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On domination numbers of zero-divisor graphs of commutative rings

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Abstract

Zero-divisor graphs of a commutative ring R, denoted $\Gamma(R)$, are well-represented in the literature. In this paper, we consider domination numbers of zero-divisor graphs. For reduced rings, Vatandoost and Ramezani characterized the possible graphs for $\Gamma(R)$ when the sum of the domination numbers of $\Gamma(R)$ and the complement of $\Gamma(R)$ is $n-1$, n, and $n+1$, where n is the number of nonzero zero-divisors of R. We extend their results to nonreduced rings, determine which graphs are realizable as zero-divisor graphs, and provide the rings that yield these graphs.

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1. Introduction

The concept of the graph of the zero-divisors of a commutative ring was first introduced by Beck in [4] when discussing the coloring of a commutative ring. In his work all elements of the ring were considered vertices of the graph. Since the seminal paper by D.F. Anderson and Livingston [3], the standard is to regard only nonzero zero-divisors as vertices of the graph, and we adhere to this standard. The *zero-divisor graph* of R, denoted $\Gamma(R)$, is the graph with $V(\Gamma(R)) = Z(R)^*$, and for distinct $r, s \in Z(R)^*$, $r - s \in E(\Gamma(R))$ if and only if $rs = 0$. Among other results, Anderson and Livingston proved that $\Gamma(R)$ is always connected and has diameter at most 3 ([3,

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Theorem 2.3]). This discovery of strong graphical structure in zero-divisor graphs has inspired researchers to continue exploring how the graphical structure of the zero-divisor graph might reveal information about the algebraic structure in the ring, a desire necessitated by the frequent lack of closure under addition in the set of zero-divisors of a ring. For general surveys of $\Gamma(R)$, see [2] and [6].

Domination has been extensively studied in the literature, though less frequently in regards to graphs constructed from rings. (See [7], [11], and [1] for some recent examples of papers on domination.) The main focus of this paper concerns the domination number of zero-divisor graphs. In particular, we generalize theorems from Section 4 of [15]. The results presented in that paper focus on what graphs are possible, and we determine which of these graphs are actually realizable as zero-divisor graphs of commutative rings. Further, whereas the results of [15] were restricted to reduced commutative rings with identity, or rings that have no nontrivial nilpotent elements, we extend these results to non-reduced rings. In some papers on zero-divisor graphs, $\Gamma(R)$ is not a simple graph in the sense that a vertex v could have a loop if and only if $v^2 = 0$. Since looped vertices do not impact the domination number of a graph, this paper will adopt the convention that all zero-divisor graphs are simple graphs. For reference, we will make copious use of the results found in [14].

Below is a summary theorem outlining the main results of this paper.

Theorem 1.1. *Let R be a commutative ring with identity.*

1. $\gamma(\Gamma(R)) = \frac{n}{2}$ *if and only if* $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ *or* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
2. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n+1$ *if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $Z(R)$ *is an ideal with* $(Z(R))^2 = 0$
3. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n$ *if and only if* $R \cong \mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, *or* $\mathbb{Z}_4[x]/(2x, x^2-2)$
4. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$ *if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$ *or* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

2. Definitions

Throughout, by a *ring* we mean a commutative ring with identity, typically denoted by R. We use $Z(R)$ to denote the set of zero-divisors of R and $Z(R)^*$ to denote the set of nonzero zero-divisors. For the set of integers modulo n and the field with n elements, we use the notations $\mathbb{Z}_n$ and $\mathbb{F}_n$, respectively. For $a \in R$, the *annihilator* of a is $\mathrm{ann}(a) = \{x \in R \mid ax = 0\}$. A ring is *local* if it has a unique maximal ideal, typically denoted by M. For a general algebra reference, see [9].

For any graph G, we denote the set of vertices of G by $V(G)$ and the set of edges by $E(G)$. We will write $v - w$ when vertices v and w are *adjacent*, or are incident to the same edge edge. By a *path* between v and w, we mean a sequence of vertices and edges $v - x_1 - x_2 - \cdots - x_n - w$, and G is *connected* if there exists a path between any two distinct vertices. The *distance* between v and w, denoted by $d(v, w)$, is the number of edges in a shortest path connecting v and w (note that $d(v, v) = 0$ and $d(v, w) = \infty$ if no such path exists). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(v, w) \mid v, w \in V(G)\}$. For a general graph theory reference, see [5].

If every pair of distinct vertices are adjacent in a graph G, then G is said to be a *complete* graph, and a complete graph on n vertices is denoted as K_n. A graph G is called *complete bipartite* if there exist sets $A, B \subset V(G)$ such that $A \cup B = V(G)$, $A \cap B = \emptyset$, for all $v_i, v_j \in A$ and $w_i, w_j \in B$, we have $v_i - v_j \notin E(G)$, $w_i - w_j \notin E(G)$, and for all $v_i \in A$ and $w_j \in B$, we have $v_i - w_j \in E(G)$. Finite complete bipartite graphs are denoted as $K_{m,n}$, where $|A| = m$ and $|B| = n$. If $|A| = 1$, then the graph $K_{1,n}$ is called a *star graph*. A graph in which at least one vertex is adjacent to every other vertex is called a *star-shaped reducible*. The graph $v_1 - v_2 - \cdots - v_n$ with no other edges or vertices is called the *path graph* on n vertices and is denoted P_n, while the graph $v_1 - v_2 - \cdots - v_n - v_1$ with no other edges or vertices is called the *cycle graph* on n vertices and is denoted C_n. To create the *corona* of graphs G and H, denoted $G \circ H$, let $V(G) = \{v_1, v_2, ..., v_n\}$. Enumerate n copies of H as $H_1, H_2, ..., H_n$. Then we create $G \circ H$ by joining v_i to every vertex in H_i with an edge for $i = 1, ..., n$.

For a graph G, a set $X \subseteq V(G)$ is a *dominating set* of G if for every $y \in V(G) \backslash X$ there exists $x \in X$ such that $x - y \in E(G)$. The *domination number* of G, denoted $\gamma(G)$, is $\gamma(G) = \min\{|X| \mid X \text{ is a dominating set of } G\}$.

This paper will also focus on the *complement* of a zero-divisor graph of a commutative ring R. Given R, the complement of $\Gamma(R)$ is denoted $\overline{\Gamma(R)}$ with $V(\Gamma(R)) = V(\overline{\Gamma(R)})$, and $a - b \in E(\overline{\Gamma(R)})$ if and only if $a - b \notin E(\Gamma(R))$; i.e., $ab \neq 0$ in R.

Throughout this paper, we only consider finite rings and will use n to denote $|V(\Gamma(R))|$; equivalently, $|Z(R)^*|$.

3. Domination numbers of zero-divisor graphs

In [15], Vatandoost and Ramezani investigated the domination number and signed domination number of reduced commutative rings. A *reduced commutative ring* is a commutative ring in which $x^2 = 0$ if and only if $x = 0$. Given R, a reduced commutative ring with identity, the results in [15] classified the realizable graphs for $\Gamma(R)$ if $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) \in \{n-1, n, n+1\}$.

The results presented below make repeated use of the excellent paper by Redmond, [14]. In this paper, Redmond classifies all possible zero-divisor graphs with 14 or fewer vertices and their associated commutative rings. Thus, given a graph with 14 or fewer vertices, it is possible to know whether it corresponds to a zero-divisor graph of a commutative ring and to which ring(s).

Our first theorem generalizes [15, Theorem 4.1], which states that for R, a reduced commutative ring with identity, $\gamma(\Gamma(R)) = \frac{n}{2}$ if and only if $\Gamma(R)$ is C_4 or $K_3 \circ K_1$. The following result from [8] and [13] is used in the proof.

Lemma 3.1. *[8, 13] For a graph Γ with even order m and no isolated vertices, $\gamma(\Gamma) = \frac{n}{2}$ if and only if the components of Γ are the cycle C_4 or the corona $H \circ K_1$, where H is a connected graph.*

Theorem 3.2. *Let R be a commutative ring with identity. Then $\gamma(\Gamma(R)) = \frac{n}{2}$ if and only if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. ($\Leftarrow$) It is easy to check that $|Z(\mathbb{Z}_3 \times \mathbb{Z}_3)^*| = 4$ and $\gamma(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$ and that $|Z(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)^*| = 6$ and $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 3$ (see Figure 1).

($\Rightarrow$) By Lemma 3.1, $\Gamma(R)$ is the cycle C_4 or the corona $H \circ K_1$, where H is a connected graph. If $\Gamma(R)$ is C_4, then $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ by [14]. Now suppose $\Gamma(R)$ is $H \circ K_1$ where H is a connected graph. Let $A = \{a_i \in Z(R)^* \mid \deg(a_i) > 1 \text{ in } \Gamma(R)\}$; i.e., A consists of the vertices from H.

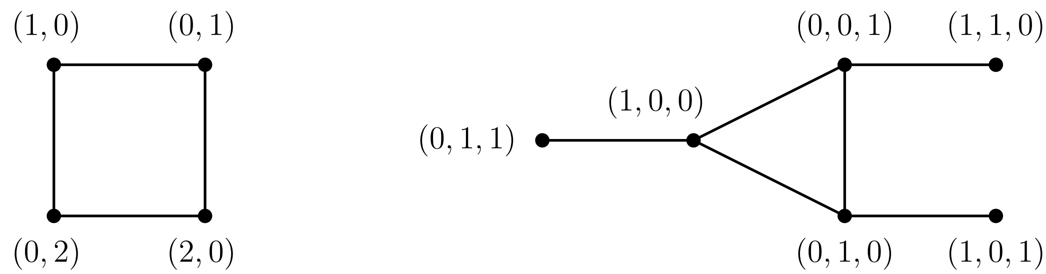

Figure 1. $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Since $\operatorname{diam}(\Gamma(R)) \leq 3$, the induced subgraph on A is complete. We consider two cases based on the size of A.

If $|A| = 2$, then $\Gamma(R)$ is the path graph P_4 with $a - b - c - d$. However, by [3, Example 2.1(b)], P_4 is not the zero-divisor graph of any commutative ring with identity. Suppose $|A| > 3$. Let $a_i \in A$ and let $\overline{a}_i \in Z(R)^*$ with $\operatorname{ann}(\overline{a}_i) \cap A = \{a_i\}$. Consider $\overline{a}_1 + a_2$. Then $\overline{a}_1 + a_2 \neq 0$. Otherwise, $\operatorname{ann}(\overline{a}_1) = \operatorname{ann}(a_2) = A \cup \{0, \overline{a}_2\}$, a contradiction. Since $a_1(\overline{a}_1 + a_2) = 0$, we have $\overline{a}_1 + a_2 \in Z(R)^*$. Since $\overline{a}_1 + a_2 \in \operatorname{ann}(a_1)\backslash\{\overline{a}_1\}$, we see that $\overline{a}_1 + a_2 \in A$. Let $b \in A\backslash\{a_1, a_2, \overline{a}_1 + a_2\}$ (since $|A| > 3$). Since the subgraph induced by A is complete, $b(\overline{a}_1 + a_2) = 0$. Thus, $b\overline{a}_1 = 0$ because $ba_2 = 0$. This is a contradiction.

Therefore, $|A| = 3$. This implies $\Gamma(R) \cong K_3 \circ K_1$. By [14], $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

The following theorem characterizes exactly when $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n + 1$. This theorem is a generalization of [15, Theorem 4.2], which states for R, a reduced commutative ring with identity, $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n + 1$ if and only if $\Gamma(R)$ is the complete graph K_n.

Theorem 3.3. *Let R be a commutative ring with identity. Then the following are equivalent.*

1. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n + 1$.
2. $\Gamma(R)$ *is the complete graph* K_n.
3. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $xy = 0$ *for all* $x, y \in Z(R)$.
4. $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $Z(R)$ *is an ideal with* $Z(R)^2 = \{0\}$.

Proof. The equivalence of 1 and 2 follows from the proof of [15, Theorem 4.2], while the equivalences of 2, 3, and 4 follow from [3, Corollary 2.7 and Theorem 2.8]. □

Note that if R is Artinian, then statement (4) of Theorem 3.3 is equivalent to $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with maximal ideal M such that $M^2 = \{0\}$.

In reference to [15], if R is reduced the following corollary holds.

Corollary 3.4. *Let R be a reduced commutative ring with identity. Then $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n + 1$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $|Z(\mathbb{Z}_2 \times \mathbb{Z}_2)^*| = 2$. By construction of $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}$, we see that $\gamma(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) + \gamma(\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)}) = 1 + 2 = 2 + 1$. Conversely, by Theorem 3.3, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$. Thus, since R is reduced, we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

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We remind the reader of a useful graph theory result.

Lemma 3.5. *[12, Theorem 13.1.3] If a simple graph G has n vertices and no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Thus, since zero-divisor graphs of commutative rings are connected and in this paper simple graphs are considered, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. This result will be utilized in the proof of Theorem 3.7 (cf. [15, Theorem 1.3]). In addition, we will use a result that relates the number of vertices, number of edges, and domination number of a graph. The next lemma follows from a theorem in [16], which states for a simple graph G with n vertices and m edges, if $\gamma(G) \geq 2$, then

$$m \leq \left\lfloor \frac{(n-\gamma(G))(n-\gamma(G)+2)}{2} \right\rfloor.$$

Lemma 3.6. *Let G be a simple graph with $n \geq 2$ vertices. Then $\gamma(\overline{G}) = n-1$ if and only if G has exactly one edge.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) If G has exactly one edge, then precisely one vertex is dominated and, thus, $\gamma(\overline{G}) = n-1$. □

We now consider when $\gamma(\Gamma(R))+\gamma(\overline{\Gamma(R)}) = n$. In the case when R is a reduced commutative ring with identity, Vatandoost and Ramezani proved $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n$ if and only if $\Gamma(R)$ is C_4 or P_3 ([15, Theorem 1.3]).

Theorem 3.7. *Let R be a commutative ring with identity. Then the following are equivalent.*

1. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n$.
2. $\Gamma(R)$ *is* C_4 *or* P_3.
3. $R \cong \mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, *or* $\mathbb{Z}_4[x]/(2x, x^2-2)$.

Proof. The equivalence of 2 and 3 follows from [14]. It is straightforward to verify ($2 \Rightarrow 1$).

($1 \Rightarrow 2$) If $\gamma(\overline{\Gamma(R)}) = \frac{n}{2}$, then $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 3.2. Observe that $\gamma(\overline{\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)}) = 2$ and $\gamma(\overline{\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}) = 2$. Thus, $\gamma(\Gamma(R))+\gamma(\overline{\Gamma(R)}) = n$ holds when $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ but not for $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is C_4.

If $\gamma(\Gamma(R)) < \frac{n}{2}$, then $\gamma(\overline{\Gamma(R)}) > \frac{n}{2}$. Therefore, $\overline{\Gamma(R)}$ has an isolated vertex by Lemma 3.5. Thus, $\gamma(\Gamma(R)) = 1$ and $\gamma(\overline{\Gamma(R)}) = n-1$. By Lemma 3.6, $\overline{\Gamma(R)}$ has exactly one edge. We now consider possible values of n. If $n = 1$, then $\gamma(\Gamma(R)) = \gamma(\overline{\Gamma(R)}) = 1$. If $n = 2$, then $\gamma(\overline{\Gamma(R)})$ consists of two isolated vertices. In both cases, $\gamma(\overline{\Gamma(R)}) \neq n-1$. These contradictions imply that $n \geq 3$.

If $n > 3$, then $\overline{\Gamma(R)}$ consists of $n-2$ isolated vertices and two vertices that are incident to a single edge. Without loss of generality, say $a_1 – a_2 \in E(\overline{\Gamma(R)})$. Let $Z(R)^* = \{a_1, a_2, \ldots, a_n\}$ with $a_i a_j = 0$ whenever $i \neq j$ and $\{i, j\} \neq \{1, 2\}$. For any a_i, $a_j \in Z(R)^*$, there exists $a_k \in Z(R)^*$ such that $a_k a_i = 0$ and $a_k a_j = 0$. Thus, $a_k(a_i + a_j) = 0$. Hence, $Z(R)$ is closed under addition. Since $|Z(R)^*| > 3$, there exists $a_i \in Z(R)^* \backslash \{a_1, a_2\}$ such that $a_1 + a_i \notin \{a_1, a_2\}$. We see

$0 = a_1(a_1 + a_i) = a_1^2 + a_1 a_i = a_1^2$, which yields $0 = a_1(a_1 + a_2) = a_1^2 + a_1 a_2 = a_1 a_2$, a contradiction.

Thus, it must be that $n = 3$. In this case the graph on the left in Figure 2 shows the only possiblity for $\overline{\Gamma(R)}$. This implies that $\Gamma(R)$ is as shown on the right in Figure 2. Hence, $\Gamma(R)$ is P_3.

Figure 2. $\overline{\Gamma(R)}$ for $n = 3$, and its associated $\Gamma(R)$.

□

We now discuss necessary and sufficient conditions for $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$. Note that when R is a reduced commutative ring with identity, [15, Theorem 1.4] states $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$ if and only if $\Gamma(R)$ is isomorphic to $K_{1,3}$ or $K_3 \circ K_1$. First, we provide two observations that will be helpful when classifying these rings.

Observation 3.8. *If $m, n > 1$, then $\gamma(K_{m,n}) + \gamma(\overline{K_{m,n}}) = 4$. If $m = 1$ or $n = 1$, then $\gamma(K_{m,n}) + \gamma(\overline{K_{m,n}}) = 3$. In addition, $\gamma(K_n) + \gamma(\overline{K_n}) = 1 + n$.*

Observation 3.9. *For a commutative ring R with identity, the equation $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$ where $|Z(R)^*| = n$ can only hold if $n > 3$.*

Note that Observation 3.9 follows from the fact that if $n = 1$ or $n = 2$, then the equation fails to hold as $\gamma(\Gamma(R)) \geq 1$ and $\gamma(\overline{\Gamma(R)}) \geq 1$. If $n = 3$, then we see that $\Gamma(R)$ is either $K_{1,2}$ or K_3 since $\Gamma(R)$ is connected. In both cases, $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) \neq n - 1$.

In the following proposition, two more possibilities for n are eliminated when $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$.

Proposition 3.10. *Let R be a commutative ring with identity. If $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$ and $\Gamma(R)$ is star-shaped reducible, then $n \notin \{5, 6\}$.*

Proof. If $n = 5$, then by [14] we have $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$. Since $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_5)$ is $K_{1,4}$, $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = 1 + 2 \neq 5 - 1$. If $n = 6$, then by again by [14], $\Gamma(R)$ is K_6. Hence, $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = 1 + 6 \neq 6 - 1$. □

We now build upon the above results.

Theorem 3.11. *Let R be a commutative ring with identity. Then the following are equivalent.*

1. $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$.
2. $\Gamma(R)$ *is* $K_{1,3}$ *or* $K_3 \circ K_1$.

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3. $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$, *or* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. By [14], we have (2 ⇔ 3).

(2 ⇒ 1) By Observation 3.8, if $\Gamma(R)$ is $K_{1,3}$, then $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = 3 = 4 - 1 = |Z(R)^*| - 1$ since $|Z(R)^*| = 4$. Similarly, if $\Gamma(R)$ is $K_3 \circ K_1$, then, as shown in Figure 3, $\gamma(\Gamma(R)) = 3$ while $\gamma(\overline{\Gamma(R)}) = 2$. Hence, $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = 3 + 2 = 6 - 1 = |Z(R)^*| - 1$ since $|Z(R)^*| = 6$.

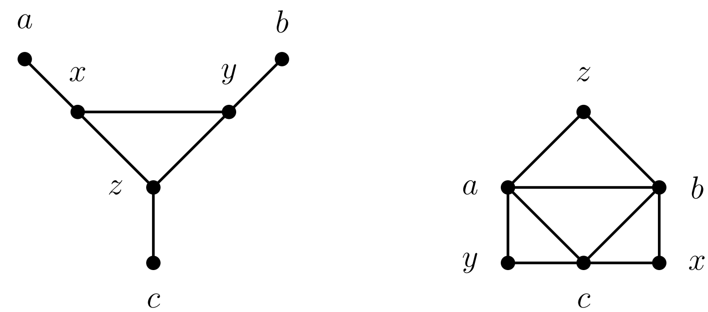

Figure 3. $K_3 \circ K_1$ and $\overline{K_3 \circ K_1}$

(1 ⇒ 2) If $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$, then $n \geq 4$ by Observation 3.9. As before, since $\Gamma(R)$ is connected, we have $\gamma(\Gamma(R)) \leq \frac{n}{2}$ by Lemma 3.5. Three cases are considered: $\gamma(\Gamma(R)) = \frac{n}{2}$, $\gamma(\Gamma(R)) = \frac{n}{2} - 1$, and $\gamma(\Gamma(R)) < \frac{n}{2} - 1$.

Case 1. Suppose that $\gamma(\Gamma(R)) = \frac{n}{2}$. Then $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 3.2. From Theorem 3.7, if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n$. Thus, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Gamma(R)$ is $K_3 \circ K_1$.

Case 2. Suppose that $\gamma(\Gamma(R)) = \frac{n}{2} - 1$. Thus $\gamma(\overline{\Gamma(R)}) = \frac{n}{2}$. By [10], we have $\gamma(\Gamma(R))\gamma(\overline{\Gamma(R)}) \leq n$. So, $(\frac{n}{2} - 1)(\frac{n}{2}) \leq n$, which implies that $\frac{n}{2} - 1 \leq 2$. Therefore, $n \leq 6$. By Observation 3.9, $n = 4, 5$, or 6. However, $\frac{n}{2} - 1 \in \mathbb{Z}$, so $n = 4$ or $n = 6$.

If $n = 4$, then $\Gamma(R)$ is one of $K_{2,2}$, K_4, or $K_{1,3}$ by [14]. Since $\gamma(K_{2,2}) = 2 \neq \frac{n}{2} - 1$, $\Gamma(R)$ is not $K_{2,2}$. Since $\gamma(\overline{K_4}) = 4$, we see that $\Gamma(R)$ is not K_4. Since $\gamma(\Gamma(K_{1,3})) + \gamma(\overline{\Gamma(K_{1,3})}) = 1 + 2 = 4 - 1$, we see that $\Gamma(R)$ is $K_{1,3}$. By [14], $R \cong \mathbb{Z}_2 \times \mathbb{F}_4$.

If $n = 6$, then $\Gamma(R)$ is one of K_6, $K_3 \circ K_1$, $K_{2,4}$, or $K_{3,3}$ by [14]. Since $\gamma(K_6) = 1 \neq \frac{n}{2} - 1$ and $\gamma(K_3 \circ K_1) = 3 \neq \frac{n}{2} - 1$, $\Gamma(R)$ is not K_6 or $K_3 \circ K_1$. By Observation 3.8, $K_{2,4}$ and $K_{3,3}$ do not satisfy $\gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1$.

Case 3. Suppose that $\gamma(\Gamma(R)) < \frac{n}{2} - 1$. Since $\gamma(\overline{\Gamma(R)}) > \frac{n}{2}$, by Lemma 3.5 $\overline{\Gamma(R)}$ has an isolated vertex, say w. Thus, in $\Gamma(R)$ the vertex w is adjacent to all other vertices. Hence, $\Gamma(R)$ is star-shaped reducible. This implies that $\gamma(\Gamma(R)) = 1$ and $\gamma(\overline{\Gamma(R)}) = n - 2$.

By Observation 3.9, $n \geq 4$, and by Proposition 3.10, we have $n \neq 5, 6$. Therefore, we have either $n = 4$ or $n \geq 7$. Since $\gamma(\Gamma(R)) \geq 1$ and $\gamma(\Gamma(R)) < \frac{n}{2} - 1$, $n = 4$ is not possible. Thus, $n \geq 7$. The remainder of the proof will show that $n \geq 7$ is not possible.

Pick a minimum dominating set D of $\overline{\Gamma(R)}$. Since $\gamma(\overline{\Gamma(R)}) = n - 2$, there are vertices $a, b \notin D$ and $Z(R)^* = D \cup \{a, b\}$. Also, there exists $d_i, d_j \in D$ such that $a - d_i, b - d_j \in E(\overline{\Gamma(R)})$. Let C_a and C_b be connected components of $\overline{\Gamma(R)}$ containing a and b, respectively. We show two things:

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1. $D\backslash(C_a \cup C_b)$ consists solely of isolated vertices in $\overline{\Gamma(R)}$.
2. There are five possible graph configurations for C_a and C_b, and hence for $\overline{\Gamma(R)}$.

First, we show $D\backslash(C_a \cup C_b)$ consists solely of isolated vertices in $\overline{\Gamma(R)}$. Pick $d \in D\backslash(C_a \cup C_b)$, and suppose $d - x \in E(\overline{\Gamma(R)})$ for some $x \in Z(R)^*$. Clearly $x \notin \{a, b\}$. Since $d \notin C_a \cup C_b$ we have $d \notin \{d_i, d_j\}$. Since $d - x, a - d_i, b - d_j \in E(\overline{\Gamma(R)})$ and $Z(R)^* = D \cup \{a, b\}$, $D' = D\backslash\{d\}$ is a dominating set of $\overline{\Gamma(R)}$. This is a contradiction since $|D'| < |D|$. Hence, each vertex of $D\backslash(C_a \cup C_b)$ is an isolated vertex of $\overline{\Gamma(R)}$.

The above shows that if C_a and C_b are the same component, then

$$\gamma(\overline{\Gamma(R)}) = n - 2 = \gamma(C_a) + n - |V(C_a)|. \quad (1)$$

In addition, if C_a and C_b are disjoint components, then

$$\gamma(\overline{\Gamma(R)}) = n - 2 = \gamma(C_a) + \gamma(C_b) + n - |V(C_a)| - |V(C_b)|. \quad (2)$$

The possible graph configurations for C_a and C_b are now investigated based on whether or not C_a and C_b are the same component or disjoint components.

Subcase 1. Assume that C_a and C_b consist of the same connected component in $\overline{\Gamma(R)}$.

If $|V(C_a)| = m \geq 5$, then $\gamma(C_a) \leq \frac{m}{2}$ by Lemma 3.5. Thus, $\gamma(\overline{\Gamma(R)}) \leq \frac{m}{2} + n - m = n - \frac{m}{2}$ by Equation 1. However, $n - \frac{m}{2} > n - 2$ since $m \geq 5$, a contradiction. Thus, $|V(C_a)| \leq 4$. Note that $|V(C_a)| > 2$ since $a, b \notin D$.

If $|V(C_a)| = 4$, then $\gamma(C_a) = 2$ since, by Equation 1, $n - 2 = \gamma(C_a) + n - 4$. Hence, C_a is either C_4 or P_4.

If $|V(C_a)| = 3$, then $\gamma(C_a) = 1$ since, by Equation 1, $n - 2 = \gamma(C_a) + n - 3$. Hence, C_a is either C_3 or P_3.

Subcase 2. Assume C_a and C_b are disjoint components of $\overline{\Gamma(R)}$. Clearly, $|V(C_a)|, |V(C_b)| \geq 2$. Assume, without loss of generality, that $|V(C_a)| = m \geq 3$. Then $\gamma(C_a) \leq \frac{m}{2}$ by Lemma 3.5. Hence, by Equation 2,

$$n - 2 = \gamma(C_a) + \gamma(C_b) + n - |V(C_a)| - |V(C_b)| \leq \frac{m}{2} + \gamma(C_b) + n - m - |V(C_b)|$$

which simplifies to

$$|V(C_b)| - \gamma(C_b) + \frac{m}{2} \leq 2$$

This inequality is impossible since $|V(C_b)| - \gamma(C_b) \geq 1$ and $\frac{m}{2} \geq \frac{3}{2}$. Similarly, $|V(C_b)|$ cannot be greater than or equal to 3. Thus, $|V(C_a)| = |V(C_b)| = 2$, and hence C_a and C_b are P_2.

The above work shows there are 5 possible configurations for $\overline{\Gamma(R)}$, as shown in Figure 4.

We show that none of these configurations for $\overline{\Gamma(R)}$ are possible. Recall that $n \geq 7$ as stated at the beginning of Case 3.

Configuration 1. The graphs of $\overline{\Gamma(R)}$ and $\Gamma(R)$ for Configuration 1 are shown in Figure 5. Let $a, b, c,$ and d be as shown in Figure 5.

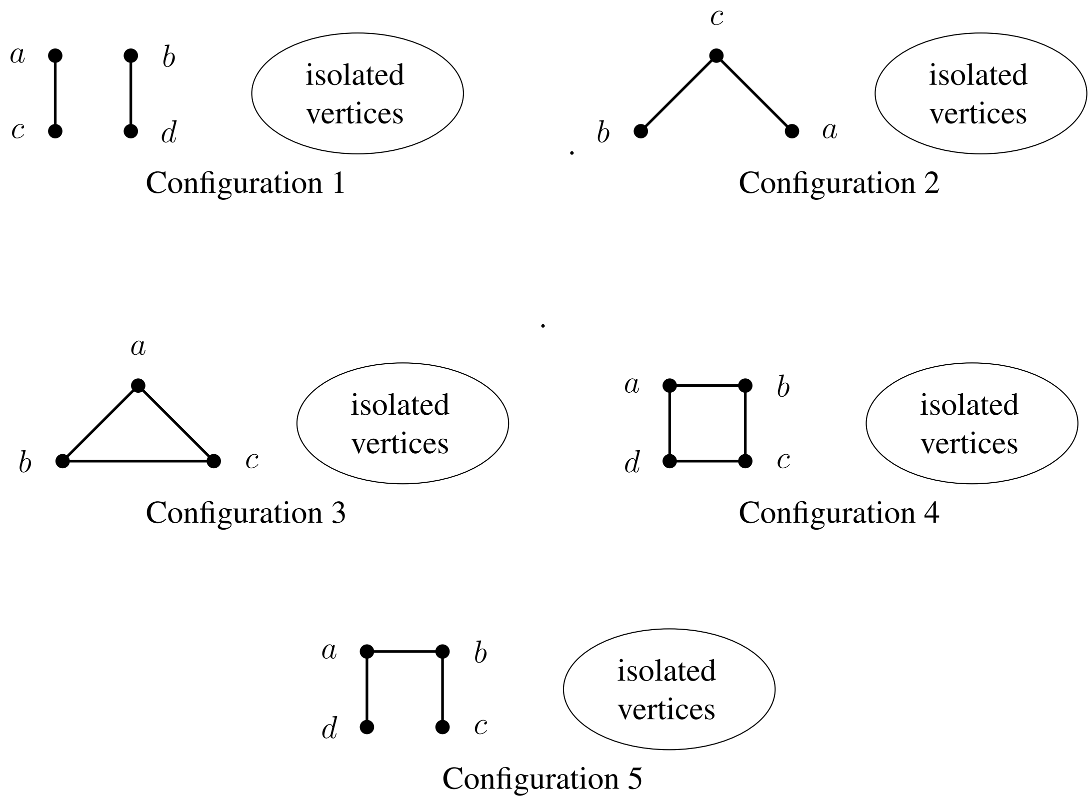

Figure 4. The five configurations for $\overline{\Gamma(R)}$

Consider $a+b$. Since $\operatorname{ann}(a) \neq \operatorname{ann}(b)$, $a \neq -b$. Thus, $a+b \neq 0$. Since there exists $l \in Z(R)^*$ with $la = lb = 0$, $l(a+b) = 0$. This implies $a+b \in Z(R)^*$. Clearly, $a+b \neq a$ and $a+b \neq b$. Observe from $\Gamma(R)$ that $c(a+b) = ca + cb = ca \neq 0$. Thus, the element $a+b$ is not in the complete subgraph portion of $\Gamma(R)$ which means $a+b \in \{a, b, c, d\}$. We see that $a+b \neq d$ since $cd = 0$. Thus, $a+b = c$. However, $0 = dc = d(a+b) = da + db = db \neq 0$, a contradiction. So, $\overline{\Gamma(R)}$ cannot take this configuration.

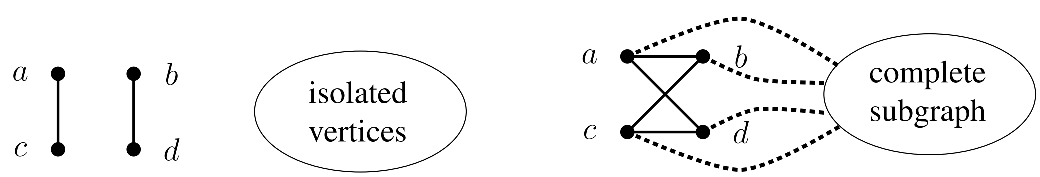

Figure 5. Configuration 1, $\overline{\Gamma(R)}$ on left and $\Gamma(R)$ on right.

Configuration 2. The graphs of $\overline{\Gamma(R)}$ and $\Gamma(R)$ for Configuration 2 are shown in Figure 6. Let a, b, and c be as shown in Figure 6.

Let l_1, l_2 be distinct vertices in the complete subgraph portion of $\Gamma(R)$ as in Figure 6. Consider the elements $l_1 + b$ and $l_2 + b$. Neither element is 0 since $\operatorname{ann}(l_1) = \operatorname{ann}(l_2) \neq \operatorname{ann}(b)$ implies $l_i \neq -b$. It can then be seen from $\Gamma(R)$ that $\{l_1 + b, l_2 + b\} \subseteq \operatorname{ann}(a) \setminus \operatorname{ann}(c) \subseteq \{a, b\}$. Clearly,

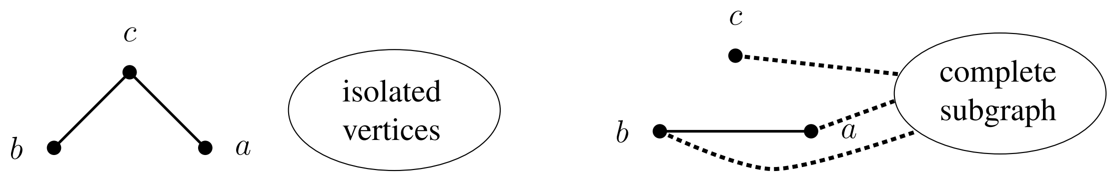

Figure 6. Configuration 2, $\overline{\Gamma(R)}$ on left and $\Gamma(R)$ on right.

l_1+b and l_2+b are not equal to b. Thus, $l_1+b=a=l_2+b$, which implies $l_1=l_2$, a contradiction. So, $\overline{\Gamma(R)}$ cannot take this configuration.

Configuration 3. The graphs of $\overline{\Gamma(R)}$ and $\Gamma(R)$ for Configuration 3 are shown in Figure 7. Let $a, b,$ and c be as shown in Figure 7.

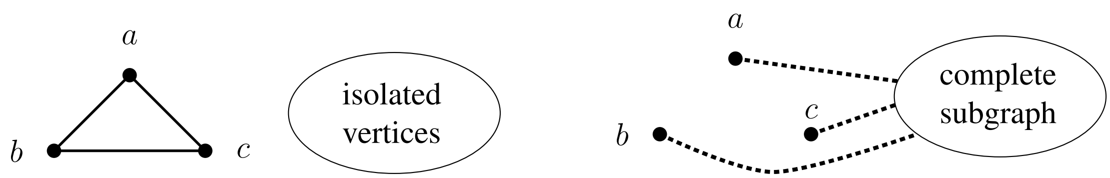

Figure 7. Configuration 3, $\overline{\Gamma(R)}$ on left and $\Gamma(R)$ on right.

Let l_1, l_2, l_3, l_4 be distinct vertices in the complete subgraph portion of $\Gamma(R)$ as in Figure 7. Then $a(l_i+b)=ab\neq 0$. Thus, l_i+b is not in the complete subgraph portion of $\Gamma(R)$. Also, since $l_i\in \mathrm{ann}(a)$ for $1\leq i\leq 4$ but $b,-b\notin \mathrm{ann}(a)$, we have $l_i+b\neq 0$ for $1\leq i\leq 4$. When $i\neq j$, we have $l_j(l_i+b)=0$, so $l_i+b\in Z(R)^*$ for $1\leq i\leq 4$. Clearly, $l_i+b\neq b$. This implies that for $1\leq i\leq 4$ we have $\{l_1+b, l_2+b, l_3+b, l_4+b\}\subseteq\{a,c\}$. Without loss of generality, $l_1+b=l_2+b$, implying that $l_1=l_2$, a contradiction. So, $\overline{\Gamma(R)}$ cannot take this configuration.

Configuration 4. The graphs of $\overline{\Gamma(R)}$ and $\Gamma(R)$ for Configuration 4 are shown in Figure 8. Let $a, b, c,$ and d be as shown in Figure 8.

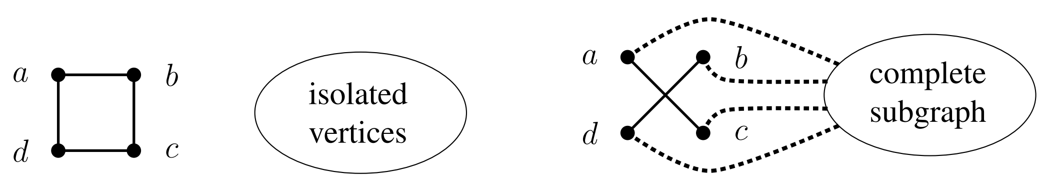

Figure 8. Configuration 4, $\overline{\Gamma(R)}$ on left and $\Gamma(R)$ on right.

None of the zero-divisor graphs with 7 vertices are isomorphic to $\Gamma(R)$ in Figure 8 since none of the realizable graphs in [14] have exactly 4 vertices of degree 4. Hence, $n\geq 8$. Let l_1, l_2, l_3, l_4 be distinct vertices in the complete subgraph portion of $\Gamma(R)$. As in the argument above for Configuration 3, $\{l_1+b, l_2+b, l_3+b, l_4+b\}\subseteq\{a,c,d\}$. This yields the same contradiction as in Configuration 3. So, $\overline{\Gamma(R)}$ cannot take this configuration.

Configuration 5. The graphs of $\overline{\Gamma(R)}$ and $\Gamma(R)$ for Configuration 5 are shown in Figure 9. Let $a, b, c,$ and d be as shown in Figure 9.

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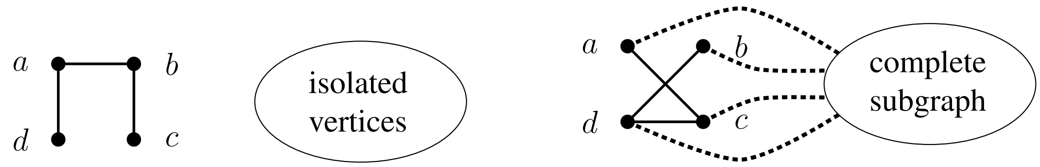

Figure 9. Configuration 5, $\overline{\Gamma(R)}$ on left and $\Gamma(R)$ on right.

Again, none of the zero-divisor graphs with 7 vertices are isomorphic to $\Gamma(R)$ in Figure 9 since none of the realizable graphs in [14] have exactly 2 vertices of degree 4 as exhibited by a and b.

Hence $n \geq 8$. Let l_1, l_2, l_3, l_4 be distinct vertices in the complete subgraph portion of $\Gamma(R)$. As in the argument above, we obtain $\{l_1 + b, l_2 + b, l_3 + b, l_4 + b\} \subseteq \{a, c, d\}$. This yields the same contradiction. So, $\overline{\Gamma(R)}$ cannot take the configuration in Configuration 5. □

We have now shown the four results given in Theorem 1.1.

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