $D+1$. Thus, $n - \eta \geq D + 1$, and therefore $D + \eta \leq n - 1 < n$. This is a contradiction.

- If $v_1 \neq v_2$, $w_1 \neq w_2$ and $d(w_1, w_2) = d(v_1, v_2) - 2$ (note that $d(w_1, w_2) \geq d(v_1, v_2) - 2$, since $P$ is a diametrical path), consider the graph $H$ induced by $P \cup C \setminus U$, where $U$ denotes the set of vertices of $P$ lying between $v_1$ and $v_2$. Using iteratively Lemma 2.2, the nullity of $H$ is 0. The number of vertices in $H$ is at least $D+1$. Thus $n - \eta \geq D + 1$ and therefore $D + \eta \leq n - 1 < n$. This is a contradiction.

(v) If $C$ and $P$ share more than one vertex and there is no edge between a vertex from $P \setminus C$ and a vertex from $C \setminus P$, denote by $v_1$ (resp. $v_2$) the vertex from $P \cap C$ which is closest to one (resp. the other) endpoint of $P$. Let $H$ be the subgraph of $G$ obtained from $C \cup P$ by the deletion of the vertices from $P$ lying between $v_1$ and $v_2$. The graph $H$ is an induced path of $G$ on at least $D + 2$ vertices. Thus, the rank of $H$ is at least $D + 1$, and then $n - \eta \geq D + 1$. Therefore, $D + \eta \leq n - 1 < n$. This is a contradiction.

(vi) If $C$ and $P$ share at leat one vertex and therefore at least one edge joins a vertex from $P \setminus C$ and a vertex from $C \setminus P$. Let $U$ denote the set of vertices in $P$ which are also in $C$ or have at least one neighbor in $C$. Let $v_1$ (resp. $v_2$) be the vertex from $U$ which is closest to one (resp. the other) endpoint of $P$. Let $H$ be the subgraph of $G$ obtained from $C \cup P$ by the deletion of the vertices from $P$ lying between $v_1$ and $v_2$. The graph $H$ is an induced path of $G$ on at least $D + 2$ vertices. Thus, the rank of $H$ is at least $D + 1$, and then $n - \eta \geq D + 1$. Therefore, $D + \eta \leq n - 1 < n$. This is a contradiction.

To conclude, $G$ necessarily contains no odd cycle, and therefore is a bipartite graph. $\square$

A well-known relationship between the diameter and the radius is $r \leq D \leq 2r$ (see e.g. [8]). Exploiting the first inequality, Lemma 2.1 and Theorem 2.2, we easily get the following corollary.

**Corollary 2.1.** *If $G$ is a connected graph on $n$ vertices with radius $r$ and nullity number $\eta$, then $\eta + r \leq n$. Moreover, if equality holds, then $G$ is bipartite.*

Note that there exist graphs for which $\eta + D = n$ and $\eta + r < n$ such as stars $S_n$ with $n \geq 3$ ($\eta = n - 2$, $D = 2$ and $r = 1$), odd paths $P_n$ with $n \geq 3$ ($\eta = 1$, $D = n - 1$ and $r = (n-1)/2$) and many other graphs.

Now, what would happen if we replace in the inequality of Lemma 2.1 $D$ by $2r$? Does the inequality remain valid? The answer is negative. Indeed, for instance, the complete bipartite graphs $K_{n-p,p}$, with $2 \leq p \leq n - 2$, have $\eta + 2r = n - 2 + 2 \times 2 = n + 2 > n$. The following proposition solves the problem.

**Proposition 2.1.** *If $G$ is a connected graph on $n$ vertices with radius $r$ and nullity number $\eta$, then $\eta + 2r \leq n + 2$.*

*Proof.* It is proved in [17] that a graph with radius $r$ contains an induced path on at least $2r - 1$ vertices. Thus $rk(G) \geq rk(P_{2r-1}) = 2r - 2$ and the result follows. $\square$
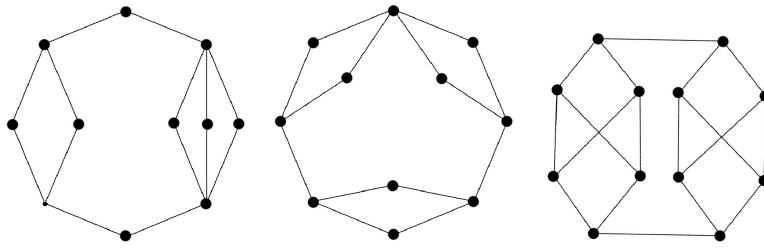
Figure 1. Examples of graphs with $\eta + 2r \leq n + 2$ and $\eta + D < n$.

Note that there exist graphs for which the bound in Lemma 2.1 is reached while that in Proposition 2.1 is not, *i.e.*, graphs satisfying $\eta + D = n$ and $\eta + 2r < n + 2$. For instance, odd paths satisfy $\eta + D = \eta + 2r = n < n + 2$.

Note also that there exist graphs for which the bound in Proposition 2.1 is reached while that in Lemma 2.1 is not, *i.e.*, graphs satisfying $\eta + 2r \leq n + 2$ and $\eta + D < n$. Three such graphs are illustrated in Figure 1: the first two graphs from the left are on $n = 11$ vertices with $\eta = 5$ and $D = r = 4$; the graph on the right is on $n = 12$ vertices with $\eta = 6$ and $D = r = 4$.

In order to improve the bound in Lemma 2.1, we used the AutoGraphiX system, a software for conjecture making in graph theory (see [3, 4, 9, 10]), to generate a conjecture next, proved. First, we recall the following lemma, that is used in the proof.

**Lemma 2.3** ([5]). *For $n \geq 4$ and $3 \leq p \leq n - 1$, $\eta\left(K_n \setminus E(S_p)\right) = 0$, where $E(S_p)$ denotes a set of $p - 1$ edges incident to a same vertex in $K_n$.*

The *maximum clique number* of a graph, denoted $\omega = \omega(G)$, is the maximum cardinality of a vertex subset of $G$ that induces a complete graph.

**Theorem 2.3.** *Let $G$ be a connected graph on $n$ vertices with diameter $D$, nullity number $\eta$ and maximum clique number $\omega$. Then $D + \eta + \omega \leq n + 2$ and the bound is best possible as shown by complete multipartite graphs and odd paths.*

*Proof.* Let $P$ be a diametrical path in $G$ and $W$ a clique on $\omega$ vertices in $G$. We also use $P$ and $W$ to denote the sets of vertices of the path $P$ and the clique $W$, respectively. Let $H$ be a subgraph of $G$ induced by $P \cup W$. Note that $P$ and $W$ can share at most 2 vertices, and therefore, $H$ contains $\omega + D - 1$, $\omega + D$ or $\omega + D + 1$ vertices depending on whether $P$ and $W$ have 0, 1 or 2 common vertices.

Since $\eta + D \leq n$, the inequality is obvious for $K_3$-free graphs. In addition, if $\omega = 3$, then the graph is not bipartite and by applying Theorem 2.2, $D + \eta \leq n - 1$ and therefore $\eta + D + \omega \leq n - 1 + 3 = n + 2$. So we can assume that $\omega \geq 4$. The inequality is also obvious when $D \leq 2$, since $\eta + \omega \leq n$, thus we can also assume that $D \geq 3$. Under those conditions, we have the following cases.

1. If $H$ is disconnected, then $rk(G) \geq rk(H) = rk(W) + rk(P) \geq \omega + D$. Thus $\eta \leq n - \omega - D < n - \omega - D + 2$, *i.e*, $D + \eta + \omega < n + 2$.

2. If $P$ and $W$ have one common vertex and there is no edge between $P$ and $W$ other than those involving the common vertex. Iterating Lemma 2.2 leads to a clique, on $\omega$ or $\omega - 1$

vertices, with eventually an isolated vertex. Thus $\eta(H) \leq 1$ and then $rk(G) \geq rk(H) \geq |P \cup W| - 1 = D + \omega - 1$. Therefore, $\eta \leq n - \omega - D + 1$, *i.e*, $D + \eta + \omega < n + 2$.

3. If $P$ and $W$ have two common vertices and there is no edge between $P$ and $W$ other than those involving the common vertices. Iterating Lemma 2.2 leads to a clique, on $\omega$, $\omega - 1$ or $\omega - 2$ vertices. Thus $\eta(H) = 0$ and then $rk(G) \geq rk(H) \geq |P \cup W| = D + \omega - 1$. Therefore, $\eta \leq n - \omega - D + 1$, *i.e*, $D + \eta + \omega < n + 2$.

4. If $P$ and $W$ have one common vertex, say $v$, and there exists at least one edge between $W$ and one or both neighbors, say $v'$ and $v''$, of $v$ on $P$, and there is no edge between $P$ and $W$ other than those involving $v$, $v'$ or $v''$.
   Iterating Lemma 2.2 leads to:

   - the clique $W - v$ with nullity zero; a clique $W$ with nullity zero;

   - the subgraph induced by $W \cup \{v'\}$ (or $W \cup \{v'\}$), which is a graph of the form of that one described in Lemma 2.3, thus with nullity zero;

   - the subgraph induced by $W \cup \{v', v''\}$ which contains a subgraph of the form of that one described in Lemma 2.3, thus with nullity at most 1.

   In all of these cases, $rk(G) \geq rk(H) \geq |P \cup W| - 1 = D + \omega - 1$. Therefore, $\eta \leq n - \omega - D + 1$, *i.e*, $D + \eta + \omega < n + 2$.

5. Let $v$, $v'$, $v''$ be the vertices on $P$ as defined in the above case, and let $v'_1$ denote the neighbor of $v'$ on $P$ other than $v$ (if it exists) and $v''_1$ the neighbor of $v''$ on $P$ other than $v$. Note that only one of $v'_1$ and $v''_1$ can have a neighbor in $W$. Indeed, if there is an edge between $v'_1$ and a vertex $w'$ from $W$ and an edge between $v'_1$ and a vertex $w''$ from $W$, the path $(P \setminus \{v, v', v'', v'_1, v''_1\}) \cup \{w', w'', v'_1, v''_1\}$ would be a path of length $D - 1$ linking the endpoints of $P$ assumed to be at distance $D$, which is a contradiction. Thus assume, without loss of genrality, that only $v'_1$ has at least one neighbor in $W$.
   Iterating Lemma 2.2 leads to:

   - the subgraph induced by $(W \setminus \{v\}) \cup \{v'\}$ with nullity at most 1 (being 1 only if $v'$ is an isolated vertex in the induced subgraph);

   - the subgraph induced by $W \cup \{v'\}$ with nullity zero;

   - the subgraph induced by $W \cup \{v', v''\}$ with nullity at most 1 (since it contains the subgraph induced by $W \cup \{v'\}$);

   - the subgraph induced by $W \cup \{v', v'_1\}$ with nullity at most 1 (since it contains the subgraph induced by $W \cup \{v'\}$);

   - the subgraph induced by $W \cup \{v', v'', v'_1\}$ with nullity at most 2 (since it contains the subgraph induced by $W \cup \{v'\}$).

   In all of these cases, $rk(G) \geq rk(H) \geq |P \cup W| - 2 = D + \omega - 2$. Therefore, $\eta \leq n - \omega - D + 2$, *i.e*, $D + \eta + \omega \leq n + 2$.

6. If $P$ and $W$ contain two common vertices and there is at least one edge between $P$ and $W$ involving vertices from $P$ other than the common vertices. Denote by $v_1$ and $v_2$ the common vertices to $P$ and $W$, and let $v'_1$ (resp. $v'_2$) the neighbor, if any, of $v_1$ (resp. $v_2$) on $P$. If an

edge exists between $W \setminus \{v_1, v_2\}$ and a vertex from $P \setminus \{v_1, v_2\}$, that vertex must be $v_1'$ or $v_2'$ (or both if there are two or more edges).

Iterating Lemma 2.2 leads to:

- the clique $W$ with nullity zero; the subgraph induced by $(W \setminus \{v_1\}) \cup \{v_2'\}$ with nullity zero (using Lemma 2.3);

- the subgraph induced by $(W \setminus \{v_2\}) \cup \{v_1'\}$ with nullity zero (using Lemma 2.3);

- the subgraph induced by $W \cup \{v_1'\}$ with nullity zero (using Lemma 2.3);

- the subgraph induced by $W \cup \{v_2'\}$ with nullity zero (using Lemma 2.3);

- the subgraph induced by $W \cup \{v_1', v_2'\}$ with nullity at most 1, since it contains the subgraph induced by $W \cup \{v_1'\}$.

In all of these cases, $rk(G) \geq rk(H) \geq |P \cup W| - 1 = D + \omega - 2$. Therefore, $\eta \leq n - \omega - D + 2$, *i.e*, $D + \eta + \omega \leq n + 2$.

This proves the inequality. Equality is reached for several graphs such as odd paths for which $\eta = 1$, $\omega = 2$ and $D = n - 1$; also for the complete multipartite graphs for which $D = 2$ and $\eta = n - \omega$ (see [1, 11]). $\qquad\square$
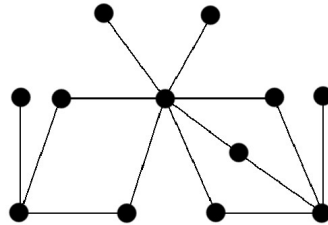


Figure 2. A graph with $D + \eta + \omega = n + 2$ ($n = 12$, $\omega = 2$, $\eta = 6$ and $D = 6$).

The characterization of the extremal graphs for the above theorem remains an open problem. Note that odd paths and complete multipartite graphs are not the only ones for which the bound in Theorem 2.3 is reached. Indeed, the graph illustrated in Figure 2 is an example of a graph for which $D + \eta + \omega = n + 2$ (here $n = 12$, $\omega = 2$, $\eta = 6$ and $D = 6$).

## 3. Results involving the number of pendant neighbors

A vertex $v$ in $G$ is called a *pendant neighbor* if there exists a vertex $u$ such that $d(u) = 1$, $d(v) \geq 2$ and $uv$ is an edge in $G$. Denote by $n_1' = n_1'(G)$ the *number of pendant neighbors* in $G$.

In this section we prove bounds on the nullity number using the number of pendant neighbors. The *matching number* of a graph $G$, denoted by $\mu = \mu(G)$, is the maximum number of disjoint edges. We first prove a lemma stating a relationship between the matching number $\mu$ and the number of pendant neighbors $n_1'$.

**Lemma 3.1.** *For any graph $G$ on $n \geq 3$ vertices with matching number $\mu$ and which contains $n_1'$ pendant neighbors, $n_1' \leq \mu$.*

*Proof.* Consider the collection $M$ of all edges of the form $vu_v$, where $v$ is a pendant neighbor and $u_v$ is a neighbor of $v$ with $d(u_v) = 1$. It is clear taht $M$ is a matching, thus $n_1' = |M| \leq \mu$. $\qquad\square$

A well-known result involving the nullity and the matching number of a tree is the following theorem.

**Theorem 3.1** ([19, 32]). *Let $T$ be a tree on $n$ vertices with nullity number $\eta$ and matching number $\mu$, then $\eta + 2\mu = n$.*

Using Lemma 3.1 and Theorem 3.1, we easily prove the following result.

**Corollary 3.1.** *Let $T$ be a tree on $n$ vertices with nullity number $\eta$ and $n_1'$ pendant neighbors, then $\eta + 2n_1' \leq n$. The bound is best possible as shown by the star $S_n$.*

The above corollary can be extended to the class of all graphs.

**Theorem 3.2.** *Let $G$ be a graph on $n$ vertices with nullity number $\eta$ and $n_1'$ pendant neighbors, then $\eta + 2n_1' \leq n$.*

*Proof.* If $n_1' = 0$, *i.e.* $G$ contains no pendant neighbor, the result is obvious.

Assume that $n_1' \geq 1$ and consider the induced subgraph $H$ of $G$ for which $rk(G) = rk(H)$ and $\eta(H) = 0$. Let $v$ be a pendant neighbor in $G$ and $u_v$ a neighbor of $v$ with $d(u_v) = 1$. Then $H$ contains $v$ and $u_v$. Indeed, if $H$ does not contain $v$ and $u_v$, then the graph $H'$ induced by the vertices of $H$ and $v$ and $u_v$ is an induced subgraph of $G$ with $rk(H') = rk(H) + 2$ (see [6, Corollary 2]) which is a contradiction with the choice of $H$; if $H$ contains $v$ but not $u_v$, then the graph $H'$ induced by the vertices of $H$ and $u_v$ is an induced subgraph of $G$ with $rk(H') = rk(H) + 2$ (see [6, Corollary 1]) which is a contradiction with the choice of $H$; if $H$ contains $u_v$ but not $v$, then $u_v$ is an isolated vertex in $H$ which is a contradiction with $\eta(H) = 0$. Therefore $H$ contains an edge of the form $vu_v$, where $v$ is a pendant neighbor and $u_v$ is a neighbor of $v$ with $d(u_v) = 1$, for every pendant neighbor $v$. Thus $2n_1' \leq rk(G) = n - \eta$ and then the inequality follows. $\qquad\square$

Note that we cannot replace $n_1'$ by $\mu$ in the above theorem since there are many graphs for which $\eta + 2\mu > n$. For instances, the complete bipartite graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ satisfies $\eta + 2\mu = n - 2 + \lfloor \frac{n}{2} \rfloor > n$ for all $n \geq 6$; the cycle $C_n$, with $n = 4k$, satisfies $\eta + 2\mu = 2 + n > n$ for all $k \geq 1$.

Recall that a vertex subset $S$ in $G$ is called a *dominating set* if any vertex in $G$ that does not belong to $S$ has at least one neighbor in $S$. The minimum cardinality of dominating set in a graph $G$ is called the *domination number* of $G$ and is denoted by $\gamma = \gamma(G)$. The following theorem is proved in [1].

**Theorem 3.3** ([1]). *Let $G$ be a connected graph on $n \geq 1$ vertices with nullity $\eta$ and domination number $\gamma$. Then $\eta + \gamma \leq n$ and the bound is best possible as shown by the complete bipartite graph $K_{n-p,p}$, $2 \leq p \leq n - 2$.*

The next result is an improvement of the bound in the above theorem.

**Theorem 3.4.** *Let $G$ be a connected graph on $n \geq 1$ vertices with nullity $\eta$, domination number $\gamma$ and $n_1'$ pendant neighbors. Then $\eta + \gamma \leq n - n_1'$. The bound is sharp for every $n \geq 3$ as shown by the star $S_n$.*

*Proof.* If $n_1' = 0$, *i.e.* $G$ contains no pendant neighbor, then the result is obvious. Assume that $n_1' \geq 1$ and consider the induced subgraph $H$ of $G$ for which $rk(G) = rk(H)$ and $\eta(H) = 0$. Let $S$ denote the vertex set of $H$. In the proof of Theorem 3.3 (in [1]), it is shown that $S$ is a dominating set in $G$. In the proof of Theorem 3.2, we showed that $H$ contains $n_1'$ disjoint edges of the form $vu_v$, where $v$ is a pendant neighbor and $u_v$ is a neighbor of $v$ with $d(u_v) = 1$, say $v_1u_1, v_2u_2, \ldots, v_{n'1}u_{n'1}$. Thus $W - \{u_1, u_2, \ldots, u_{n_1'}\}$ is also a dominating set in $G$, therefore $\gamma \leq rk(H) - n_1' = n - \eta - n_1'$, and the bound follows.

For the star $S_n$, $n \geq 3$, we have $\eta = n - 2$, $\gamma = 1$ and $n_1' = 1$, the equality is reached. □

As already mentioned, the above theorem is an improvement of Theorem 3.3. In another attempt to strengthen the theorem using the AutoGraphiX system, we could state the following conjecture, which is similar to Theorem 2.3.

**Conjecture 1.** *Let $G$ be a connected graph on $n \geq 1$ vertices with nullity $\eta$, maximum clique number $\omega$ and domination number $\gamma$. Then $\eta + \omega + \gamma \leq n + 2$ and the bound is best possible as shown by the complete multipartite graphs.*

Related to the above conjecture, we prove the next proposition.

**Proposition 3.1.** *Let $G$ be a graph on $n \geq 1$ vertices with nullity $\eta$, maximum clique number $\omega$ and domination number $\gamma$. Then $\eta + \omega + \gamma \leq 2n + 1$ with equality if and only if $G$ is the empty graph $\overline{K}_n$. If $G \not\cong \overline{K}_n$, then $\eta + \omega + \gamma \leq 2n - 1$ with equality if and only if $G$ is the union of $K_2$ and $n - 2$ isolated vertices.*

*Proof.* Let $W$ be a maximum clique in $G$. Let $S = (V(G) \setminus W) \cup \{w\}$, where $w$ is any vertex in $W$. It is obvious that $S$ is a dominating set in $G$. Thus $\gamma \leq n - \omega + 1$. In addition, $\eta \leq n$ with equality if and only if $G \cong \overline{K}_n$. Thus $\eta + \gamma + \omega \leq 2n + 1$ with equality if and only if $G \cong \overline{K}_n$.

If $G$ is not the empty graph, let $H$ be the largest subgraph of $G$ with no isolated vertices. In fact $H$ is the graph obtained from $G$ by removing all isolated vertices from $G$.

If $H \cong K_2$, it is obvious that the bound is reached. So assume that $H$ contains at least two edges. Let $n_0$ denote the number of isolated vertices in $G$. Note that $n_0 \leq n - 3$ since $H$ contains at least 3 vertices. If $H$ is a clique, then $\eta + \gamma + \omega = n_0 + n_0 + 1 + n - n_0 = n + n_0 + 1 \leq 2n - 2$. If $H$ is not a clique, let $H_1, H_2, \ldots, H_p$ denote the connected components of $H$ (if $H$ is connected then $p = 1$). For each componente $H_i$, $\gamma(H_i) + \omega(H_i) \leq n_i$ (see [2]), where $n_i$ denotes the number of edges in the componenet. We have

$$\gamma(H) + \omega(H) = \sum_{i=1}^{p} \gamma(H_i) + \max_{1 \leq i \leq p} \omega(H_i) \leq \sum_{i=1}^{p} (\gamma(H_i) + \omega(H_i)) \leq \sum_{i=1}^{p} n_i = n - n_0.$$

It is easy to see that $\eta \leq n - 2$, $\omega = \omega(H)$ and $\gamma = \gamma(H) + n_0$. Thus $\eta + \gamma + \omega \leq n - 2 + \gamma(H) + n_0 + \omega(H) \leq n - 2 + n - n_0 + n_0 = 2n - 2$.

In conclusion, if $G \not\cong \overline{K}_n$, then $\eta + \omega + \gamma \leq 2n - 1$ with equality if and only if $G$ is the union of $K_2$ and $n - 2$ isolated vertices. □

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