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On Ramsey $(C_4, K_{1,n})$-minimal graphs

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Abstract

Let F, G and H be any simple graphs. The notation $F \to (G, H)$ means for any red-blue coloring on the edges of graph F, there exists either a red copy of G or a blue copy of H. If $F \to (G, H)$, then graph F is called a Ramsey graph for (G, H). Additionally, if the graph F satisfies that $F - e \nrightarrow (G, H)$ for any edge e of F, then graph F is called a Ramsey (G, H)-minimal. The set of all Ramsey (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$. In this paper, we construct a new class of Ramsey $(C_4, K_{1,n})$-minimal graphs.

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1. Introduction

In this paper, all graphs are simple graphs. A cycle and a star of order n is denoted by C_n and $K_{1,n-1}$, respectively. For any three graphs F, G and H, the notation of $F \to (G, H)$ to mean that for any red-blue coloring on the edges of F, there exists a red copy of G or a blue copy of H.

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Definition 1.1. A graph F is called a *Ramsey graph* for a pair of graphs (G, H) if F satisfies that $F \to (G, H)$.

Definition 1.2. A graph F is called a *Ramsey* (G, H)*-minimal* if F satisfies the following these conditions:

(i) $F \to (G, H)$, and
(ii) $F - e \nrightarrow (G, H)$, for any $e \in E(F)$.

The set of all Ramsey (G, H)-minimal graphs will be denoted by $\mathcal{R}(G, H)$.

The pair (G, H) is called a *Ramsey-finite* if $\mathcal{R}(G, H)$ is finite. Otherwise, the pair (G, H) is called *Ramsey-infinite*. The study on the Ramsey minimal graphs was initiated by Burr et al. [1]. In general, finding the Ramsey (G, H)-minimal graph is both a challenging and interesting problem to be solved. There are several papers that are dedicated to some Ramsey classes for specific graphs G and H. For instance, Burr et al. [2] showed that for an arbitrary graph G, the pair (mK_2, G) is Ramsey-finite. In 1980, Burr et al. [3] proved that if $H = K_{1,n}$ and G is any 2-connected graph, then the set $\mathcal{R}(G, H)$ is infinite. Nešetřil and Rödl [4] proved that if both G and H are 3-connected or if G and H are forests and neither of which is a union of stars, then the pair (G, H) is Ramsey-infinite. Over decades later, Borowiecki et al. [5] characterized all graphs in $\mathcal{R}(K_3, K_{1,2})$. Mushi and Baskoro [6] gave necessary and sufficient conditions for all members of $\mathcal{R}(3K_2, K_{1,n})$ for $n \geq 3$. Additionally, for $3 \leq n \leq 7$ they were able to list all Ramsey $(3K_2, K_{1,n})$-minimal graphs of order at most 10 vertices.

Here are some of the latest related papers that discuss the pair of a cycle and a star. Nisa et al. [7] constructed some graphs in $\mathcal{R}(C_6, K_{1,2})$. Nabila and Baskoro [8] gave some Ramsey $(K_{1,2}, C_n)$-minimal graphs for $n \in \{5, 6, 8\}$ and constructed the Ramsey $(C_n, K_{1,2})$ graphs for $n \in \{10, 12, 14, 16, 18\}$. Hadiputra and Silaban [9] showed a new class of graphs using C_4-*paths* and some edge additions are the members of an infinite family in $\mathcal{R}(K_{1,2}, C_4)$. Moreover, Nabila et al. [10] gave some finite and infinite classes of Ramsey $(C_4, K_{1,n})$-minimal graphs for any $n \geq 3$ in the form of path graphs.

In this paper, we give a new class of graph called theta-trees, which are constructed using an edge-weighted tree.

2. Main Results

In this section, we derive some sufficient conditions of Ramsey (G, H)-minimal graphs for G is a cycle C_4 on four vertices and H is a star $K_{1,n}$ with $n \geq 2$. Based on these sufficient conditions, we give some classes of Ramsey $(C_4, K_{1,n})$-minimal graphs.

For any edge-weighted tree T on m edges, define a theta-tree graph $\theta[T]$ as follows.

Definition 2.1. Let T be an edge-weighted tree on m edges $e_1, e_2, \ldots, e_m$ with $a_i \in \mathbb{N}$ is the weight of e_i for each i. The *theta-tree graph* based on T, denoted by $\theta[T]$, is a graph constructed from T by replacing each edge $e_i = (x_i, y_i)$ by a union of a_i paths of length 2 whose internal vertices are disjoint. This internally disjoint union of a_i paths connects x_i and y_i.

From the Definition 2.1, we have $V(\theta[T]) = V(T) \cup A_1 \cup A_2 \cup \cdots \cup A_m$, where $A_i = \{u_{i,j} |\ i \in [1, m], j \in [1, a_i]\}$. All members of each A_i are called *internal vertices* in $\theta[T]$ graph. For example, in Figure 1 we give the theta-tree graph $\theta[T]$ obtained from a tree T on 6 edges.

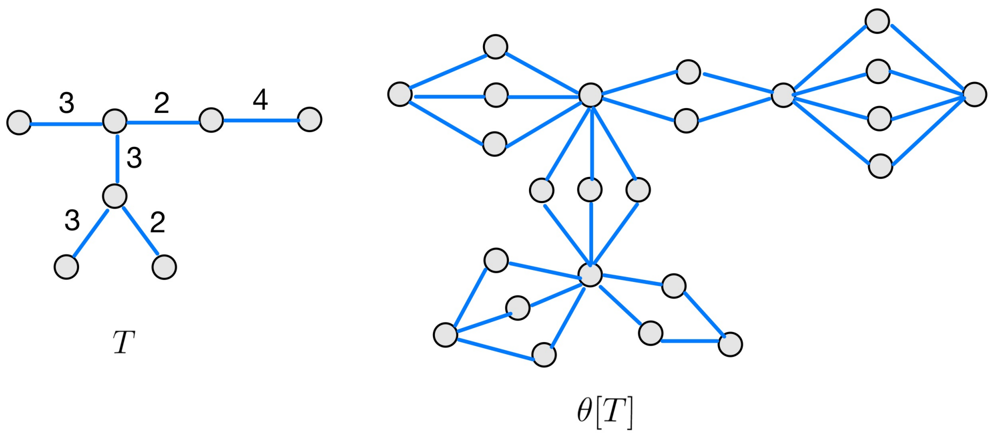

Figure 1. An edge-weighted tree T on 6 edges and the corresponding graph $\theta[T]$.

2.1. Sufficient conditions

In this section, we present some sufficient conditions for an edge-weighted tree T on m edges such that the theta-tree graph $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.

Let T be an edge-weighted tree with m edges $e_1, e_2, \ldots, e_m$ and a_i is the weight of e_i for each i. The *sum* of graph T, denoted by $\text{sum}(T)$, is defined as the sum of all edge weights, namely $\text{sum}(T) = \sum_{i=1}^{m} a_i$.

Theorem 2.1. *Let n, m be natural numbers and T be a weighted tree on m edges $e_1, e_2, \ldots, e_m$ with weights $a_1, a_2, \ldots, a_m$, respectively, where $n, m \geq 1$ and $2 \leq a_i \leq 2n$. If the following statements hold*

(a) sum$(T) = (m+1)n$,
(b) sum$(T') < (l+1)n$ for each proper subtree T' of T induced by any l edges,

then $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.

Proof. Let T be a weighted tree on m edges $e_1, e_2, \ldots, e_m$. Let $G \cong \theta[T]$. Let $\sum_{i=1}^{m} a_i = (m+1)n$, we will show that $G \to (C_4, K_{1,n})$. Consider any red-blue coloring α on the edges of G containing no blue copy of $K_{1,n}$. Let B be the set of all blue edges in G by the coloring α. Since there is no blue copy of $K_{1,n}$ in G, then

$$|B| \leq (m+1)(n-1). \quad (1)$$

This is true since the maximum blue star is a $K_{1,n-1}$, and it must have a center v_i for some $i \in [1, m+1]$. Assume G has no red copy of C_4 by α coloring, then the number of blue edges incident with A_i is at least $a_i - 1$ where at most one internal vertex in A_i is incident with two red edges, for each $i \in [1, m]$. Thus,

$$|B| \geq \sum_{i=1}^{m}(a_i - 1) = -m + \sum_{i=1}^{m} a_i = (m+1)(n-1) + 1. \quad (2)$$

From Eq. (1) and (2) we have a contradiction. It means that if G has no blue copy of $K_{1,n}$, then G must contain a red copy of C_4. Therefore, $G \to (C_4, K_{1,n})$.

Next, we will show that $G - e \nrightarrow (C_4, K_{1,n})$ for any edge e. Let $e = ab \in G$, where a is a non-internal vertex and b is an internal vertex in A_j for some $j \in [1, m]$. Define a red-blue coloring on the edges of G such that the edge incident to b is colored by red and the remaining edges are colored by red and blue with satisfying the following three conditions:

(i) each internal vertex in A_i is incident with at most one blue edge, such that the number of blue edges incident with A_i is exactly $a_i - 1$ for each $i \neq j$,
(ii) each internal vertex in A_j is incident with at most one blue edge, such that the number of blue edges incident with A_j is exactly $a_j - 2$, and
(iii) the number of blue edges incident to each non-internal vertex is exactly $(n - 1)$.

This above coloring can always be done since from the conditions (i), (ii), (iii), and requirement (b), we obtain

$$|B| = a_j - 2 + \sum_{i=1 \text{ and } i \neq j}^{m} (a_i - 1) = (m + 1)(n - 1). \tag{3}$$

where B is the set of all blue edges in G by the above coloring. By condition (iii) we have no blue $K_{1,n}$ in G. By conditions (i) and (ii) we have exactly one internal vertex incident with two red edges. Therefore, there is no red C_4 in G. Thus, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. Note that the (b) condition is required; otherwise, the existence of a subtree T' of size l satisfying sum$(T') = (l + 1)n$ would make $G[T'] \to (C_4, K_{1,n})$. □

2.2. Special sequences of theta-tree graphs

From now on, let T be an edge-weighted tree on m edges $e_1, e_2, \ldots, e_m$ with the weights $a_1, a_2, \ldots, a_m$ and $a_1 \geq a_2 \geq \ldots \geq a_m$. In this section, we give some sequences $a_1, a_2, \ldots, a_m$ such that the theta-tree graph $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.

Theorem 2.2. *Let $j \geq 1$ be fixed. If $a_i = n + j$ for $i \in [1, m]$ where $n \geq 2$ with $n = mj$, then $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.*

Proof. Let T be an edge-weighted tree on m edges $e_1, e_2, \ldots, e_m$ with the weights $a_i = n + j$ for each $i \in [1, m]$. In order to show that $\theta[T]$ is a Ramsey$(C_4, K_{1,n})$-minimal graph, it is enough to prove that T satisfies the two conditions of Theorem 2.1. The first condition of Theorem 2.1 is satisfied, since sum$(T) = (n + j) + \cdots + (n + j) = (m + 1)n$.

Now, consider any proper subtree T' of T induced by any l edges. Then, we have that sum$(T') = l(n+j)$. It is easy to verify that, in any case, sum$(T') < (l+1)n$ if $1 \leq l \leq m-1$. So, the second condition of Theorem 2.1 is satisfied. Therefore, $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

Theorem 2.3. *If $a_1 = 2n - m + 1$ and $a_i = n + 1$ for $i \in [2, m]$ where $n \geq 2$ and $2 \leq m \leq n$, then $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph.*

Proof. Let T be an edge-weighted tree on m edges $e_1, e_2, \ldots, e_m$ where $2 \leq m \leq n$. Let a_i be the weight of e_i with $a_1 = 2n - m + 1$ and $a_i = n + 1$ for each $i \in [2, m]$. In order to show that $\theta[T]$ is a Ramsey$(C_4, K_{1,n})$-minimal graph, it is enough to prove that T satisfies the two conditions of Theorem 2.1. The first condition of Theorem 2.1 is satisfied, since sum$(T) = (2n - m + 1) + (n + 1) + \ldots + (n + 1) = (m + 1)n$.

Now, consider any proper subtree T' of T induced by any l edges. Then, we have that sum$(T') = l(n + 1)$ or sum$(T') = (2n - m + 1) + (l - 1)(n + 1)$. It is easy to verify that, in any case, sum$(T') < (l + 1)n$ if $2 \leq m \leq n$. So, the second condition of Theorem 2.1 is satisfied. Therefore, $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

Theorem 2.4. *If one of the following statements:*

(i) $a_1 = 2n - m - k + 1, a_2 = n + k + 1, a_i = n + 1$ *for* $i \in [3, m]$ *where* $n \geq 6$, $3 \leq m \leq n - 2k - 2$, *and* $1 \leq k \leq \frac{n-m}{2} - 1$,

(ii) $a_1 = 2n - m - k, a_2 = n + k_1 + 1, a_3 = n + k_2 + 2, a_i = n + 1$ *for* $i \in [4, m]$ *where* $n \geq 6$, $4 \leq m \leq n - 2k - 3$, $k_1 > k_2$, $k = k_1 + k_2$, *and* $1 \leq k \leq \frac{n-m-1}{2} - 1$,

holds, then $\theta[T]$ *is a Ramsey* $(C_4, K_{1,n})$*-minimal graph.*

Proof. Let T be an edge-weighted tree on m edges $e_1, e_2, \ldots, e_m$. In order to show that $\theta[T]$ is a Ramsey$(C_4, K_{1,n})$-minimal graph, it is enough to prove that T satisfies the two conditions of Theorem 2.2. If (i) holds, then sum$(T) = (2n-m-k+1)+(n+k+1)+(n+1)+\ldots+(n+1) = (m + 1)n$. Thus, the first condition of Theorem 2.1 is satisfied. Now, consider any proper subtree T' of T induced by any l edges. Then sum$(T') \leq (l-2)(n+1)+(n+k+1)+(2n-m-k+1)$. The upper bound is achieved if the edges e_1 and e_2 are in T'. It is easy to verify that sum$(T') < (l+1)n$ if $3 \leq m \leq n - 2k - 2$. So, the second condition of Theorem 2.1 is satisfied.

If (ii) holds, then sum$(T) = (2n-m-k)+(n+k_1+1)+(n+k_2+2)+(n+1)\ldots+(n+1) = (m + 1)n$. Thus, the first condition of Theorem 2.1 is satisfied. Now, consider any proper subtree T' of T induced by any l edges. Then sum$(T') \leq (l - 3)(n + 1) + (n + k_2 + 2) + (n + k_1 + 1) + (2n - m - k)$. The upper bound is achieved if the edges e_1, e_2, and e_3 are in T'. It is easy to verify that sum$(T') < (l + 1)n$ if $4 \leq m \leq n - 2k - 3$. So, the second condition of Theorem 2.1 is satisfied. Therefore, $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

Theorem 2.5. *If* $a_1 = 2n - \frac{1}{2}(m - 1)m$ *and* $a_i = n + m - (i - 1)$ *for* $i \in [2, m]$ *where* $n \geq 6$ *and* $2 \leq m \leq \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor$, *then* $\theta[T]$ *is a Ramsey* $(C_4, K_{1,n})$*-minimal graph.*

Proof. Let T be an edge-weighted tree on m edges $e_1, e_2, \ldots, e_m$ with the weights $a_1 = 2n-\frac{1}{2}(m-1)m$ and $a_i = n+m-(i-1)$ for each $i \in [2, m]$. In order to show that $\theta[T]$ is a Ramsey$(C_4, K_{1,n})$-minimal graph, it is enough to prove that T satisfies the two conditions of Theorem 2.1. The first condition of Theorem 2.1 is satisfied, since sum$(T) = (2n - \frac{1}{2}(m - 1)m) + (n + m - 1) + \cdots + (n + 1) = (m + 1)n$.

Now, consider any proper subtree T' of T induced by any l edges. Then, we have that sum$(T') \leq (2n - \frac{1}{2}(m - 1)m) + \cdots + (n + 2) = mn - 1$. It is easy to verify that, in any case, sum$(T') < (l + 1)n$ if $1 \leq l \leq m - 1$. So, the second condition of Theorem 2.1 is satisfied. Therefore, $\theta[T]$ is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

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