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On D-distance (anti)magic labelings of shadow graph of some graphs

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Abstract

Let G be a graph with vertex set $V(G)$ and diameter diam(G). Let $D \subseteq \{0, 1, 2, 3, \ldots, diam(G)\}$ and $\varphi : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ be a bijection. The graph G is called D-*distance magic*, if $\sum_{s \in N_D(t)} \varphi(s)$ is a constant for any vertex $t \in V(G)$. The graph G is called (α, β)-D-*distance antimagic*, if $\{\sum_{s \in N_D(t)} \varphi(s) : t \in V(G)\}$ is a set $\{\alpha, \alpha + \beta, \alpha + 2\beta, \ldots, \alpha + (|V(G)| - 1)\beta\}$. In this paper, we study D-distance (anti)magic labelings of shadow graphs for $D = \{1\}$, $\{0, 1\}$, $\{2\}$, and $\{0, 2\}$.

Keywords: D-distance (anti)magic labeling, D-distance (anti)magic graph, shadow graph
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1. Introduction

We follow the terminologies and notations introduced in [11, 12, 15]. Let G be a simple graph with vertex set $V(G)$ and diameter diam(G). For two vertices $s, t \in V(G)$, the *distance* between s and t is denoted by $d(s, t)$. Let D be a set of distances in $\{0, 1, 2, 3, \ldots, diam(G)\}$, and $\varphi : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ be a bijection. The *neighborhood* of a vertex $t \in V(G)$ under D is $N_D(t) = \{s \in V(G) : d(s, t) \in D\}$, and its *weight* is $w_D(t) = \sum_{s \in N_D(t)} \varphi(s)$. If $D = \{1\}$,

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$N_{\{1\}}(t) = N(t) = \{s \in V(G) : st \in E(G)\}$ and $w_{\{1\}}(t) = w(t) = \sum_{s \in N(t)} \varphi(s)$. If $D = \{0, 1\}$, $N_{\{0,1\}}(t) = \{t\} \cup N(t)$ and $w_{\{0,1\}}(t) = \varphi(t) + w(t)$.

In the two next definitions, in case the graph G is a disconnected graph, diam(G) is the maximum diameter of its components.

Definition 1. [11] *A bijection $\varphi : V(G) \to \{1, 2, \ldots, |V(G)|\}$ is called a D-distance magic* (DM) *labeling of a graph G, if $w_D(t) = \sum_{s \in N_D(t)} \varphi(s)$ is a constant k for every vertex $t \in V(G)$. A graph which admits a D-*DM *labeling is called a D-*DM *graph*

The constant k is called *vertex sum* of the labeling φ. If $D = \{1\}$, a $\{1\}$-DM labeling and a $\{1\}$-DM graph are called a DM labeling and a DM graph, respectively [16]. These notions were independently introduced in [10, 17].

Definition 2. [15] *Let $\varphi : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection.*
i). If $w_D(s) \neq w_D(t)$ for every $s, t \in V(G)$, then φ is called D-distance antimagic (DA) *labeling of G and G is called a D-*DA *graph.*
*ii). If $\{w_D(t) : t \in V(G)\}$ is $\{\alpha, \alpha + \beta, \alpha + 2\beta, \ldots, \alpha + (|V(G)| - 1)\beta\}$, where $\beta \geq 0$ and $\alpha > 0$ are fixed integers, then φ is called an (α, β)-D-*DA *labeling of G, and G is called an (α, β)-D-*DA *graph*

If $D = \{1\}$, a $\{1\}$-DA labeling (resp. a $\{1\}$-DA graph) is called a *DA labeling* (resp. a *DA graph*) [7]. If $D = \{1\}$, an (α, β)-$\{1\}$-DA labeling (resp. an (α, β)-$\{1\}$-DA graph) is called an (α, β)-*DA labeling* (resp. an (α, β)-*DA graph*) [1].

Many results on these subjects have been published. Some results on D-DM labeling can be seen in [2, 4, 12, 13, 14, 16], results on D-DA labeling can be seen in [1, 3, 5, 8, 13, 14], recent results on $\{0, 2\}$-DM labeling on shadow graph of some graphs can be seen in [9], and the complete results can be seen in [6].

Let G be a graph with no isolated vertices. The shadow graph of a graph G, denoted by $D_2(G)$, is the graph constructed from $2G$ by joining each vertex in the second component to the neighbors of the corresponding vertex in the first component. We denote the first component by G with vertex set $V(G) = \{u_i : 1 \leq i \leq |V(G)|\}$ and the second one by G' with the corresponding vertex set $V(G') = \{u'_i : 1 \leq i \leq |V(G')|\}$. From the definition of $D_2(G)$, every vertex u and u' has the same neighbors, namely $N(u) = N(u')$, in $D_2(G)$, and $d(u, u') = 2$. Examples of shadow graphs of P_4 and C_4 are given in Figures 1 (a) and 1 (b), respectively.

In this paper, we give some necessary conditions for $D_2(G)$ to be D-DM as well as D-DA, where G is a regular graph. Also, we prove the existence and nonexistence of the D-DM labeling and the (α, β)-D-DA labeling of shadow graph of cycles and complete bipartite graphs for $D = \{1\}$, $\{0, 1\}$, $\{2\}$, and $\{0, 2\}$.

2. Main Results

Our first result shows the relationship between a D-DM graph and an $(\alpha, 1)$-D'-DA graph for some $D, D' \in \{1, 2, 3, \ldots, diam(G)\}$.

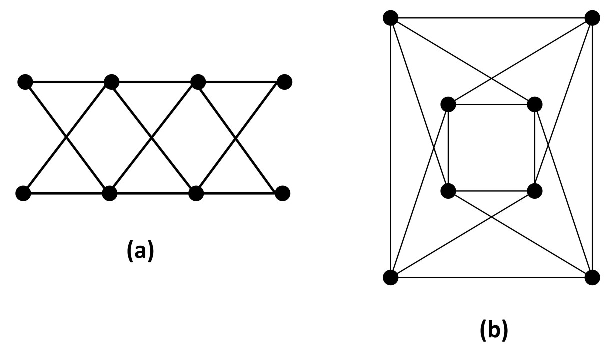

Figure 1. The graphs $D_2(P_4)$ and $D_2(C_4)$.

Lemma 2.1. *Let G be a graph with p vertices and diameter diam(G). Let $D^* \subseteq \{1, 2, 3, \dots, diam(G)\}$ and $D = D^* \cup \{0\}$.*
*i). If G is a D^*-*DM *graph with vertex sum k, then G is a $(k + 1, 1)$-D-*DA *graph.*
*ii). If G is a D-*DM *graph with vertex sum k, then G is a $(k - p, 1)$-D^*-*DA *graph.*

Proof. i). Let φ be a D^*-DM labeling of G with vertex sum k. Then $w_{D^*}(t) = \sum_{s \in N_{D^*}(t)} \varphi(s) = k$ for every $t \in V(G)$. Now, $\{w_D(t) : t \in V(G)\} = \{\varphi(t) + \sum_{s \in N_{D^*}(t)} \varphi(s) : t \in V(G)\} = \{\varphi(t) + k : t \in V(G)\}$. Since $\varphi(t) \in \{1, 2, 3, \dots, p\}$, then $\{w_D(t) : t \in V(G)\} = \{k+1, k+2, k+3, \dots, k+p\}$.
ii). Let φ be a D-DM labeling of G with vertex sum k. Then $\{w_{D^*}(t) : t \in V(G)\} = \{\sum_{s \in N_D(t)} \varphi(s) - \varphi(t) : t \in V(G)\} = \{k - \varphi(t) : t \in V(G)\} = \{k - p, k - p + 1, k - p + 2, \dots, k - 1\}$. □

The next results show that the graph $D_2(G)$ has no a D-DA labeling as well as a D-DM labeling for some D.

Lemma 2.2. *Let G a graph with no isolated vertices.*
i). The graph $D_2(G)$ is not a DA *graph and it is not a $\{0, 1\}$-*DM *graph.*
*ii). The graph $D_2(G)$ is not a $\{0, 2\}$-*DA *graph and it is not a $\{2\}$-*DM *graph.*

Proof. i). Assume that $D_2(G)$ is a DA graph with a DA labeling φ. Let us consider vertices u dan u'. Since $N(u) = N(u')$, then $w(u) = \sum_{v \in N(u)} \varphi(v) = \sum_{v \in N(u')} \varphi(v) = w(u')$. It is a contradiction to the fact that $w(u) \neq w(u')$.

Next, suppose that $D_2(G)$ is an $\{0, 1\}$-DM graph with a $\{0, 1\}$-DM labeling φ. Then $\varphi(u) + \sum_{v \in N(u)} \varphi(v) = w_{\{0,1\}}(u) = w_{\{0,1\}}(u') = \varphi(u') + \sum_{v \in N(u')} \varphi(v)$. Since $N(u) = N(u')$, then $\varphi(u) = \varphi(u')$. It is a contradiction, since φ is a bijection.

ii). Notice that $N_{\{0,2\}}(u) = N_{\{0,2\}}(u') = \{u, u'\} \cup \{v \in V(G) : d(u, v) = 2\} \cup \{v' \in V(G') : d(u', v') = 2\}$, $N_{\{2\}}(u) = \{u'\} \cup \{v \in V(G) : d(u, v) = 2\} \cup \{v' \in V(G') : d(u', v') = 2\}$, and $N_{\{2\}}(u') = \{u\} \cup \{v \in V(G) : d(u, v) = 2\} \cup \{v' \in V(G') : d(u', v') = 2\}$. By similar argument as in the first part, we can show that $D_2(G)$ is not $\{0, 2\}$-DA and it is not $\{2\}$-DM. □

The following results provide some necessary conditions for $D_2(G)$ to be a D-DM graph or an (α, β)-D-DA graph for some D.

Lemma 2.3. *Let G be a graph with p vertices, $|N(u)| = r_1$, and $|N_{\{2\}}(u)| = r_2$ for each $u \in V(G)$.*
i). If $D_2(G)$ is a DM *graph, then its vertex sum is $k = r_1(2p+1)$.*
*ii). If $D_2(G)$ is a $\{0,2\}$-*DM *graph, then its vertex sum is $k = (r_2+1)(2p+1)$.*

Proof. i). The graph $D_2(G)$ has $2p$ vertices and $|N(u)| = 2r_1$ for each $u \in V(D_2(G))$. If $D_2(G)$ is DM with vertex sum is k, then $2pk = 2r_1(1+2+3+\cdots+2p) = 2pr_1(2p+1)$.
ii). For each $u \in V(D_2(G))$, $|N_{\{0,2\}}(u)| = |\{u,u'\}| + |\{v \in V(G) : d(u,v) = 2\}| + |\{v' \in V(G') : d(u',v') = 2\}| = 2r_2 + 2$. So, If $D_2(G)$ is $\{0,2\}$-DM with vertex sum is k, then $2pk = (2r_2+2)(1+2+3+\cdots+2p) = 2p(r_2+1)(2p+1)$. □

Theorem 2.1. *Let G be a graph with p vertices, $|N(u)| = r_1$, and $|N_{\{2\}}(u)| = r_2$ for each $u \in V(G)$.*
*i). If $D_2(G)$ is an (α_1, β_1)-$\{0,1\}$-*DA *graph, then β_1 is odd and for $r_1 << p$, $\beta_1 \leq 2r_1 - 1$.*
*ii). If $D_2(G)$ is an (α_2, β_2)-$\{2\}$-*DA *graph, then β_2 is odd and for $r_2 << p$, $\beta_2 \leq 2r_2 - 1$.*

Proof. i). Notice that, for every $u \in V(D_2(G))$, $|N_{\{0,1\}}(u)| = |\{u\} \cup N(u)| = 2r_1 + 1$. Next, let $D_2(G)$ be an (α_1, β_1)-$\{0,1\}$ DA graph. Then $\{w_{\{0,1\}}(u) : u \in V(D_2(G))\} = \{\alpha_1, \alpha_1+\beta_1, \alpha_1 + 2\beta_1, \ldots, \alpha_1 + (2p-1)\beta_1\}$. The sum of all vertex weights is $\alpha_1 + (\alpha_1+\beta_1) + (\alpha_1+2\beta_1) + \cdots + (\alpha_1 + (2p-1)\beta_1) = 2p\alpha_1 + \beta_1 p(2p-1)$. This sum contains $2r_1+1$ times each vertex label, since $|N_{\{0,1\}}(u)| = 2r_1 + 1$ for every $u \in V(D_2(G))$. So,

$$2p\alpha_1 + \beta_1 p(2p-1) = (2r_1+1)(1+2+\cdots+2p) = (2r_1+1)p(2p+1)$$

or

$$2\alpha_1 + \beta_1(2p-1) = (2r_1+1)(2p+1). \tag{1}$$

Since $(2r_1+1)(2p+1)$ is an odd integer and $2\alpha_1$ is an even integer, then $\beta_1(2p-1)$ must be an odd integer. Hence, β_1 is an odd integer.

Next, the minimum possible (vertex)weight is $1+2+3+\cdots+(2r_1+1)$ and its maximum is $2p + (2p-1) + (2p-2) + (2p-3) + \cdots + (2p-2r_1)$. Hence, $\alpha_1 \geq (r_1+1)(2r_1+1)$ and $\alpha_1 + (2p-1)\beta_1 \leq (2r_1+1)(2p-r_1)$. So,

$$\beta_1 \leq 2r_1 + 1 - \frac{2r_1(2r_1+1)}{2p-1}. \tag{2}$$

For a small r_1 and a large p, then $0 < \frac{2r_1(2r_1+1)}{2p-1} < 1$. Hence, $\beta_1 \leq 2r_1 - 1$, since β_1 is an odd integer.

ii). For every $u \in V(D_2(G))$, $|N_{\{2\}}(u)| = |\{u'\} \cup \{v \in V(G) : d(u,v) = 2\} \cup \{v' \in V(G') : d(u',v') = 2\}| = 2r_2 + 1$. By the same argument as in the first part, we have the desire results. □

Lemma 2.4. *Let G be a graph with p vertices and d be a positive integer.*
*i). If φ_1 is an (α_1, β_1)-$\{0,1\}$-*DA *labeling of $D_2(G)$, then $|\varphi_1(u) - \varphi_1(u')| = d\beta_1$ for every pair u and u' in $V(D_2(G))$.*
*ii). If φ_2 is an (α_2, β_2)-$\{2\}$-*DA *labeling of $D_2(G)$, then $|\varphi_2(u) - \varphi_2(u')| = d\beta_2$ for every pair u and u' in $V(D_2(G))$.*

Proof. i). For every pair u and u' in $V(D_2(G))$, $w_{\{0,1\}}(u) = \varphi_1(u) + \sum_{v\in N(u)} \varphi_1(v) = \alpha_1 + d_1\beta_1$ and $w_{\{0,1\}}(u') = \varphi_1(u') + \sum_{v\in N(u')} \varphi_1(v) = \alpha_1 + d_2\beta_1$ for some $d_1, d_2 \in \{0, 1, 2, 3, \dots, 2p-1\}$. Since $\sum_{v\in N(u)} \varphi_1(v) = \sum_{v\in N(u')} \varphi_1(v)$, then $\varphi_1(u) - \varphi_1(u') = (d_1 - d_2)\beta_1 = d\beta_1$ or $\varphi_1(u') - \varphi_1(u) = (d_2 - d_1)\beta_1 = -d\beta_1$.
ii). For every pair u and u' in $V(D_2(G))$, $w_{\{2\}}(u) = \varphi_2(u') + \sum_{v\in S\cup S'} \varphi_2(v) = \alpha_2 + d_3\beta_2$ and $w_{\{2\}}(u') = \varphi_2(u) + \sum_{v'\in S\cup S'} \varphi_2(v') = \alpha_2 + d_4\beta_2$ for some $d_3, d_4 \in \{0, 1, 2, 3, \dots, 2p-1\}$, where $S = \{v \in V(G) : d(u, v) = 2\}$ and $S' = \{v' \in V(G') : d(u', v') = 2\}$. Since, $\sum_{v\in S\cup S'} \varphi_2(v) = \sum_{v'\in S\cup S'} \varphi_2(v')$, then $|\varphi_2(u') - \varphi_2(u)| = |(d_3 - d_4)|\beta_2$. □

Next, we consider m copies of the graph $D_2(G)$, namely $mD_2(G)$, where $G = C_n$ and $K_{n,n}$. Notice that $mD_2(G) \cong D_2(mG)$. By Lemma 2.2, the graphs $mD_2(C_n)$ and $mD_2(K_{n,n})$ are not DA and $\{0, 2\}$-DA. Also, they are not $\{0, 1\}$-DM and $\{2\}$-DM. In the next theorem, we show that $mD_2(C_n)$ has DM and $\{0, 2\}$-DM labelings for every integer $m \geq 1$ and $n \geq 3$.

Theorem 2.2. *For every integer $m \geq 1$ and $n \geq 3$, the graph $mD_2(C_n)$ is* DM *and* $\{0, 2\}$-DM.

Proof. Let $V(mD_2(C_n)) = \{u_{i,j}, u'_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(mD_2(C_n)) = \{u_{i,j}u_{i+1,j}, u'_{i,j}u'_{i+1,j}, u'_{i,j}u_{i+1,j}, u'_{i+1,j}u_{i,j}, : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_{n,j}u_{1,j}, u'_{n,j}u'_{1,j}, u'_{n,j}u_{1,j}, u'_{1,j}u_{n,j} : 1 \leq j \leq m\}$. For $1 \leq j \leq m$, let $A_j = \{\{u_{i,j}, u'_{i,j}\} : 1 \leq i \leq n\}$ and $B_j = \{\{(j-1)n + i, 2nm + 1 - (j-1)n - i\} : 1 \leq i \leq n\}$. It is clear that for $k \neq l$, $A_k \cap A_l = \varnothing$ and $B_k \cap B_l = \varnothing$. Also, $\mathcal{A} = \cup_{j=1}^{m} A_j = V(D_2(C_n))$ and $\mathcal{B} = \cup_{j=1}^{m} B_j = \{1, 2, 3, \dots, 2nm\}$. Hence, $\mathcal{A}$ is a partion of $\{1, 2, 3, \dots, 2mn\}$ and $\mathcal{B}$ is a partition of $V(D_2(C_n))$.

Next, one can check that, for $1 \leq j \leq m$, and any bijection $\varphi : \mathcal{A} \to \mathcal{B}$ we have $w(u_{1,j}) = w(u'_{1,j}) = \varphi(u_{n,j}) + \varphi(u'_{n,j}) + \varphi(u_{2,j}) + \varphi(u'_{2,j}) = 4mn + 2$, $w(u_{i,j}) = w(u'_{i,j}) = \varphi(u_{i-1,j}) + \varphi(u'_{i-1,j}) + \varphi(u_{i+1,j}) + \varphi(u'_{i+1,j}) = 4mn + 2$ for $1 \leq i \leq n-1$, and $w(u_{n,j}) = w(u'_{n,j}) = \varphi(u_{n-1,j}) + \varphi(u'_{n-1,j}) + \varphi(u_{1,j}) + \varphi(u'_{1,j}) = 4mn + 2$. Therefore, φ is a DM labeling of $mD_2(C_n)$ with vertex sum $4mn + 2$.

Now, we show that $\varphi : \mathcal{A} \to \mathcal{B}$ is also a $\{0, 2\}$-DM labeling of $mD_2(C_n)$. To do this, we consider three the following cases:
Case $n = 3$. For $1 \leq j \leq m$ and $1 \leq i \leq 3$, $w_{\{0,2\}}(u_{i,j}) = w_{\{0,2\}}(u'_{i,j}) = \varphi(u_{i,j}) + \varphi(u'_{i,j}) = 6m + 1$. Thus, φ is a $\{0, 2\}$-DM labeling of $mD_2(C_3)$ with vertex sum $6m + 1$.
Case $n = 4$. For $1 \leq j \leq m$, $w_{\{0,2\}}(u_{1,j}) = w_{\{0,2\}}(u'_{1,j}) = w_{\{0,2\}}(u_{3,j}) = w_{\{0,2\}}(u'_{3,j}) = \varphi(u_{1,j}) + \varphi(u'_{1,j}) + \varphi(u_{3,j}) + \varphi(u'_{3,j}) = 16m + 2$, and $w_{\{0,2\}}(u_{2,j}) = w_{\{0,2\}}(u'_{2,j}) = w_{\{0,2\}}(u_{4,j}) = w_{\{0,2\}}(u'_{4,j}) = \varphi(u_{2,j}) + \varphi(u'_{2,j}) + \varphi(u_{4,j}) + \varphi(u'_{4,j}) = 16m + 2$. Thus, φ is a $\{0, 2\}$-DM labeling of $mD_2(C_4)$ with vertex sum $16m + 2$.
Case $n \geq 5$. For $1 \leq j \leq m$, $w_{\{0,2\}}(u_{1,j}) = w_{\{0,2\}}(u'_{1,j}) = \varphi(u_{1,j}) + \varphi(u'_{1,j}) + \varphi(u_{3,j}) + \varphi(u'_{3,j}) + \varphi(u_{n-1,j}) + \varphi(u'_{n-1,j}) = 6mn + 3$, $w_{\{0,2\}}(u_{2,j}) = w_{\{0,2\}}(u'_{2,j}) = \varphi(u_{2,j}) + \varphi(u'_{2,j}) + \varphi(u_{4,j}) + \varphi(u'_{4,j}) + \varphi(u_{n,j}) + \varphi(u'_{n,j}) = 6mn + 3$, $w_{\{0,2\}}(u_{i,j}) = w_{\{0,2\}}(u'_{i,j}) = \varphi(u_{i,j}) + \varphi(u'_{i,j}) + \varphi(u_{i+2,j}) + \varphi(u'_{i+2,j}) + \varphi(u_{i-2,j}) + \varphi(u'_{i-2,j}) = 6mn + 3$ for $3 \leq i \leq n-2$, $w_{\{0,2\}}(u_{n-1,j}) = w_{\{0,2\}}(u'_{n-1,j}) = \varphi(u_{n-1,j}) + \varphi(u'_{n-1,j}) + \varphi(u_{1,j}) + \varphi(u'_{1,j}) + \varphi(u_{n-3,j}) + \varphi(u'_{n-3,j}) = 6mn + 3$, and $w_{\{0,2\}}(u_{n,j}) = w_{\{0,2\}}(u'_{n,j}) = \varphi(u_{n,j}) + \varphi(u'_{n,j}) + \varphi(u_{2,j}) + \varphi(u'_{2,j}) + \varphi(u_{n-2,j}) + \varphi(u'_{n-2,j}) = 6mn + 3$. Thus, φ is a $\{0, 2\}$-DM labeling of $mD_2(C_n)$, $n \geq 5$, with vertex sum $6mn + 3$. □

As an example, let consider the case $m = 1$. In this case, we redefine vertex and edge sets of $D_2(C_n)$ as follows: $V(D_2(C_n)) = \{u_i, u'_i : 1 \leq i \leq n\}$ and $E(D_2(C_n)) = \{u_iu_{i+1}, u'_iu'_{i+1} :$

$1 \leq i \leq n-1\} \cup \{u_n u_1, u'_n u'_1\} \cup \{u'_i u_{i+1}, u'_{i+1} u_i : 1 \leq i \leq n-1\} \cup \{u'_n u_1, u'_1 u_n\}$. Also, $\mathcal{A} = \{\{u_i, u'_i\} : 1 \leq i \leq n\}$ and $\mathcal{B} = \{\{i, 2n+1-i\} : 1 \leq i \leq n\}$.

Next, let $\varphi(\{u_i, u'_i\}) = \{i, 2n+1-i\}$ for $1 \leq i \leq n$. Then $w(u_1) = w(u'_1) = [\varphi(u_n)+\varphi(u'_n)] + [\varphi(u_2)+\varphi(u'_2)] = [n+n+1] + [2+2n-1] = 4n+2$, $w(u_i) = w(u'_i) = [\varphi(u_{i-1})+\varphi(u'_{i-1})] + [\varphi(u_{i+1})+\varphi(u'_{i+1})] = [i-1+2n+1-i+1] + [i+1+2n+1-i-1] = 4n+2$ for $1 \leq i \leq n-1$, and $w(u_n) = w(u'_n) = [\varphi(u_{n-1})+\varphi(u'_{n-1})] + [\varphi(u_1)+\varphi(u'_1)] = [n-1+n+2] + [1+2n] = 4n+2$. Hence, φ is a DM labeling of $D_2(C_n)$ with vertex sum $4n+2$.

Next, we show that φ is also a $\{0,2\}$-DM labeling $D_2(C_n)$.
Case $n = 3$. $w_{\{0,2\}}(u_i) = w_{\{0,2\}}(u'_i) = \varphi(u_i) + \varphi(u'_i) = 7$ for $1 \leq i \leq 3$.
Case $n = 4$. $w_{\{0,2\}}(u_1) = w_{\{0,2\}}(u'_1) = w_{\{0,2\}}(u_3) = w_{\{0,2\}}(u'_3) = \varphi(u_1) + \varphi(u'_1) + \varphi(u_3) + \varphi(u'_3) = 18$ and $w_{\{0,2\}}(u_2) = w_{\{0,2\}}(u'_2) = w_{\{0,2\}}(u_4) = w_{\{0,2\}}(u'_4) = \varphi(u_2) + \varphi(u'_2) + \varphi(u_4) + \varphi(u'_4) = 18$.
Case $n \geq 5$. $w_{\{0,2\}}(u_1) = w_{\{0,2\}}(u'_1) = \varphi(u_1)+\varphi(u'_1)+\varphi(u_3)+\varphi(u'_3)+\varphi(u_{n-1})+\varphi(u'_{n-1}) = 6n+3$, $w_{\{0,2\}}(u_2) = w_{\{0,2\}}(u'_2) = \varphi(u_2)+\varphi(u'_2)+\varphi(u_4)+\varphi(u'_4)+\varphi(u_n)+\varphi(u'_n) = 6n+3$, $w_{\{0,2\}}(u_i) = w_{\{0,2\}}(u'_i) = \varphi(u_i)+\varphi(u'_i)+\varphi(u_{i+2})+\varphi(u'_{i+2})+\varphi(u_{i-2})+\varphi(u'_{i-2}) = 6n+3$ for $3 \leq i \leq n-2$, $w_{\{0,2\}}(u_{n-1}) = w_{\{0,2\}}(u'_{n-1}) = \varphi(u_{n-1})+\varphi(u'_{n-1})+\varphi(u_1)+\varphi(u'_1)+\varphi(u_{n-3})+\varphi(u'_{n-3}) = 6n+3$, and $w_{\{0,2\}}(u_n) = w_{\{0,2\}}(u'_n) = \varphi(u_n)+\varphi(u'_n)+\varphi(u_2)+\varphi(u'_2)+\varphi(u_{n-2})+\varphi(u'_{n-2}) = 6n+3$. Hence, φ is a $\{0,2\}$-DM labeling of $D_2(C_3)$, $D_2(C_4)$, and $D_2(C_n)$, $n \geq 5$, with vertex sum 7, 18 , and $6n+3$, respectively.

Next, we consider the (α,β)-$\{0,1\}$-DA and (α,β)-$\{2\}$-DA labelings of the graph $mD_2(C_n)$.

Lemma 2.5. *Let $m \geq 1$ and $n \geq 3$ be integers.*
*i). If the graph $mD_2(C_n)$ is (α_1,β_1)-$\{0,1\}$-*DA*, then $\beta_1 = 1$, $\alpha_1 = 4nm+3$ and $\beta_1 = 3$, $\alpha_1 = 2nm+4$.*
*ii). If the graph $mD_2(C_n)$ is (α_2,β_2)-$\{2\}$-*DA *graphs, then $\beta_2 = 1$, $\alpha_2 = 4nm+3$ and $\beta_2 = 3$, $\alpha_2 = 2nm+4$.*

Proof. By Theorem 2.1 and equation (1), we have the desire results. □

As a consequence of Lemma 2.1 and Theorem 2.2, we have the following result.

Corollary 2.1. *i). For every integer $m \geq 1$ and $n \geq 3$, the graph $mD_2(C_n)$ is $(4nm+3,1)$-$\{0,1\}$-*DA.
*ii a). For every integer $m \geq 1$, the graph $mD_2(C_3)$ is $(1,1)$-$\{2\}$-*DA.
*ii b). For every integer $m \geq 1$, the graph $mD_2(C_4)$ is $(8m+2,1)$-$\{2\}$-*DA.
*ii c). For every integer $m \geq 1$ and $n \geq 5$, the graph $mD_2(C_n)$ is $(4mn+3,1)$-$\{2\}$-*DA.

Lemma 2.6. *If $mn \equiv 1,2 \ (mod\ 3)$, then the graph $mD_2(C_n)$ is not $(\alpha_1,3)$-$\{0,1\}$-*DA *and it is not $(\alpha_2,3)$-$\{2\}$-*DA *for some integer α_1 and α_2.*

Proof. Due to Lemma 2.4, if φ is an $(\alpha,3)$-$\{0,1\}$ (resp. $(\alpha,3)$-$\{2\}$)-DA labeling of $mD_2(C_n)$, then $|\varphi(u) - \varphi(u')| = 3d$ for some positive integer d and for every pair $u, u' \in V(mD_2(C_n))$. Hence, $2nm \equiv 0$ (mod 3) or $nm \equiv 0$ (mod 3). □

Next, let us consider the graph $D_2(C_n)$, where $n \equiv 0$ (mod 3). It is not easy for us to prove whether $D_2(C_n)$ is $(2n+4,3)$-$\{0,1\}$-*DA* or not. We only have the following results. By equation (2), the graph $D_2(C_3)$ is not $(10,3)$-$\{0,1\}$-DA. Let $D_2(C_6)$ is $(16,3)$-$\{0,1\}$-*DA*, then $\{w(u) : u \in V(D_2(C_6))\} = \{16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49\}$. Since there is a unique way to express 16 and 49 as a sum of five numbers in the set $\{1, 2, 3, \ldots, 12\}$, that is $16 = 1+2+3+4+6$ and $49 = 7+9+10+11+12$, and due to Lemma 2.4, we have two possibilities to label of vertices of $D_2(C_6)$ as in the Figure 2. We can verify that the labelings do not lead to a $(16,3)$-$\{0,1\}$-DA labeling of $D_2(C_6)$. So, the graph $D_2(C_6)$ is not $(16,3)$-$\{0,1\}$-DA. By the same arguments, $D_2(C_3)$ is not $(10,3)$-$\{2\}$-DA and $D_2(C_6)$ is not $(16,3)$-$\{2\}$-DA.

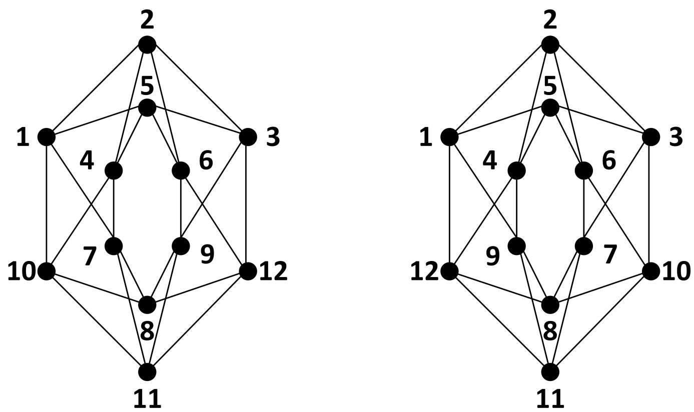

Figure 2. The possibilities to label of $D_2(C_6)$

Problem 1. *Decide if there exists a* $(2n+4,3)$-$\{0,1\}$ $(resp.\ (2n+4,3)-\{2\})$-DA *labeling of* $D_2(C_n)$ *for every integer* $9 \le n \equiv 0$ *(mod* 3*).*

Next, we consider the shadow graph $mD_2(K_{n,n})$. In the next result, we show that $mD_2(K_{n,n})$ is a DM graph as well as a $\{0,2\}$-DM graph.

Theorem 2.3. *For every integer* $m, n \ge 1$*, the graph* $G = mD_2(K_{n,n})$ *is* DM *and* $\{0,2\}$-DM.

Proof. For $1 \le j \le m$, let $V(G) = V_{1,j} \cup V_{2,j} \cup V'_{1,j} \cup V'_{2,j}$ and $E(G) = V_{1,j}V_{2,j} \cup V'_{1,j}V'_{2,j} \cup V_{2,j}V'_{1,j} \cup V_{1,j}V'_{2,j}$, where $V_{1,j} = \{u_{i,j} : 1 \le i \le n\}$, $V_{2,j} = \{v_{i,j} : 1 \le i \le n\}$, $V'_{1,j} = \{u'_{i,j} : 1 \le i \le n\}$, $V'_{2,j} = \{v'_{i,j} : 1 \le i \le n\}$, and $V_{i,j}V_{k,l}$ means that every vertex in $V_{i,j}$ is adjacent to each vertex in $V_{k,l}$ and vice versa. Next, for $1 \le j \le m$, let $S_{1,j} = \{(j-1)n+i : 1 \le i \le n\}$, $S_{2,j} = \{mn+(j-1)n+i : 1 \le i \le n\}$, $S_{3,j} = \{3mn+1-(j-1)n-i : 1 \le i \le n\}$, and $S_{4,j} = \{4mn+1-(j-1)n-i : 1 \le i \le n\}$. It is clear that, for $1 \le j \le m$, $S_{1,j} \cup S_{2,j} \cup S_{3,j} \cup S_{4,j} = \{1, 2, 3, \ldots, 4mn\}$, $\sum_{s \in S_{1,j}} s = \frac{1}{2}n(2n(j-1)+n+1)$, $\sum_{s \in S_{2,j}} s = \frac{1}{2}n(2mn+2n(j-1)+n+1)$, $\sum_{s \in S_{3,j}} s = \frac{1}{2}n(6mn-2n(j-1)-n+1)$, and $\sum_{s \in S_{4,j}} s = \frac{1}{2}n(8mn-2n(j-1)-n+1)$.

Next, for $1 \le j \le m$, label each vertex in $V_{1,j}$ by every number in $S_{1,j}$, each vertex in $V_{2,j}$ by every number in $S_{2,j}$, each vertex in $V'_{1,j}$ by every number in $S_{4,j}$, and each vertex in $V'_{2,j}$ by every number in $S_{3,j}$. Then, for $1 \le i \le n$ and $1 \le j \le m$, $w(u_{i,j}) = w(u'_{i,j}) =$

$\frac{1}{2}n(2mn + 2n(j-1) + n + 1) + \frac{1}{2}n(6mn - 2n(j-1) - n + 1) = n(4mn + 1)$, and $w(v_{i,j}) = w(v'_{i,j}) = \frac{1}{2}n(2n(j-1)+n+1)+\frac{1}{2}n(8mn-2n(j-1)-n+1) = n(4mn+1)$. Thus, $mD_2(K_{n,n})$ is a DM graph.

Next, we show that the labeling is also a $\{0,2\}$-DM labeling of $mD_2(K_{n,n})$. For $1 \le i \le n$ and $1 \le j \le m$, $w(u_{i,j}) = w(u'_{i,j}) = \frac{1}{2}n(2n(j-1)+n+1)+\frac{1}{2}n(8mn-2n(j-1)-n+1) = n(4mn+1)$, and $w(v_{i,j}) = w(v'_{i,j}) = \frac{1}{2}n(2mn+2n(j-1)+n+1)+\frac{1}{2}n(6mn-2n(j-1)-n+1) = n(4mn+1)$. Hence, $mD_2(K_{n,n})$ is a $\{0,2\}$-DM graph. □

Next, we provide an illustration of the proof of Theorem 2.3 for $m = 1$. First, redefine vertex and edge sets of $D_2(K_{n,n})$ as follows: $V(D_2(K_{n,n})) = V_1 \cup V_2 \cup V'_1 \cup V'_2$ and $E(D_2(K_{n,n})) = V_1V_2 \cup V'_1V'_2 \cup V'_1V_2 \cup V'_2V_1$, where $V_1 = \{u_i : 1 \le i \le n\}$, $V_2 = \{v_i : 1 \le i \le n\}$, $V'_1 = \{u'_i : 1 \le i \le n\}$, $V'_2 = \{v'_i : 1 \le i \le n\}$. Also, $S_1 = \{1, 2, 3, \ldots, n\}$, $S_2 = \{n+1, n+2, n+3, \ldots, 2n\}$, $S_3 = \{3n, 3n-1, 3n-2, \ldots, 2n+1\}$, and $S_4 = \{4n, 4n-1, 4n-2, \ldots, 3n+1\}$. Obviously, $\sum_{s\in S_1} s = \frac{1}{2}n(n+1)$, $\sum_{s\in S_2} s = \frac{1}{2}n(3n+1)$, $\sum_{s\in S_3} s = \frac{1}{2}n(5n+1)$, and $\sum_{s\in S_4} s = \frac{1}{2}n(7n+1)$.

Finally, label every vertex in V_1 by each member of S_1, every vertex in V_2 by each member of S_2, every vertex in V'_1 by each member of S_4, and every vertex in V'_2 by each member of S_3. Under this labeling, for $1 \le i \le n$, $w(u_i) = w(u'_i) = \frac{1}{2}n(3n+1) + \frac{1}{2}n(5n+1) = n(4n+1)$, and $w(v_i) = w(v'_i) = \frac{1}{2}n(n+1) + \frac{1}{2}n(7n+1) = n(4n+1)$. So, $D_2(K_{n,n})$ is a DM graph. Next, for $1 \le i \le n$, $w(u_i) = w(u'_i) = \frac{1}{2}n(n+1) + \frac{1}{2}n(7n+1) = n(4n+1)$, and $w(v_i) = w(v'_i) = \frac{1}{2}n(3n+1) + \frac{1}{2}n(5n+1) = n(4n+1)$. So, $D_2(K_{n,n})$ is a $\{0,2\}$-DM graph.

As a consequence of Lemma 2.1 and Theorem 2.3, we have the following result.

Corollary 2.2. *The graph* $mD_2(K_{n,n})$ *is* $(n(4mn+1)+1, 1)$-$\{0,1\}$-DA *and* $(n(4mn-4m+1), 1)$-$\{2\}$-DA *for every integer* $m, n \ge 1$.

By a similar argument as in the proof of Lemma 2.6, we have the following lemma.

Lemma 2.7. *If* $mn \equiv 1, 2 \ (mod\ 3)$*, then the graph* $mD_2(K_{n,n})$ *is not* $(\alpha_1, 3)$-$\{0,1\}$-DA *and it is not* $(\alpha_2, 3)$-$\{2\}$-DA *for some integer* α_1 *and* α_2.

The problem related to these results are as follows.

Problem 2. *Decide if there exists a* $(\alpha, 3)$-$\{0,1\}$ $(resp.\ (\alpha, 3)$-$\{2\})$-DA *labeling of* $(D_2(K_{n,n}))$ *for every integer* $n \equiv 0 \ (mod\ 3)$.

Problem 3. *For every integer* $m \ge 1$ *and* $n_1 \ne n_2 \ge 1$*, find a* DM *labeling and an* (α, β)-$\{0,1\}$-DA *labeling of the graph* $mD_2(K_{n_1,n_2})$.

3. Conclusion

In this paper, we study D-DM labeling and (α, β)-D-DA labeling of shadow graphs for $D \in \{\{1\}, \{0,1\}, \{2\}, \{0,2\}\}$. We provide some necessary conditions for the shadow graph of a regular graph to be D-DM or (α, β)-$\{D\}$-DA. Also, we prove the existence and nonexistence of D-DM labeling and (α, β)-$\{D\}$-DA labeling for the graphs $mD_2(C_n)$ and $mD_2(K_{n,n})$. Our results also give an example if $D_2(G)$ is D-DM, G needs not to be D-DM. Namely, $D_2(C_n)$ is DM for every $n \ge 3$, however, C_n is not DM for $n \ne 4$.

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