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On matching number, decomposition and representation of well-formed graph

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Abstract

In this paper, we find a special type of non-traceable cubic bridge graph called *well-formed graph* whose central fragment is isomorphic to a hairy cycle and whose branches are pairwise isomorphic. We then show that a well-formed graph can be partition into isomorphic subgraph. Some properties of a well-formed graph such as perfect matching, matching number, decomposition and some parameters for pictorial representation are also provided.

Keywords: cubic graph, non-traceable cubic bridge graph (NTCBG), well-formed graph, hairy cycle
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1. Introduction

In [5], Nieva and Nocum introduced a new classification of cubic graph known as non-traceable cubic bridge graph (NTCBG) with two main components, namely the *central fragment* and *branches*. They discussed some of its properties, including its chromatic number and clique number. It was shown that this family of NTCBG satisfies the conjecture of Zoeram and Yaqubi [6]. They also showed that the family of hairy cycle $\mathcal{H}_k$ is a central fragment. Some properties of NTCBG such as minimum leaf, chromatic and clique number were determined. They also found out that NTCBG has spanning $k-$trees where $3 \leq k \leq \left\lfloor \frac{n+2}{6} \right\rfloor$ [5].

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2. Preliminaries

Throughout this article we only consider finite simple undirected graph without loops or multiple edges. A simple graph where each vertex has degree 3 is called a cubic graph. Cubic graph can be classified into three different types, namely 1-connected, 2-connected and 3-connected cubic graphs. A cubic graph that contains a minimum of one edge so that its removal will disconnect the graph is said to be a 1-connected cubic graph or cubic bridge graph. Moreover, a graph contains a minimum of two or three edges whose removal will disconnect the graph is known as a 2-connected cubic graph or 3-connected cubic graph, respectively.

For graph-theoretic terms that have not been defined but are used in the paper, see [1, 3]. A cubic bridge graph G is said to be a *non-traceable cubic bridge graph* if G does not contain a Hamiltonian path. Otherwise, if G contains a Hamiltonian path, then it is a *traceable cubic bridge graph*. A *central fragment* of a graph G denoted as $\mathscr{C}$, where $\mathscr{C}$ is a subgraph of G, satisfies the following condition; $\mathscr{C}$ is connected, a bridge graph, non-traceable and must contain vertices of degree 1 and of degree 3 only. A cubic graph denoted by $\mathscr{B}_i$ is said to be a *branch* of a NTCBG. In [5], the NTCBG can be obtained by taking the union of the central fragment $\mathscr{C}$ and a branch $\mathscr{B}_i$ where one edge of $\mathscr{B}_i$ is subdivided into two to produce a path of length two such that the new vertex is also an end-vertex of the central fragment. The graph formed from $\mathscr{B}_i$ with one additional vertex is called the constructed $\widehat{\mathscr{B}_i}$ of $\mathscr{B}_i$. For notation purposes and to avoid confusion we reserve the letter $\mathcal{G}$ to denote non-traceable cubic bridge graph.

A set of edges in a graph G is independent if no two edges in the set are adjacent in G. The edges in an independent set of edges of G are called a *matching* in G. A matching of maximum size in G is a *maximum matching* in G. The edge independence number $\alpha'(G)$ of G is the number of edges in a maximum matching of G. In fact, $\alpha'(G)$ is sometimes referred to as the *matching number* of G. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. A graph G is said to be *decomposable* into subgraphs $H_1, H_2, ..., H_k$ if $\{E(H_1), E(H_2), ..., E(H_k)\}$ is a partition of $E(G)$. Such a partition produces a *decomposition* of G. If each H_i is isomorphic to a graph H, then the graph G is H-*decomposable* and the decomposition is an H-*decomposable*.

The cycle C_n with one pendant edge attached to each cycle vertex is called a *hairy cycle* for all $n \in \mathbb{N}$ with $n \geq 3$, and is denoted by $\mathcal{H}_k = C_n \odot 1K_1$. In the definition, the $\odot$ indicates that we attach a copy of the K_1 to each cycle vertex of C_n. The *union* $G = G_1 \oplus G_2$ of graph G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$ [2, 4].

Here are some important theorems that are necessary in order to prove our results.

Theorem 2.1. *[5] Let $\mathcal{B}_i$, $i \in \mathbb{N}$ be a branch of NTCBG. Then $|V(\mathcal{B}_i)| = 2(b-1)$ where $b \geq 3$.*

Theorem 2.2. *[5] Let $\mathcal{G}$ be an NTCBG with k_c-leaves. Then $\sum |V(\mathscr{B}_i)| \geq 4k_c$.*

Theorem 2.3. *[5] Let $\mathcal{H}_k$ be a central fragment then $|V(\mathcal{H}_k)| = |E(\mathcal{H}_k)| = 2k$, where k is the order and size of C_k, respectively.*

Theorem 2.4. *A nontrivial connected graph G is P_3-decomposable if and only if G has even size.*

One of the earliest result in graph theory is the Petersen theorem about the bridgeless cubic graph stated as follows.

Theorem 2.5. *Every cubic bridgeless graph contains a perfect matching.*

Theorem 2.6. *(The First Theorem of Graph Theory). If G is a graph of size m, then* $\sum_{v \in V} deg(v) = 2m$.

3. Well-formed Graph and its Matching Number

Among the numerous possible NTCBG that can be created, particular attention is directed towards examining those that exhibit visual symmetry. Visual symmetry in this context refers to NTCBG where each branch is structurally identical to one another, and the central fragment is isomorphic to some hairy cycle. In this study we explore a class of NTCBG that satisfies this condition.

Definition 1. *Let $\mathcal{G}$ be an NTCBG. If the central fragment of $\mathcal{G}$ is isomorphic to $\mathcal{H}_k$ and all branches of $\mathcal{G}$ are pairwise isomorphic, then $\mathcal{G}$ is said to be a well-formed graph.*

For example, the graph $\mathcal{G}_1$ in Figure 1 is well-formed since the central fragment of $\mathcal{G}_1$ is isomorphic to $\mathcal{H}_3$ and all the branches of $\mathcal{G}_1$ are pairwise isomorphic. In contrast, although the central fragment $\mathcal{G}_2$ in Figure 1 is isomorphic to a hairy cycle, it is not well-formed because not all of its branches are pairwise isomorphic. Also, $\mathcal{G}_3$ is not well-formed since its central fragment is not isomorphic to any $\mathcal{H}_k$ although its branches are pairwise isomorphic.

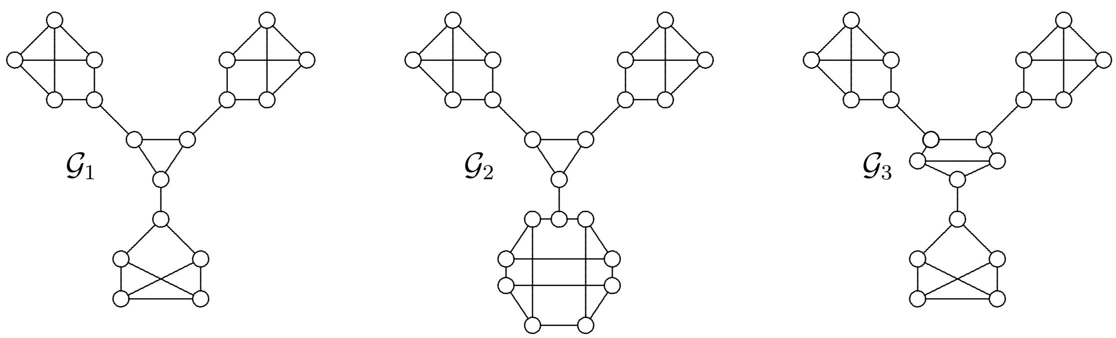

Figure 1. Well-formed graph $\mathcal{G}_1$ and not well-formed graphs $\mathcal{G}_2$ and $\mathcal{G}_3$

One question that possible arises here is determining all orders for which an NTCBG has a well-formed representation. Recall that any NTCBG can be written as $\mathcal{G} = \mathscr{C} \oplus \mathscr{B}_i$, where $\mathscr{B}_i$ need to be derived as constructed $\widehat{\mathscr{B}_i}$ to produce a valid union operation between $\mathscr{C}$ and $\mathscr{B}_i$. Note that the order of the vertex set in $\mathscr{C}$ and $\mathscr{B}_i$ is always equal to the order of the vertex set of $\mathscr{C} \oplus \widehat{\mathscr{B}_i}$ since for every $\mathscr{B}_i$ the additional vertex in $\widehat{\mathscr{B}_i}$ is an end-vertex of $\mathscr{C}$. Thus, we write $\mathcal{G} = \mathscr{C} \oplus \mathscr{B}_i$ instead of $\mathcal{G} = \mathscr{C} \oplus \widehat{\mathscr{B}_i}$. If a graph $\mathcal{G}$ is well-formed then by definition its central fragment is isomorphic to some hairy cycle and all $\mathscr{B}_i$ are pairwise isomorphic.

Remark 1. Theorem 2.3 implies that the order of the cycle in $\mathcal{H}_k$ is equal to the number of leaves in $\mathcal{H}_k$. Thus, we can rewrite $|V(\mathcal{H}_k)|$ as $2k_c$, where $k_c \leq 3$ is the number of leaves in $\mathcal{H}_k$.

Remark 2. By Theorem 2.1 and 2.2, if the branches of an NTCBG are all isomorphic we can say that $\sum |V(\mathcal{B}_i)| = k_c 2(b-1)$ where k_c is the number of leaves of the central fragment.

The next theorem will be useful in determining whether an NTCBG of a given order can be represented as a well-formed graph. While this theorem is necessary, this is not sufficient, and its formally stated as follows.

Theorem 3.1. *For any well-formed graph $\mathcal{G}$, $|V(\mathcal{G})| = 2k_c b$ where $k_c \leq 3$ is the number of leaves in $\mathcal{H}_k$ and $b \geq 3$.*

Proof. Suppose $\mathcal{G}$ is a well-formed graph. By Definition 1, the central fragment of $\mathcal{G}$ is isomorphic to $\mathcal{H}_k$ and each branch is pairwise isomorphic. It follows from Theorem 2.3 and Remark 1 that $|V(\mathcal{H}_k)| = 2k_c$, where $k_c \leq 3$ is the number of leaves in $\mathcal{H}_k$. By Theorem 2.1 and Remark 2, $\sum |V(\mathscr{B}_i)| = k_c[2(b-1)]$ where $k_c \leq 3$ is the number of leaves in $\mathcal{H}_k$ and $b \geq 3$. Thus we have

$$
\begin{aligned}
|V(G)| &= |V(\mathcal{H}_k)| + \sum |V(\mathscr{B}_i)| \\
|V(G)| &= 2k_c + k_c[2(b-1)] \\
|V(G)| &= 2k_c + 2k_c[(b-1)] \\
|V(G)| &= 2k_c(1 + (b-1)) \\
|V(G)| &= 2k_c b
\end{aligned}
$$

Moreover, for any well-formed NTCBG $\mathcal{G}$, $|V(\mathcal{G})| = 2k_c b$ where $k_c \leq 3$ is the number of leaves in $\mathcal{H}_k$ and $b \geq 3$. □

Lemma 3.1. *Every hairy cycle $\mathcal{H}_k$ has perfect matching.*

Proof. Suppose we have a hairy cycle $\mathcal{H}_k$. Note that $\mathcal{H}_k$ has a cycle $C_k = [c_1, c_2, ..., c_k, c_1]$ and end-vertices $a_1, a_2, ..., a_k$. Also, every pendant edge of $\mathcal{H}_k$ produces a matching $[a_i, c_i]$ where a_i is a pendant vertex and c_i is a vertex in C_k. By definition of hairy cycle, the set of pendant edges is equal to $[a_1, c_1], [a_2, c_2], ..., [a_k, c_k]$. Note that the set of pendant edge incident to all the vertices in $\mathcal{H}_k$ and produces a matching. Therefore, $\mathcal{H}_k$ has perfect matching. □

Theorem 3.2. *Any well-formed graph in which every branch is bridgeless contains a perfect matching.*

Proof. Let $\mathcal{G}$ be a well-formed graph. By Definition 1, $\mathcal{G}$ has a central fragment $\mathcal{H}_k$ and pairwise isomorphic branch $\mathscr{B}_i$. By Lemma 3.1, the set of pendant edge in $\mathcal{H}_k$ is a perfect matching. Since every $\mathscr{B}_i$ is bridgeless, by Theorem 2.5 each $\mathscr{B}_i$ also has perfect matching. Thus it remains to be shown that the union $\mathcal{H}_k \oplus \mathscr{B}_i$ also has perfect matching. Recall that a constructed $\widehat{\mathscr{B}_i}$ is formed by subdividing any edge of a $\mathscr{B}_i$ to produce a path of length 2 such that the new vertex is precisely one of the end-vertex of the central fragment. Since each of the pairwise isomorphic branches is bridgeless, it follows that every edge in a cubic bridgeless graph is contained in some perfect matching. Hence, for any edge chosen in $\mathscr{B}_i$, the constructed $\widehat{\mathscr{B}_i}$ still contains a perfect matching that does not include the chosen edge. Now, consider the union $\mathcal{H}_k \oplus \mathscr{B}_i$. Note that each end-vertex of $\mathcal{H}_k$ which is part of its matching is connected to a new vertex in constructed $\widehat{\mathscr{B}_i}$. Here, the edge of $\mathscr{B}_i$ that was subdivided to obtain the new vertex of the constructed $\widehat{\mathscr{B}_i}$ is not part of a perfect matching of $\mathscr{B}_i$. Thus, each pendant edge of $\mathcal{H}_k$ is not adjacent to any perfect matching of $\mathscr{B}_i$. Therefore, the perfect matching of $\mathcal{H}_k \oplus \mathscr{B}_i$ exists, and is composed of the perfect matching of $\mathcal{H}_k$ and the perfect matching of $\mathscr{B}_i$. □

For illustration, suppose we have an NTCBG with central fragment $\mathcal{H}_3$ and branches K_4 as shown in Figure 2. By Theorem 2.5, K_4 contains perfect matching, and likewise $\mathcal{H}_3$ contains a perfect matching by Lemma 3.1. It follows that by definition we have a constructed $\widehat{K_4}$. Hence we have the following theorem.

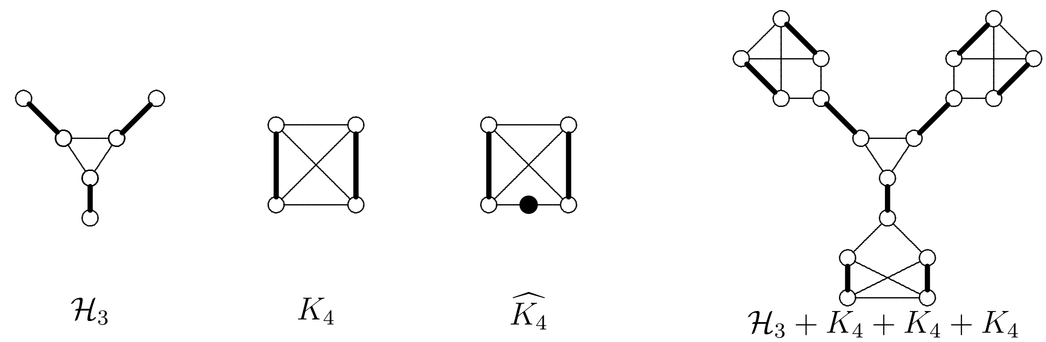

Figure 2. Perfect Matching $\mathcal{H}_3$ union 3 copies of K_4

Theorem 3.3. *Let $\mathcal{G}$ be a well-formed graph of order n and size m whose branch are bridgeless. Then the matching number of $\mathcal{G}$ is $\alpha'(G) = \frac{n}{2}$ and $\alpha'(G) = \frac{m}{3}$.*

Proof. Let $\mathcal{G}$ be a well-formed graph of order n whose branches are bridgeless, Then by Theorem 3.2 $\mathcal{G}$ has perfect matching. By definition the matching number of any perfect matching is $\frac{n}{2}$. Thus, $\alpha'(\mathcal{G}) = \frac{n}{2}$.

Now, by Theorem 2.6, since every vertex of NTCBG is of degree 3 then the size is given by $m = \frac{3n}{2}$, we can write this as $n = \frac{2m}{3}$. Since $\alpha'(\mathcal{G}) = \frac{n}{2}$ it follows that $\alpha'(\mathcal{G}) = \frac{\frac{2m}{3}}{2} = \frac{m}{3}$. Therefore, $\alpha'(\mathcal{G}) = \frac{m}{3}$. □

4. Decomposition of Well-formed Graph

The notion of subgraph is widely studied in graph theory. Numerous conjectures and open problems are studied regarding collections of subgraphs of a given graph where each edge of that graph belongs to exactly one subgraph in the collection. The collection of subgraphs with this property are often divided into two categories, depending on whether the subgraphs are required to be spanning subgraphs of a given graph or not. From this, an interesting problem arises: Does there exist an NTCBG $\mathcal{G}$ with a partition $H_1, ..., H_k$ of $V(\mathcal{G})$ such that all induced subgraphs are pairwise isomorphic, that is, $\mathcal{G}[H_i] \cong \mathcal{G}[H_j]$ for all $i, j \in \mathbb{N}$?

Recall the notion of hairy cycle introduced by Barrientos [2]. A hairy cycle is constructed by attaching one copy of K_1 to each vertex in a cycle to form a pendant edge. Hence, if we have a cycle $C_k = [c_1, c_2, ..., c_k, c_1]$ and K_1, then $\mathcal{H}_k$ has pendant edges $[a_1, c_1], [a_2, c_2], ..., [a_k, c_k]$. This graph labeling is also discussed in the proof of Lemma 3.1. The next result will discuss the decomposition of hairy cycle.

Lemma 4.1. *Any $\mathcal{H}_k$ can be decomposed as $P_3 = [a_i, c_i, c_{i+1}]$, where $1 \leq i \leq k$.*

Proof. By Theorem 2.3, the size $\mathcal{H}_k$ is even. Thus by Theorem 2.4, it is P_3-decomposable. Now, we will show that P_3-decomposition is composed of $[a_i, c_i, c_{i+1}]$ where $1 \leq i \leq k$. Let $C_k = [c_1, c_2, ..., c_{k-1}, c_k, c_1]$ and $a_1, a_2, ..., a_{k-1}, a_k$ be the end-vertex of each pendant edge attached to each vertex in the cycle. Without loss of generality, we can name all the pendant edges as $[a_1, c_1], [a_2, c_2], ..., [a_{k-1}, c_{k-1}], [a_k, c_k]$. Also, note that every non-empty graph is P_2-decomposable, so that C_k is decomposable as $\{[c_1, c_2], [c_2, c_3], ..., [c_{k-1}, c_k]\}$. By following the labeling of the central fragment, if we take the union of pendant edges and the decomposition of C_k with common vertex it is clear that $P_3 = [a_i, c_i, c_{i+1}]$ is the decomposition of $\mathcal{H}_k$, where $1 \leq i \leq k$. □

Remark 3. Any $\mathcal{H}_k$ can be decomposed as k-copies of $P_3 = [a_i, c_i, c_{i+1}]$, where $1 \leq i \leq k$.

Theorem 4.1. *Let $\mathcal{G}$ be a NTCBG. Then $\mathcal{G}$ is well-formed if and only if $\mathcal{G}$ is $(P_3 \oplus \mathscr{B}_i)$ - decomposable.*

Proof. Suppose $\mathcal{G}$ is well-formed graph. Then we have a central fragment $\mathcal{H}_k$ and k copies of isomorphic branch $\mathscr{B}_i$. By Lemma 4.1, $\mathcal{H}_k$ can decomposable as $P_3 = [a_i, c_i, c_{i+1}]$ where $1 \leq i \leq k$. Recall that for each $\mathscr{B}_i$ the constructed $\widehat{\mathscr{B}_i}$ has a new vertex equal to exactly one end-vertex of the central fragment. Thus every constructed $\widehat{\mathscr{B}_i}$ contains a new vertex a_i where $1 \leq i \leq k$. Taking the union of $P_3 = [a_i, c_i, c_{i+1}]$ and $\mathscr{B}_i$, we have $P_3 \oplus \mathscr{B}_i$ for all $i = 1, 2, ..., k$. Thus, $\mathcal{G}$ is $P_3 \oplus \mathscr{B}_i$ - decomposable.

Conversely, suppose $\mathcal{G}$ is $(P_3 \oplus \mathscr{B}_i)$ - decomposable. It is clear that we have k copies $(P_3 \oplus \mathscr{B}_i)$. Furthermore, there are also k copies of P_3 and k copies $\mathscr{B}_i$. It remains to be shown that the union of k copies of P_3 is isomorphic to $\mathcal{H}_k$. By Lemma 4.1, we have a set of P_3 that is, $[a_1, c_1, c_2], [a_2, c_2, c_3], ..., [a_{k-1}, c_{k-1}, c_k]$ and $[a_k, c_k, c_1]$. Taking the union of these k copies of P_3 yields $C_k = [c_1, c_2, ..., c_k, c_1]$ and $[a_1, c_1], [a_2, c_2], ..., [a_k, c_k]$. Note that this is exactly the representation of the hairy cycle $\mathcal{H}_k$. Hence, the union of k-copies of P_3 is isomorphic to $\mathcal{H}_k$. Therefore, $\mathcal{G}$ is well-formed if and only if $\mathcal{G}$ is $(P_3 \oplus \mathscr{B}_i)$ - decomposable. □

If $\mathcal{G}$ is $(P_3 \oplus \mathscr{B}_i)$ - decomposable graph for some $P_3 \oplus \mathscr{B}_i$, then certainly $P_3 \oplus \mathscr{B}_i$ is a subgraph of $\mathcal{G}$ and the size of $P_3 \oplus \mathscr{B}_i$ divides the size of $\mathcal{G}$. Although this last condition is necessary, it is not sufficient. If k be the number of copies of $P_3 \oplus \mathscr{B}_i$ in the decomposition of $\mathcal{G}$, then $k = |V(C_k)|$ in $\mathcal{H}_k$.

Theorem 4.2. *Let $\mathcal{G}$ be an NTCBG. If $\mathcal{G}$ is well-formed, then $\mathcal{G}$ is $k(P_3 \oplus \mathscr{B}_i)$ - decomposable where k is the order of C_k.*

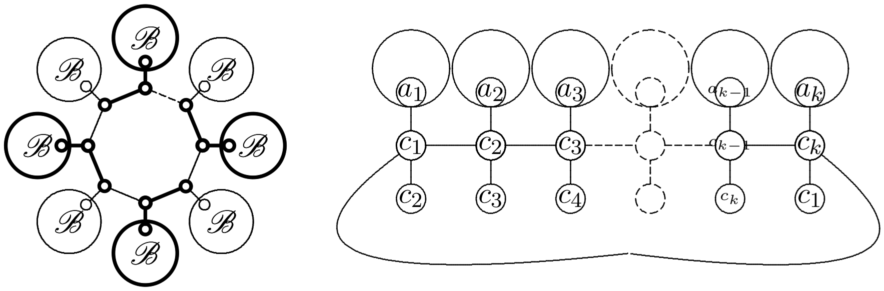

Figure 3. Illustration of Theorem 4.1

Proof. Let $\mathcal{G}$ be an NTCBG such that $\mathcal{G}$ is a well-formed graph. Then $\mathcal{G}$ contains a central fragment $\mathcal{H}_k$ and all branch $\mathscr{B}_i$ are pairwise isomorphic. By Theorem 4.1, $\mathcal{G}$ is $(P_3 \oplus \mathscr{B}_i)$-decomposable. Note that, by Remarks 3, $\mathcal{H}_k$ has k-copies of $P_3 = [a_i, c_i, c_{i+1}]$. Since every branch $\mathscr{B}_i$ is connected to P_3, we have a k copies of $(P_3 \oplus \mathscr{B}_i)$. Therefore, $\mathcal{G}$ is $k(P_3 \oplus \mathscr{B}_i)$ - decomposable where k is the order of C_k. □

Illustration 1. *Here is an illustration for Theorem 4.2. Note that the graph $\mathcal{G}$ shown in Figure 4 has a hairy cycle $\mathcal{H}_3$. This implies that we can have 3-copies of $P_3 \oplus \mathscr{B}_i$.*

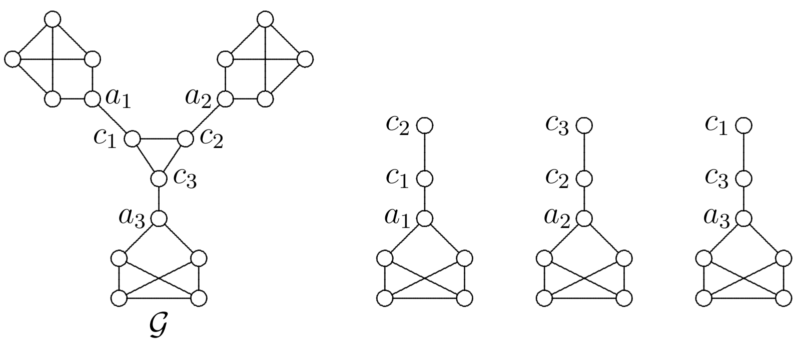

Figure 4. A well-formed graph $\mathcal{G}$ with Hairy Cycle $\mathcal{H}_3$ and branch K_4

5. Well-formed graph pictorial representation

The notion of representing NTCBG is quite difficult because of their extensive isomorphisms. For instance, there are numerous distinct branches of an NTCBG of order 6. In this section, we endeavor to find methods of determining whether an NTCBG of a given order is well-formed. The next theorem gives a sufficient condition for this, but it does not provide information about the uniqueness of said well-formed representation.

Theorem 5.1. *Let $\mathcal{G}$ be an NTCBG of order n such that the central fragment of $\mathcal{G}$ has k_c leaves. Then $\mathcal{G}$ has a well-formed representation if and only if there exists a factor pair $\mathcal{F} = \{[k_c, b] \mid k_c \times b = \frac{n}{2}$ where $3 \leq k_c, b \leq \frac{n}{2}\}$.*

Proof. Let $\mathcal{G}$ be an NTCBG of order n such that the central fragment of $\mathcal{G}$ has k_c leaves. Suppose there exists a factor pair $\mathcal{F} = \{[k_c, b] \mid k_c \times b = \frac{n}{2}$ where $3 \leq k_c, b \leq \frac{n}{2}\}$. It follows that $n = 2k_c b$. Clearly, $\mathcal{G}$ has a well-formed representation by Theorem 3.1.

Conversely, suppose $\mathcal{G}$ is a well-formed NTCBG of order n. By Theorem 3.1, $|V(\mathcal{G})| = 2k_c b$, k_c, b are both greater than or equal to 3. It follows that $n = 2k_c b \Rightarrow \frac{n}{2} = k_c b$. Therefore there exists a factor pairs $[k_c, b]$ of $\frac{n}{2}$ such that $k_c, b \geq 3$. □

Illustration 2. *Show that a graph $\mathcal{G}$ of order 32 has well-formed representation.*

Solution: By Theorem 3.1 $\mathcal{G}$ is well-formed. Also, by Theorem 5.1 we have: $k_c b = \frac{32}{2}$ which implies that $k_c b = 16$. The factor pairs of 16 are $\{[1, 16], [2, 8], [4, 4], [8, 2], [16, 1]\}$ among which $[4, 4]$ satisfies $\mathcal{F} = \{[k_c, b] \mid k_c \times b = \frac{32}{2}$ where $3 \leq k_c, b \leq \frac{32}{2}\}$, that is, $[4, 4]$. Thus G has a well-formed representation. Furthermore, since we have $k_c = 4$ and $b = 4$, it follows that the central fragment is of order $|V(\mathcal{H}_4)| = 2k_c = 2(4) = 8$ and there are 4 pairwise isomorphic branches of order $|V(\mathscr{B}_i)| = 2(b-1) = 2(4-1) = 6$ as shown in Figure 5.

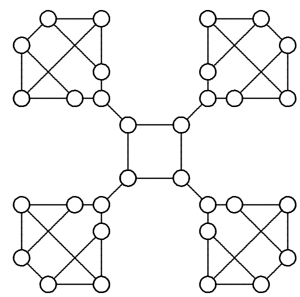

Figure 5. Well-formed representation of $\mathcal{G}$ of order 32

Corollary 5.1. *Let $\mathcal{F}$ be the set of factor pairs of a graph $\mathcal{G}$ of order n such that $\mathcal{F} = \{[k_c, b] \mid k_c \times b = \frac{n}{2}, 3 \leq k_c, b \leq \frac{n}{2}\}$. Then the cardinality of factor pairs in $\mathcal{F}$ is the number of ways such that $\mathcal{G}$ has a well-formed representation.*

6. Summary and Conclusion

In this paper, we introduced a special type of NTCBG called *well-formed graph*. Here we gave the isomorphic subgraph partition of NTCBG. Furthermore, the perfect matching, matching number, decomposition and some parameters for pictorial representation are also determined.

From the previously discussed results, one may consider a rather interesting problem, namely that of determining the family of constructed $\widehat{\mathscr{B}_i}$'s (up to isomorphism) that is formed by selecting different edges of a given starting $\mathscr{B}_i$. As an example, we will show that if our branch is $\mathscr{B}_i = K_4$, then all the constructed $\widehat{K_4}$ are pairwise isomorphic. Recall the defintion of the construction $\widehat{\mathscr{B}_i}$ of $\mathscr{B}_i$, where that we pick any edge in $\mathscr{B}_i$ and subdivide it into two to produce a path of length two such that the new vertex is precisely one of the end-vertices of the central fragment. Clearly, we have here six edges of K_4 to choose from. Graphs H_1 and H_2 in Figure 6 show two of the six graphs that can be obtained from different choices of edge in K_4. It can be easily verified that graphs obtained by subdividing other edges are represented by rotations of either H_1 or H_2, and are thus isomorphic to either of these graphs. Specifically, subdividing edge $[a, c]$ or $[b, c]$ according to the labeling used in Figure 6 will result in a constructed $\widehat{K_4}$ that is a rotation of H_1, so that said constructed $\widehat{K_4}$ is isomorphic to H_1. Likewise, subdividing edge $[c, d]$ or $[b, d]$ according to the same labeling will result in a constructed $\widehat{K_4}$ that is a rotation of H_2, so that said constructed $\widehat{K_4}$ is isomorphic to H_2.

Moreover, we can see that H_1 and H_2 in Figure 6 are isomorphic by defining the mapping $\phi : V(H_1) \longmapsto V(H_2)$ by $\phi = a \longmapsto a, b \longmapsto d, c \longmapsto c, d \longmapsto b, e \longmapsto e$. Finally, since graph isomorphism is an equivalence relation, a graph is isomorphic to H_1 if and only if it is isomorphic to H_2. Therefore all six constructed $\widehat{\mathscr{B}_i}$ are pairwise isomorphic.

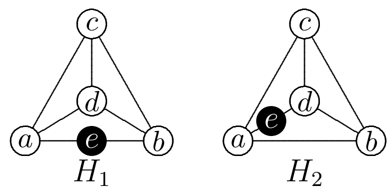

Figure 6. Uniqueness of $\widehat{K_4}$ up to isomorphism

Notice that by definition of edge-transitive graph, if a cubic graph $\mathscr{B}_i$ is edge-transitive, then all the constructed $\widehat{\mathscr{B}_i}$ are pairwise isomorphic. To date, the only known cubic graphs (up to isomorphism) that have edge-transitive properties are symmetric and semi-symmetric cubic graphs. Hence we arrive at the following conjecture.

Conjecture 1. *Symmetric and semi-symmetric cubic graphs are the only branches of NTCBG such that all constructed branches are pairwise isomorphic.*

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