Electronic Journal of Graph Theory and Applications **11** (2) (2023), 447–454

Electronic Journal of Graph Theory and Applications

On z-cycle factorizations with two associate classes where z is $2a$ and a is even

Joshua Lambert, Michael Tiemeyer

Department of Mathematical Sciences, Georgia Southern University, USA

joshualambert@georgiasouthern.edu, mtiemeyer@georgiasouthern.edu

Abstract

Let $K = K(a, p; \lambda_1, \lambda_2)$ be the multigraph with: the number of parts equal to p; the number of vertices in each part equal to a; the number of edges joining any two vertices of the same part equal to λ_1; and the number of edges joining any two vertices of different parts equal to λ_2. The existence of C_4-factorizations of K has been settled when a is even; when $a \equiv 1 \pmod 4$ with one exception; and for very few cases when $a \equiv 3 \pmod 4$. The existence of C_z-factorizations of K has been settled when $a \equiv 1 \pmod z$ and λ_1 is even, and when $a \equiv 0 \pmod z$. In this paper, we give a construction for C_z-factorizations of K for $z = 2a$ when a is even.

Keywords: cycles, factorization, associate classes
Mathematics Subject Classification : 05C38
DOI: 10.5614/ejgta.2023.11.2.9

1. Introduction

Let $K = K(a, p; \lambda_1, \lambda_2)$ denote the graph formed from p vertex-disjoint copies of the multigraph $\lambda_1 K_a$–each edge in K_a appearing λ_1 times–by joining each pair of vertices in different copies with λ_2 edges. The vertex set, $V(K)$, is always chosen to be $\mathbb{Z}_a \times \mathbb{Z}_p$, with parts $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_p$; naturally, each part induces a copy of $\lambda_1 K_a$. We say the vertex (i, j) is on *level i* and in *part j*. An edge is said to be a *mixed edge* if it joins vertices in different parts, and is said to be a *pure edge* (in part j) if it joins two vertices in the jth part.

Received: 5 December 2022, Revised: 28 May 2023, Accepted: 6 August 2023.

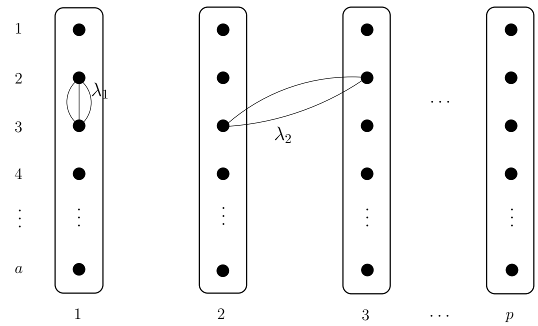

Figure 1. $K = K(a, p; \lambda_1, \lambda_2)$

Let C_z denote a cycle of length z. A C_z-factorization is a 2-factorization such that each component of each 2-factor is a cycle of length z; each 2-factor of a C_z-factorization is known as a C_z-factor. C_z-factorizations are also known as *resolvable C_z-decompositions*. A $C_{\{z_1,z_2,...,z_k\}}$-factorization is a 2-factorization such that each 2-factor is a C_w-factor where $w \in \{z_1, z_2, ..., z_k\}$.

There has been considerable interest recently in C_z-decompositions of various graphs, such as complete graphs and complete multipartite graphs. In the resolvable case, these results are collectively known as addressing the Oberwolfach problem. More recently, the existence problem for C_z-decompositions of K for $z \in \{3, 4\}$ has been solved [3, 4, 5]. Such decompositions are known as *C_z-group-divisible designs with two associate classes*, following the notation of Bose and Shimamoto who considered the existence problem for K_z-group divisible designs. The reason for this name is that the structure can be thought of as partitioning ap symbols, or vertices, into p sets of size a in such a way that symbols that are in the same set in the partition occur together in λ_1 blocks, and are known as *first associates*, whereas symbols that are in different sets in the partition occur together in λ_2 blocks, and are known as *second associates* [1].

C_z-factorizations of K have also been of interest [5]. Recently the existence of a C_4-factorization of K has been completely settled when a is even [2] and when $a \equiv 1 \pmod 4$ with one difficult exception [8, 9]. Some work has also been doen for the case where $a \equiv 3 \pmod 4$ [6]. A general construction for C_z-factorizations of K when z is even, $a \equiv 1 \pmod z$, and λ_1 is even, and when $a \equiv 0 \pmod z$ has also been given [10]. In this paper, we give a construction for C_z-factorizations of K for $z = 2a$ when a is even.

Open problems include a construction for C_z-factorizations of K for $z = 2a$ when a is odd, which is proving to be more difficult. Also, considering different cycle lengths, $z = ka$ for $k > 2$, in the C_z-factorization is a worthy endeavor; the authors suspect that the parity of k may play a role in the difficulty of the constructions.

Lemma 1.1. *Let $z = 2a$ where a is even. If there exists a C_z-factorization of $K(a, p; \lambda_1, \lambda_2)$, then:*

1. p is even,
2. λ_1 is even, and
3. $\lambda_2 > 0$.

Proof. Since the number of z-cycles in each C_z-factor is the number of vertices divided by z, z must divide ap, and since $a = z/2$, $p \equiv 0 \pmod 2$.

Each vertex is joined with λ_1 edges to each of the $(a-1)$ other vertices in its own part and with λ_2 edges to each of the $a(p-1)$ vertices in the other parts; so the degree of each vertex is:

$$d_K(v) = \lambda_1(a-1) + \lambda_2 a(p-1).$$

Clearly, since K has a C_z-factorization, it is regular of even degree. The second term is even since a is even. The first term must therefore be even, so since $(a-1)$ is odd, λ_1 must be even. Since $a < z$, each C_z-factor must contain mixed edges; hence $\lambda_2 > 0$. □

Lemma 1.2. *Let $z = 2a$ where a is even. If there exists a C_z-factorization of $K(a, p; \lambda_1, \lambda_2)$, then $\lambda_1 \leq \lambda_2 a(p-1)$.*

Proof. Since $a < z$, each C_z-factor contains at most $(a-1)$ pure edges in each part. So each C_z-factor contains at most $(a-1)p$ pure edges. Since there are $\lambda_1\binom{a}{2}p$ pure edges, the number of C_z-factors in any C_z-factorization is at least:

$$\frac{\lambda_1\binom{a}{2}p}{(a-1)p} = \frac{\lambda_1 a}{2}.$$

Each C_z-factor has ap edges, of which at most $(a-1)p = ap - p$ are pure, so there are at least p mixed edges in any C_z-factor. Then the number of mixed edges in any C_z-factorization is at least:

$$\frac{\lambda_1 ap}{2}.$$

Therefore, this number must be at most the number of mixed edges, $\lambda_2\binom{p}{2}a^2$, in K:

$$\frac{\lambda_1 ap}{2} \leq \lambda_2\binom{p}{2}a^2,$$

so

$$\lambda_1 \leq \lambda_2 a(p-1).$$

□

Lemma 1.3. *Let a be even. There exists a cyclical decomposition of K_a into edge-disjoint Hamiltonian paths such that the ends of the paths are vertices i and $i + a/2$ for $i \in \mathbb{Z}_{a/2}$.*

Proof. Let $i \in \mathbb{Z}_{a/2}$. The ith such Hamiltonian path is

$$h_i = \big(i, i+1, i+(a-1), i+2, i+(a-2), \ldots, i+(a/2-1), i+(a/2+1), i+(a/2)\big)$$

See Figure 2. Note that

$$K_a = \bigcup_{i \in \mathbb{Z}_{a/2}} h_i$$

and the ends of the Hamiltonian paths are always i and $i + a/2 \pmod{a}$. Let

$$H_a = \{h_i | i \in \mathbb{Z}_{a/2}\}$$

□

Theorem 1.1. *[7] Suppose $z > 2$. There exists a C_z-factorization of $K(a, p; 0, 1)$ if and only if $K \neq K(6, 2; 0, 1)$ where $z = 6$.*

Theorem 1.2. *[2] Let a be even. There exists a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$.*

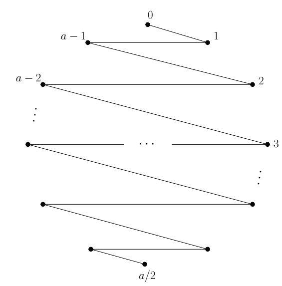

Figure 2. Hamiltonian path of K_a.

2. The main result - z is $2a$

Theorem 2.1. *Let $z = 2a$ where $a > 2$ is even. There exists a C_{2a}-factorization of $K = K(a, p; \lambda_1, \lambda_2)$ if and only if*

1. *p is even,*
2. *λ_1 is even,*
3. *$\lambda_2 > 0$, and*
4. *$\lambda_1 \leq \lambda_2 a(p-1)$*

Proof. The necessity of these conditions follows from Lemmas 1.1 and 1.2. So now assume that K satisfies conditions (1–4). If $\lambda_1 = 0$, then the required factorization is given by Theorem 1.1. So we may also assume that $\lambda_1 > 0$.

Given part size a, there are a mixed differences, 0, 1, ... $a-1$, between the levels of the vertices in each part. Given two parts, m and n, an edge of mixed difference 0 would join the vertex on level ℓ in part m to the vertex on level ℓ in part n. An edge of mixed difference d would join a vertex on level ℓ in part m to the vertex on level $(\ell + d) \pmod{a}$ in part n. For $d \in \mathbb{Z}_a$, $m, n \in \mathbb{Z}_p$, $m < n$, let

$$M(d, m, n) = \{((\ell, m), (\ell + d, n)) \mid \ell \in \mathbb{Z}_a\}$$

be the set of a mixed edges of difference d between parts m and n. See Figure 3 for an example showing all the mixed edges of mixed difference 1 between a pair of parts of size $a = 6$.

For $d \in \mathbb{Z}_a$, $\ell \in \mathbb{Z}_{a/2}$, and $m, n \in \mathbb{Z}_p$, $m < n$, let

$$M_2(d, m, n, \ell) = \{((\ell, m), (\ell + d, n)), ((\ell + a/2, m), (\ell + a/2 + d \text{ (mod } a), n))\}$$

be the set of two mixed edges of $M(d, m, n)$ on parts m and n such that the ends of the edges are on levels ℓ and $\ell + a/2$ in part m and on levels $\ell + d$ and $\ell + a/2 + d \pmod{a}$ in part n. Notice that

$$M(d, m, n) = \bigcup_{\ell \in \mathbb{Z}_{a/2}} M_2(d, m, n, \ell).$$

Since p is even, there exists a 1-factorization of $\lambda_2 K_p$, denoted F, consisting of $\lambda_2(p-1)$ 1-factors. Let f_s be the s^{th} 1-factor of F where

$$F = \{f_s \mid s \in \mathbb{Z}_{\lambda_2(p-1)}\}.$$

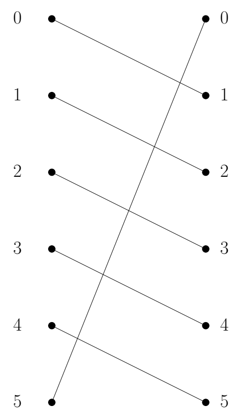

Figure 3. The mixed edges of difference 1 between a pair of parts of size $a = 6$.

For $s \in \mathbb{Z}_{\lambda_2(p-1)}$, let

$$M_2(s,d,\ell) = \{M_2(d,m,n,\ell) \mid (m,n) \in f_s, m < n\}$$

be the set of p mixed edges of difference d distributed across the paired parts of K defined by the 1-factor f_s where the ends of the edges are on levels ℓ and $\ell + a/2$ in part m and on levels $\ell + d$ and $\ell + a/2 + d \pmod a$ in part n. Also let

$$M_2(s,d) = \bigcup_{\ell \in \mathbb{Z}_{a/2}} M_2(s,d,\ell)$$

be the set of all mixed edges of difference d distributed across the paired parts of K defined by the 1-factor f_s. Notice that $M_2(s,d)$ is a 1-factor of K and that

$$\pi(s,d) = M_2(s,d) \cup M_2(s,d+1)$$

is a 2-factor of K, specifically, it is a C_{2a}-factor of K. In fact, these C_{2a}-factors can be used to produce a C_{2a}-factorization of $K(a,p;0,\lambda_2)$, namely:

$$\bigcup_{s \in \mathbb{Z}_{\lambda_2(p-1)}} \bigcup_{\{d=2x \mid x \in \mathbb{Z}_{a/2}\}} \pi(s,d).$$

However, we have pure edges to use too, since $\lambda_1 > 0$ by assumption, which is accomplished by spreading the edges of the $2a$-cycles in $\pi(s,d)$ among a C_{2a}-factors p edges at a time. Each such C_{2a}-factor contains the p mixed edges of $M_2(s,d,\ell)$ for some $d \in \mathbb{Z}_a$, $\ell \in \mathbb{Z}_{a/2}$ together with a Hamiltonian path in each part. More specifically, for each $i \in \mathbb{Z}_a$ and $k \in \mathbb{Z}_p$, using Lemma 1.3, let $h_i(k)$ be the Hamiltonian path of a cyclical, edge-disjoint Hamiltonian path decomposition of K_a on the vertex set $\mathbb{Z}_a \times \{k\}$ where the ends of the path are i and $i + a/2 \pmod a$.

For $i \in \mathbb{Z}_{a/2}$, $d \in \mathbb{Z}_a$, $m, n \in \mathbb{Z}_p$, $m < n$, and $s \in \mathbb{Z}_{\lambda_2(p-1)}$, let

$$P(s,d,i) = \left\{h_i(m) \cup h_{i+d}(n) \cup M_2(d,m,n,i) \mid (m,n) \in f_s\right\}$$

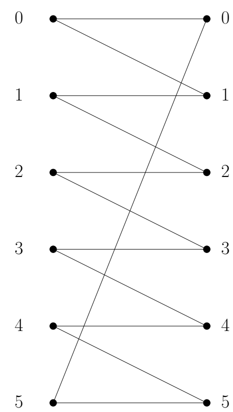

Figure 4. The mixed edges of differences 0 and 1 forming a 12-cycle.

be such a C_{2a}-factor of K; see Figure 5 for an example. Notice that

$$\bigcup_{i\in\mathbb{Z}_{a/2}} P(s,d,i)$$

contains

(a) each pure edge in each part exactly once, and
(b) precisely the mixed edges in $M_2(s,d)$.

Also notice that

$$P(s,d) = \left(\bigcup_{i\in\mathbb{Z}_{a/2}} P(s,d,i)\right) \cup \left(\bigcup_{i\in\mathbb{Z}_{a/2}} P(s,d+1,i)\right)$$

contains

(c) each pure edge in each part exactly twice, and
(d) precisely the mixed edges in $\pi(s,d)$.

Let $S = \{(s,d) \mid s \in \mathbb{Z}_{\lambda_2(p-1)}, d \in \mathbb{Z}_a, d \text{ is even}\}$. Let $S_1 \subseteq S$ have size $\frac{\lambda_1}{2}$. Notice that by condition 4. of the theorem, $\lambda_1 \leq \lambda_2 a(p-1)$, so $|S_1| = \frac{\lambda_1}{2} \leq \frac{\lambda_2 a(p-1)}{2} = |S|$, so such a set $|S_1|$ exists. Then

$$\bigcup_{(s,d)\in S_1} P(s,d)$$

is a set of $\frac{\lambda_1 a}{2}$ C_{2a}-factors that contains each pure edge $2|S_1| = \lambda_1$ times by (c), and uses precisely the mixed edges in

$$\bigcup_{(s,d)\in S_1} \pi(s,d)$$

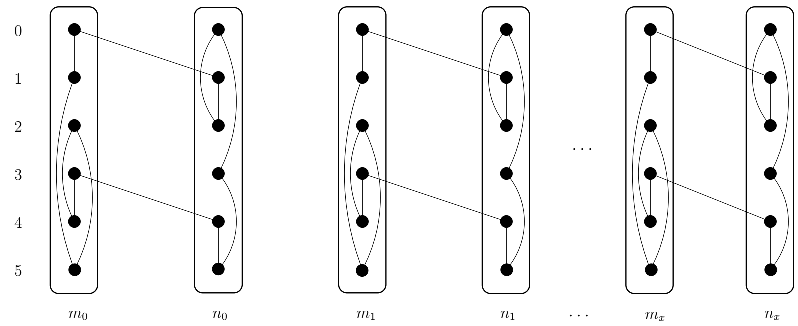

Figure 5. An example of P(s,d,i).

by (d). Therefore, the required C_{2a}-factorization of K is defined by

$$P = \left(\bigcup_{(s,d)\in S_1} P(s,d) \right) \cup \left(\bigcup_{(s,d)\in S\setminus S_1} \pi(s,d) \right).$$

Notice that

$$\begin{aligned} |P| &= a|S_1| + |S \setminus S_1| \\ &= \frac{\lambda_1 a}{2} + \frac{\lambda_2 a(p-1)}{2} - \frac{\lambda_1}{2} \\ &= \frac{\lambda_1(a-1)}{2} + \frac{\lambda_2 a(p-1)}{2} \end{aligned}$$

as required.

□

References

[1] R.C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *Journal of the American Statistical Association*, **47** (1952), 151–184.

[2] E.J. Billington and C.A. Rodger, Resolvable 4-cycle group divisible designs with two associate classes: part size even, *Discrete Math.*, **308** (2008), 303–307.

[3] H.L. Fu, C.A. Rodger, and D.G. Sarvate, The existence of group divisible designs with first and second associates, having block size 3, *Ars Combin.*, **54** (2000), 33–50.

[4] H.L. Fu and C.A. Rodger, 4-cycle group-divisible designs with two associate classes, *Combin. Probab. Comput.* **10** (2001), 317–343.

[5] H.L. Fu and C.A. Rodger, Group divisible designs with two associate classes: $n = 2$ or $m = 2$, *J. Combin. Theory (A)*, **83** (1998), 94–117.

[6] C. Goss, M. Tiemeyer, and R. Waller, On C_4-factorizations with two associate classes where $a \equiv 3 \pmod 4$ and small, *Congr. Numer.* **219** (2014), 97–128; MR3308539.

[7] J. Liu, A generalization of the Oberwolfach problem and C_t-factorizations of complete equipartite grpahs, *J. Combin. Designs*, **8** (2000), 42–49.

[8] C.A. Rodger and M.A. Tiemeyer, C_4-Factorizations with Two Associate Classes, *Australas. J. Combin.* **40** (2008), 217–228.

[9] C.A. Rodger and M.A.Tiemeyer, C_4-factorizations with two associate classes, λ_1 is odd, *Australas. J. Combin.* **50** (2011), 259–278.

[10] M.A. Tiemeyer, On z-cycle factorizations with two associate classes where z is even and a is 0 or 1 mod z, *J. Combin. Math. and Comb. Computing* **105** (2018), 11–19.