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On irregularity strength of disjoint union of friendship graphs

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Abstract

We investigate the vertex total and edge total modification of the well-known *irregularity strength of graphs*.
We have determined the exact values of the total vertex irregularity strength and the total edge irregularity strength of a disjoint union of friendship graphs.

Keywords: vertex irregular total k-labeling, edge irregular total k-labeling, total vertex irregularity strength, total edge irregularity strength, friendship graph
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1. Introduction

Chartrand *et al.* [8] introduced labelings of the edges of a graph G with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [6, 19]. Karoński *et al.* [12] conjectured that the edges of every connected graph of order

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at least 3 can be assigned labels from $\{1,2,3\}$, such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different.

Motivated by irregular assignments Bača *et al.* [5] defined a *vertex irregular total k-labeling* of a (p,q)-graph $G=(V,E)$ to be a labeling of the vertices and edges of G

$$\phi : V \cup E \to \{1,2,\ldots,k\}$$

such that the *total vertex-weights*

$$wt(x) = \phi(x) + \sum_{xy \in E} \phi(xy)$$

are different for all vertices, that is, $wt(x) \neq wt(y)$ for all different vertices $x, y \in V$. Furthermore, they defined the *total vertex irregularity strength*, $tvs(G)$, of G as the minimum k for which G has a vertex irregular total k-labeling.

It is easy to see that irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G.

If an edge labeling $f : E \to \{1,2,\ldots,s(G)\}$ provides the irregularity strength $s(G)$, then we extend this labeling to total labeling ϕ in such a way

$$\begin{aligned} \phi(xy) &= f(xy) \quad \text{for every } xy \in E(G),\\ \phi(x) &= 1 \qquad\quad \text{for every } x \in V(G). \end{aligned}$$

Thus, the total labeling ϕ is a vertex irregular total labeling and for graphs with no component of order ≤ 2 has $tvs(G) \leq s(G)$.

Nierhoff [14] proved that for all (p,q)-graphs G with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G) \leq p-1$. From this result it follows that

$$tvs(G) \leq p-1. \tag{1}$$

In [5] several bounds and exact values of $tvs(G)$ were determined for different types of graphs (in particular for stars, cliques and prisms). Among others, the authors proved that for every (p,q)-graph G with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$,

$$\left\lceil \frac{p+\delta(G)}{\Delta(G)+1} \right\rceil \leq tvs(G) \leq p + \Delta(G) - 2\delta(G) + 1. \tag{2}$$

In the case of r-regular graphs (2) gives

$$\left\lceil \frac{p+r}{r+1} \right\rceil \leq tvs(G) \leq p - r + 1. \tag{3}$$

For graphs with no component of order ≤ 2, Bača *et al.* in [5] strengthened also these upper bounds, proving that $tvs(G) \leq p - 1 - \left\lceil \frac{p-2}{\Delta(G)+1} \right\rceil$. These results were then improved by Przybylo in [18] for sparse graphs and for graphs with large minimum degree. In the latter case were proved the bounds $tvs(G) < 32\,\frac{p}{\delta(G)} + 8$ in general and $tvs(G) < 8\,\frac{p}{r} + 3$ for r-regular (p, q)-graphs. Anholcer *et al.* [4] established a new upper bound of the form

$$tvs(G) \leq 3\left\lceil \frac{p}{\delta(G)} \right\rceil + 1. \tag{4}$$

Wijaya *et al.* [21] determined an exact value of the total vertex irregularity strength for complete bipartite graphs. Wijaya *et al.* [20] found the exact values of tvs for wheels, fans, suns and friendship graphs. Nurdin *et al.* determined exact values of tvs for several types of trees and for disjoint union of paths in [17] and [15], respectively. Ahmad *et al.* [2] found exact values of tvs for Jahangir graphs and circulant graphs.

Now we consider a total k-labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ with the associated total edge-weight

$$wt(xy) = \phi(x) + \phi(xy) + \phi(y).$$

Bača *et al.* in [5] define a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ to be an *edge irregular total k-labeling* of the graph $G = (V, E)$ if for every two different edges xy and $x'y'$ of G one has $wt(xy) \neq wt(x'y')$. The *total edge irregularity strength*, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

In [5] we can find that

$$tes(G) \geq \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}, \tag{5}$$

where $\Delta(G)$ is the maximum degree of G, and also there are determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs.

Recently Ivančo and Jendroľ [9] proved that for any tree T the $tes(T) = \max\left\{ \left\lceil \frac{|E(T)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(T)+1}{2} \right\rceil \right\}$. Moreover, they posed the following conjecture.

Conjecture 1. [9] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = \max\left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}. \tag{6}$$

The Ivančo and Jendroľ's conjecture has been verified for complete graphs and complete bipartite graphs in [10] and [11], for the Cartesian product of two paths in [13], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [7], for the categorical product of a cycle and a path in [1] and for the categorical product of two paths in [3].

The main aim of this paper is determined the exact values of the total vertex irregularity strength and the total edge irregularity strength of a disjoint union of friendship graphs.

2. Total vertex irregularity strength of disjoint union of friendship graphs

The *friendship graph* F_n is a set of n triangles having a common central vertex, and otherwise disjoint. The friendship graph F_n has $2n+1$ vertices ($2n$ vertices of degree 2 and one vertex of degree $2n$) and $3n$ edges.

Nurdin *et al.* [16] proved the following lower bound of tvs for any graph G.

Theorem 2.1. [16] *Let G be a connected graph having n_i vertices of degree i ($i = \delta, \delta+1, \delta+2, \dots, \Delta$), where δ and Δ are the minimum and the maximum degree of G, respectively. Then*

$$tvs(G) \geq \max\left\{ \left\lceil \frac{\delta + n_\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta+2} \right\rceil, \dots, \left\lceil \frac{\delta + \sum\limits_{i=\delta}^{\Delta} n_i}{\Delta+1} \right\rceil \right\}. \tag{7}$$

The next theorem determines the exact value of the total vertex irregularity strength for disjoint union of arbitrary friendship graphs.

Theorem 2.2. *Let F_{n_j} be a friendship graph with n_j triangles, $n_j \geq 3$ and $1 \leq j \leq m$, $m \geq 2$. Let $G \cong \bigcup\limits_{j=1}^{m} F_{n_j}$ be a disjoint union of the friendship graphs F_{n_j}. Then*

$$tvs(G) = \left\lceil \frac{2 + 2\sum\limits_{j=1}^{m} n_j}{3} \right\rceil. \tag{8}$$

Proof. The disjoint union of the friendship graphs has $2\sum\limits_{j=1}^{m} n_j$ vertices of degree 2, say, u_i^j, v_i^j, $1 \leq j \leq m$, $1 \leq i \leq n_j$, and vertices of degree $2n_j$, say, c^j, $1 \leq j \leq m$. From inequality (7) it follows that

$$tvs(G) \geq \left\lceil \frac{2 + 2\sum\limits_{j=1}^{m} n_j}{3} \right\rceil. \tag{9}$$

For our convenient, we order the friendship graphs F_{n_j} such that $n_1 \leq n_2 \leq \cdots \leq n_m$. Let $E(G) = \{c^j v_i^j, c^j u_i^j : 1 \leq j \leq m, 1 \leq i \leq n_j\} \cup \{v_i^j u_i^j : 1 \leq j \leq m, 1 \leq i \leq n_j\}$ be the edge set of $\bigcup\limits_{j=1}^{m} F_{n_j}$.

Put $k = \left\lceil \frac{2+2\sum_{j=1}^{m} n_j}{3} \right\rceil$. To show that k is an upper bound for total vertex irregularity strength of disjoint union of the friendship graphs we describe a total k-labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ as follows:

$\phi(c^j) = k$ for $1 \leq j \leq m$,

$$\phi(u_i^j) = \begin{cases} 1, & \text{if } 1 \leq i \leq n_j - 1 \text{ and } 1 \leq j \leq m \\ 2\left(1 - m - k + \sum\limits_{s=1}^{m} n_s\right) + j, & \text{if } i = n_j \text{ and } 1 \leq j \leq m \end{cases}$$

$$\phi(v_i^j) = \begin{cases} 1 - m - k + \sum\limits_{s=1}^{m} n_s + \left\lceil \frac{1+i-j+\sum_{s=1}^{j-1} n_s}{2} \right\rceil, & \text{if } 1 \leq i \leq n_j - 1 \\ & \text{and } \ 1 \leq j \leq m \\ 2\left(1 - k + \sum\limits_{s=1}^{m} n_s\right) - m + j, & \text{if } i = n_j \text{ and } 1 \leq j \leq m \end{cases}$$

$$\phi(c^j u_i^j) = \begin{cases} \left\lceil \frac{1+i-j+\sum_{s=1}^{j-1} n_s}{2} \right\rceil, & \text{if } 1 \leq i \leq n_j - 1 \text{ and } 1 \leq j \leq m \\ k, & \text{if } i = n_j \text{ and } 1 \leq j \leq m \end{cases}$$

$$\phi(u_i^j v_i^j) = \begin{cases} \left\lceil \frac{2+i-j+\sum_{s=1}^{j-1} n_s}{2} \right\rceil, & \text{if } 1 \leq i \leq n_j - 1 \text{ and } 1 \leq j \leq m \\ k, & \text{if } i = n_j \text{ and } 1 \leq j \leq m \end{cases}$$

$\phi(c^j v_i^j) = k$ for $1 \leq j \leq m$ and $1 \leq i \leq n_j$.
Under the labeling ϕ for total vertex-weights we have:

$$wt(u_i^j) = \begin{cases} 3 + i - j + \sum\limits_{s=1}^{j-1} n_s, & \text{if } 1 \leq i \leq n_j - 1 \text{ and } 1 \leq j \leq m \\ 2(\sum\limits_{s=1}^{m} n_s - m) + 2 + j, & \text{if } i = n_j \text{ if } 1 \leq j \leq m \end{cases}$$

$$wt(v_i^j) = \begin{cases} 3 - m + i - j + \sum\limits_{s=1}^{m} n_s + \sum\limits_{s=1}^{j-1} n_s, & \text{if } 1 \leq i \leq n_j - 1 \\ & \text{and } 1 \leq j \leq m \\ 2 - m + j + 2\sum\limits_{s=1}^{m} n_s, & \text{if } i = n_j \text{ and } 1 \leq j \leq m \end{cases}$$

$$wt(c^j) = (n_j + 2)k + \sum_{i=1}^{n_j - 1} \left\lceil \frac{1+i-j+\sum_{s=1}^{j-1} n_s}{2} \right\rceil \text{ for } 1 \leq j \leq m.$$

It is a routine matter to verify that all vertex and edge labels are at most k and the total vertex-weights are different for all pairs of distinct vertices. In fact,

$$tvs\left(\bigcup_{j=1}^{m} F_{n_j}\right) \leq \left\lceil \frac{2+2\sum_{j=1}^{m} n_j}{3} \right\rceil. \tag{10}$$

Combining (10) with the lower bound given by (9), we conclude that

$$tvs\left(\bigcup_{j=1}^{m} F_{n_j}\right) = k.$$

□

Using the previous theorem we can get the following corollary.

Corollary 2.1. *Let F_n be a friendship graph with n triangles, $n \geq 3$ and let mF_n be the disjoint union of m copies of F_n, $m \geq 2$. Then*

$$tvs(mF_n) = \left\lceil \frac{2(mn+1)}{3} \right\rceil. \tag{11}$$

Proof. Since for the disjoint union of m copies of the friendship graph F_n we have that $\delta(mF_n) = 2$ and number of vertices of degree δ is $n_\delta = 2mn$ then from inequality (7) it follows that

$$tvs(mF_n) \geq \left\lceil \frac{2(mn+1)}{3} \right\rceil. \tag{12}$$

According the proof of previous theorem from (10) it follows that

$$tvs(mF_n) \leq \left\lceil \frac{2(mn+1)}{3} \right\rceil. \tag{13}$$

Combining (12) and (13) produces the desired result. □

The result from Theorem 2.2 adds further support to a recent conjecture.

Conjecture 2. [16] *Let G be a connected graph having n_i vertices of degree i ($i = \delta, \delta+1, \delta+2, \dots, \Delta$), where δ and Δ are the minimum and the maximum degree of G, respectively. Then*

$$tvs(G) = \max\left\{ \left\lceil \frac{\delta+n_\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta+n_\delta+n_{\delta+1}}{\delta+2} \right\rceil, \dots, \left\lceil \frac{\delta+\sum_{i=\delta}^{\Delta} n_i}{\Delta+1} \right\rceil \right\}.$$

3. Total edge irregularity strength of disjoint union of friendship graphs

The following theorem determines the exact value of the total edge irregularity strength for disjoint union of arbitrary friendship graphs.

Theorem 3.1. *Let F_{n_j} be a friendship graph with n_j triangles, $n_j \geq 3$ and $1 \leq j \leq m$, $m \geq 2$. Let $G \cong \bigcup_{j=1}^{m} F_{n_j}$ be a disjoint union of the friendship graphs F_{n_j}. Then*

$$tes(G) = 1 + \sum_{j=1}^{m} n_j. \tag{14}$$

Proof. The disjoint union of the friendship graphs, $\bigcup_{j=1}^{m} F_{n_j}$, has $3\sum_{j=1}^{m} n_j$ edges. From (5) is given the following lower bound on the total edge irregularity strength.

$$tes\left(\bigcup_{j=1}^{m} F_{n_j}\right) \geq \left\lceil \frac{2 + 3\sum_{j=1}^{m} n_j}{3} \right\rceil = 1 + \sum_{j=1}^{m} n_j. \tag{15}$$

Put $k = 1 + \sum_{j=1}^{m} n_j$. In view that k is an upper bound on the total edge irregularity strength of $\bigcup_{j=1}^{m} F_{n_j}$ it suffices to prove the existence of a total k-labeling $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ such that

$$\varphi(x) + \varphi(xy) + \varphi(y) \neq \varphi(x') + \varphi(x'y') + \varphi(y')$$

for every $xy, x'y' \in E$ with $xy \neq x'y'$.

For vertices and edges of $\bigcup_{j=1}^{m} F_{n_j}$ let

$$\begin{aligned}
\varphi(u_i^j) = \varphi(u_i^j v_i^j) &= 1, && \text{for } 1 \leq i \leq n_j \text{ and } 1 \leq j \leq m \\
\varphi(v_i^j) = \varphi(c^j u_i^j) &= i + \sum_{s=1}^{j-1} n_s, && \text{for } 1 \leq i \leq n_j \text{ and } 1 \leq j \leq m \\
\varphi(c^j) = \varphi(c^j v_i^j) &= k, && \text{for } 1 \leq i \leq n_j \text{ and } 1 \leq j \leq m.
\end{aligned}$$

Observe that the total edge-weights under the labeling φ constitute the sets

$W_1 = \{wt(u_i^j v_i^j) = 2 + i + \sum_{s=1}^{j-1} n_s : 1 \leq i \leq n_j, 1 \leq j \leq m\} = \{3, 4, \ldots, k+1\}$,

$W_2 = \{wt(c^j u_i^j) = k + 1 + i + \sum_{s=1}^{j-1} n_s : 1 \leq i \leq n_j, 1 \leq j \leq m\} = \{k+2, k+3, \ldots, 2k\}$,

$W_3 = \{wt(c^j v_i^j) = 2k + i + \sum_{s=1}^{j-1} n_s : 1 \le i \le n_j, 1 \le j \le m\} = \{2k+1, 2k+2, \dots, 3k-1\}.$

It is not difficult to see that the function φ is the required total k-labeling such that the total edge-weights are different for all edges.

Thus we have that

$$tes\left(\bigcup_{j=1}^{m} F_{n_j}\right) \le 1 + \sum_{j=1}^{m} n_j.$$

This concludes the proof. □

From Theorem 3.1 it is easy to get the following corollary.

Corollary 3.1. *Let F_n be a friendship graph with n triangles, $n \ge 3$ and let mF_n be the disjoint union of m copies of F_n, $m \ge 2$. Then*

$$tes(mF_n) = mn + 1. \tag{16}$$

Proof. The disjoint union of m copies of the friendship graph F_n has $3mn$ edges and its maximum degree is $2n$. Hence, from (5) it follows that

$$tes(mF_n) \ge \max\left\{\left\lceil \frac{3mn+2}{3} \right\rceil, \left\lceil \frac{2n+1}{2} \right\rceil\right\}. \tag{17}$$

For $m \ge 2$, (17) gives $tes(mF_n) \ge mn + 1$. From the proof of Theorem 3.1 it follows the existence of total $(mn+1)$-labeling φ where under labeling φ total edge-weights are different for all edges. Thus we arrive at the desired result. □

Our result on total edge irregularity strength of disjoint union of friendship graphs adds further support to Conjecture 1.

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