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Size multipartite Ramsey numbers for stripes versus small cycles

Chula Jayawardene[a], Edy Tri Baskoro[b], Lilanthi Samarasekara[c], Syafrizal Sy[d]

[a] *Department of Mathematics, University of Colombo, Colombo, Sri Lanka*
[b] *Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia*
[c] *Department of Mathematics, University of Colombo, Colombo, Sri Lanka*
[d] *Universitas Andalas, Padang, Indonesia*

c_jayawardene@yahoo.com, ebaskoro@math.itb.ac.id, lilanthi@maths.cmb.ac.lk,syafrizalsy@gmail.com

Abstract

For simple graphs G_1 and G_2, the size Ramsey multipartite number $m_j(G_1, G_2)$ is defined as the smallest natural number s such that any arbitrary two coloring of the graph $K_{j\times s}$ using the colors red and blue, contains a red G_1 or a blue G_2 as subgraphs. In this paper, we obtain the exact values of the size Ramsey numbers $m_j(nK_2, C_m)$ for $j \geq 2$ and $m \in \{3, 4, 5, 6\}$.

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1. Introduction

All graphs $G = (V, E)$ considered in this paper are finite graphs without loops and multiple edges. The order of the graph $G = (V, E)$ is denoted by $|V|$ and the number of edges in the graph is denoted by $|E|$. A n stripe of a graph G is defined as a set of n edges without a common vertex. The complete graph on n vertices is denoted by K_n. The graphs $K_{j,s}$ and $K_{j\times s}$ represent complete bipartite graph with partite sets of size j and s and the complete multipartite graph consisting of j partite sets having exactly s vertices in each partite set, respectively.

Let G and H be two finite graphs. If for every two coloring (red and blue) of the edges of a complete graph K_n, there exists a copy of G in the first color (red) or a copy of H in the second color (blue), we denote it by $K_n \to (G, H)$. The Ramsey number $r(G, H)$ is defined as the smallest

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positive integer n such that $K_n \to (G, H)$. The classical Ramsey number $r(s, t)$ is defined as $r(K_s, K_t)$. The exact determination of these numbers (see [5] for a survey) becomes increasingly difficult from the nearly trivial $r(3, 3) = 6$ to the stubbornly resistant $r(5, 5)$ (at present known to be between 43 and 49). One of the first variations of the classical Ramsey numbers was introduced by Erdös, Faudree, Rousseau and Shelph [3]. Some of the most striking results involving the size Ramsey numbers are by Rousseau et al. ([1], [4] [6] and [13]). In the last decade, using the idea of the original classical Ramsey numbers and of the size Ramsey numbers, the notion of size multipartite Ramsey numbers were introduced by Burger and Vuuren [2] and Syafrizal et al. [7] by considering the two colorings of a $K_{j\times s}$ by fixing the size j of the uniform multipartite sets. More precisely, $m_j(G, H)$ is the smallest number s, so that for every two coloring (red and blue) of the edges of $K_{j\times s}$, there necessarily is a copy of G in the first color (red) or a copy of H in the second color (blue). Ramsey numbers of small paths versus certain classes of graphs have been studied by Syafrizal Sy, Baskaro et al. in [8] [9] [10] [11] and [12]. Motivated by these findings, we have attempted in this paper to find size multipartite Ramsey numbers for stripes versus small cycles using Bondy's Lemma and some results of Hamiltonian graphs.

2. Some useful lemmas

Lemma 2.1. *Suppose G contains a matching H of size $n - 1$ such that u and v are two vertices of $G[V(G) \setminus V(H)]$. Suppose G contains no matching of size n. Then for any edge (a, b) of H either u is not adjacent to a or v is not adjacent to b.*

Proof. If the conclusion of the above lemma is false, by removing (a, b) from H and introducing the two edges (u, a) and (v, b) to the remaining $n - 2$ edges of H, we would get a nK_2 in G, a contradiction. □

Lemma 2.2. *$m_3(nK_2, C_3) \geq n$ and $m_3(nK_2, C_5) \geq n$.*

Proof. To show that $m_3(nK_2, C_3) \geq n$ for $n \geq 1$ and $m_3(nK_2, C_5) \geq n$, for $n \geq 1$ consider the coloring given by $K_{3\times(n-1)} = H_R \oplus H_B$, generated by coloring all edges between the first two partite sets of $K_{3\times(n-1)}$ by blue and all the other edges by red. Then, the graph has no blue C_3, no blue C_5 and no red nK_2. Hence, $m_3(nK_2, C_3) \geq n$ and $m_3(nK_2, C_5) \geq n$. □

Notation Let $V(K_{t\times n}) = \{v_{i,k} \mid i \in \{1, 2, ..., n\}$ and $k \in \{1, 2, ..., t\}\}$ and for a fixed $k \in \{1, 2, ..., t\}$ let $V_k = \{v_{i,k} \mid i \in \{1, 2, ..., n\}\}$ represent the k^{th} partite set. Given any $i \in \{1, 2, ..., n\}$, let $G_R[i]$ and $G_B[i]$ denote the red induced graph and the blue induced graph generated by $\{v_{i,k} \mid k \in \{1, 2, ..., t\}\}$ respectively.

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Lemma 2.3. $m_3(nK_2, C_3) = n$.

Proof. To show that $m_3(nK_2, C_3) \leq n$ consider a coloring given by $K_{3\times n} = H_R \oplus H_B$. If each $G_R[i]$, $1 \leq i \leq n$ contains a red edge then these edges will constitute a red nK_2; otherwise $G_B[i]$ will be isomorphic to a blue C_3 for some i where $1 \leq i \leq n$. Therefore, $m_3(nK_2, C_3) \leq n$. Hence the lemma follows. □

Lemma 2.4. $m_2(nK_2, C_4) \geq n + 1$.

Proof. To show that $m_2(nK_2, C_4) \geq n + 1$ for $n \geq 1$ consider the coloring given by $K_{2\times n} = H_R \oplus H_B$, generated by coloring all edges of any $K_{2\times(n-1)}$ subgraph by red and all the other edges by blue. Then, the graph has no blue C_4 and no red nK_2. Hence, $m_2(nK_2, C_4) \geq n + 1$. □

Lemma 2.5. $m_2(nK_2, C_4) = n + 1$.

Proof. We know that $m_2(K_2, C_4) = 2$ and $m_2(2K_2, C_4) = 3$. To first prove the inequality $m_2(nK_2, C_4) \leq n + 1$ for $n \geq 3$, we will use induction on n. Clearly the result is true for $n = 3$. Assume that $m_2(pK_2, C_4) \leq p + 1$ for all $p < n$. Consider the red nK_2 free coloring given by $K_{2\times(n+1)} = H_R \oplus H_B$. Suppose that H_B has no blue C_4. Using the induction hypothesis and the previous lemma, since $m_2((n-1)K_2, C_4) = n$, we may assume that the induced subgraph on $\{v_{i,k} \mid k \in \{1, 2\}$ and $i \in \{2, ..., n + 1\}\}$ contains a red $(n-1)K_2$. Without loss of generality, assume $v_{1,1}$, $v_{1,2}$, $v_{2,1}$ and $v_{2,2}$ are not adjacent to any edges of the $(n-1)K_2$ in red. But then in order to avoid a red nK_2, $(v_{1,1}, v_{1,2})$, $(v_{1,1}, v_{2,2})$, $(v_{2,1}, v_{1,2})$ and $(v_{2,1}, v_{2,2})$ have to be blue edges. So $(v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{1,1})$ will be a blue C_4, a contradiction. Therefore, $m_2(nK_2, C_4) \leq n + 1$. Hence by previous lemmma, $m_2(nK_2, C_4) = n + 1$. □

Lemma 2.6. $m_3(K_2, C_5) = 2$ *and* $m_3(nK_2, C_5) = n$ *for* $n \geq 2$.

Proof. We know that $m_3(K_2, C_5) = 2$ and $m_3(2K_2, C_5) = 2$. To first prove the inequality $m_3(nK_2, C_5) \leq n$ for $n \geq 3$, we will use induction on n. Clearly the result is true for $n = 3$. Assume that $m_3(pK_2, C_5) \leq p$ for all $p < n$. Consider the red nK_2 free coloring given by $K_{3\times n} = H_R \oplus H_B$. Suppose that H_B has no blue C_5. If each $G_R[i]$, $1 \leq i \leq n$ contains a red edge then these edges will constitute a red nK_2; a contradiction. So without loss of generality $G_B[1]$ will be isomorphic to a blue C_3. By induction and Lemma 2.2 the induced subgraph on $\{v_{i,k} \mid k \in \{1, 2, 3\}$ and $i \in \{3, ..., n\}\}$ contains a red $(n-2)K_2$. Inspection shows that in order to avoid a blue C_5 consisting of vertices $v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}$ and $v_{2,2}$, without loss of generality $(v_{2,1}, v_{1,3})$ has to be a red edge. As there is no red nK_2, we get $(v_{2,2}, v_{1,1})$, $(v_{2,3}, v_{1,2})$ and $(v_{2,2}, v_{2,3})$ are blue edges. So $(v_{1,3}, v_{1,2}, v_{2,3}, v_{2,2}, v_{1,1}, v_{1,3})$ will be the blue C_5, a contradiction. Therefore, $m_3(nK_2, C_5) \leq n$. Hence the lemma follows. □

Lemma 2.7. $m_2(nK_2, C_6) \geq n + 2$ *for* $n \geq 1$.

Proof. To show that $m_2(nK_2, C_6) \geq n + 2$ for $n \geq 1$ consider the coloring given by $K_{2\times(n+1)} = H_R \oplus H_B$, generated by coloring only edges adajacent to $v_{1,1}$ and $v_{2,1}$ in blue. Then, the graph has no blue C_6 and no red nK_2. Hence, $m_2(nK_2, C_6) \geq n + 2$. □

Lemma 2.8. $m_2(nK_2, C_6) = n + 2$.

Proof. We know that $m_2(K_2, C_6) = 3$ and $m_2(2K_2, C_6) = 4$. To prove $m_2(nK_2, C_6) \leq n + 2$ for $n \geq 3$, we will use induction on n. Clearly the result is true for $n = 3$. Assume that $m_2(pK_2, C_6) \leq p + 2$ for all $p < n$. Consider the red nK_2 free coloring given by $K_{2\times(n+2)} = H_R \oplus H_B$. Suppose that H_B has no blue C_6. Using the induction hypothesis and the previous lemma, since $m_2((n-1)K_2, C_5) = n + 1$, we may assume that the induced subgraph on $\{v_{i,k} \mid k \in \{1, 2\}$ and $i \in \{2, ..., n + 2\}\}$ contains a red $(n - 1)K_2$. Without loss of generality, assume $v_{1,1}$, $v_{1,2}$, $v_{2,1}$, $v_{2,2}$, $v_{3,1}$ and $v_{3,2}$ are not adjacent to any edges of the $(n - 1)K_2$ in red. But then in order to avoid a red nK_2, $(v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, v_{1,1})$ will be the blue C_6, a contradiction. Therefore, $m_2(nK_2, C_6) \leq n + 2$. Hence by previous lemma, $m_2(nK_2, C_6) = n + 2$. □

3. Size Ramsey numbers related to stripes versus three cycles

Lemma 3.1. *Suppose* $j \geq 4$ *and* $n \geq 2$. *Given that* $m_j((n-1)K_2, C_3) = \left\lceil \frac{2n-2}{j-1} \right\rceil$ *it follows that* $m_j(nK_2, C_3) \leq \left\lceil \frac{2n}{j-1} \right\rceil$.

Proof. Let $j \geq 4$. Assume that $m_j((n-1)K_2, C_3) = \left\lceil \frac{2n-2}{j-1} \right\rceil$ is true. To prove $m_j(nK_2, C_3) \leq \left\lceil \frac{2n}{j-1} \right\rceil$, consider any red nK_2 free, red and blue coloring $K_{j\times s} = H_R \oplus H_B$ where $s = \left\lceil \frac{2n}{j-1} \right\rceil$. Assume that H_B contains no blue C_3. Then the subgraph $K_{j\times s_0}$ where $s_0 = \left\lceil \frac{2n-2}{j-1} \right\rceil$ has no blue C_3, so it has a red $(n-1)K_2$. Let $W = V((n-1)K_2)$ and let W_i be the subgraph consisting of the disjoint edges of the matching $(n - 1)K_2$ that belong to $G[V(K_{j\times s}) \setminus V_i]$; $1 \leq i \leq j$. The graph contains two vertices in any $V(K_{(j-1)\times s}) \setminus V(W_i)$ since $s(j - 1) - 2(n - 1) \geq \left(\frac{2n}{j-1}\right)(j - 1) - 2(n - 1) \geq 2$,. We know that, if there are three vertices $u \in U$, $v \in V$ and $w \in W'$ where $u, v, w \notin W$, and U, V and W' are distinct partite sets of $K_{j\times s}$ then we would get a blue C_3, a contradiction. Therefore, by the repeated use of the previous statement we would get that there exists vertices $u, v \in U$ and $w, y \in V$ where U and V are distinct partite sets of $K_{j\times s}$ such that $\{u, v, w, y\} \cap W = \phi$.

Since $j \geq 4$, we can select a red edge (a, b) of the red $(n - 1)K_2$ such that $a, b \notin U \cup V$. As H_B is C_3 free a is incident to at least two vertices of $\{u, v, w, y\}$ in red. By applying the Lemma 2.1, we get b is incident to all the vertices of $\{u, v, w, y\}$ in blue. But then (b, y, v, b) is a blue C_3, a contradiction. Hence the result. □

Theorem 3.1. *If $j \geq 2$, then*

$$m_j(nK_2, C_3) = \begin{cases} \infty & \text{if } j = 2 \\ \left\lceil \dfrac{2n}{j-1} \right\rceil & \text{otherwise} \end{cases}$$

Proof. Since a bipartite graph contains no odd cycle $m_2(nK_2, C_3) = \infty$. Also we know that $m_j(nK_2, C_3) = \left\lceil \frac{2n}{j-1} \right\rceil$ when $j = 3$ (see Lemma 2.3).

To show that $m_j(nK_2, C_3) \geq \left\lceil \frac{2n}{j-1} \right\rceil$ for $j \geq 4$ and $n \geq 2$, assume that, $\left\lceil \frac{2n}{j-1} \right\rceil > 1$. Consider the coloring given by $K_{j \times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{2n}{j-1} \right\rceil - 1$, generated by coloring only edges incident to the vertices of the first partite set of $K_{j \times s}$ by blue. Then, the graph has no blue C_3, and $s(j-1) = \left(\left\lceil \frac{2n}{j-1} \right\rceil - 1 \right) (j-1) < 2n$. Therefore, the graph contains no red nK_2.

Assume $j \geq 4$ and $n \geq 2$. Now we are left to show $m_j(nK_2, C_3) \leq \left\lceil \frac{2n}{j-1} \right\rceil$. The result is clearly true for $n = 1$ because $m_j(K_2, C_3) = 1$ for all $j \geq 4$.

Thus, we would get $m_j(nK_2, C_3) \leq \left\lceil \frac{2n}{j-1} \right\rceil$, by applying induction on n and using Lemma 3.1 along with $m_j(nK_2, C_3) \geq \left\lceil \frac{2n}{j-1} \right\rceil$. Hence the theorem follows.

□

4. Size Ramsey numbers related to stripes versus four cycles

Lemma 4.1. *Suppose $j \geq 3$ and $n \geq 3$. Given that $m_j((n-1)K_2, C_4) = \left\lceil \frac{2n-1}{j} \right\rceil$ it follows that $m_j(nK_2, C_4) \leq \left\lceil \frac{2n+1}{j} \right\rceil$.*

Proof. Let $j \geq 3$ and $n \geq 3$. Assume that $m_j((n-1)K_2, C_4) = \left\lceil \frac{2n-1}{j} \right\rceil$ is true. To prove $m_j(nK_2, C_4) \leq \left\lceil \frac{2n+1}{j} \right\rceil$, consider any red-blue, red nK_2 free coloring given by $K_{j \times s} = H_R \oplus H_B$ where $s = \left\lceil \frac{2n+1}{j} \right\rceil$. Assume that the coloring contains no blue C_4. Then the subgraph $K_{j \times s_0}$

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where $s_0 = \left\lceil \frac{2n-1}{j} \right\rceil$ has no blue C_4, so it has a red $(n-1)K_2$. Let $W = V((n-1)K_2)$. Since $sj - 2(n-1) \geq \left(\frac{2n+1}{j}\right) j - 2(n-1) \geq 3$, there exists three vertices u, v and w belonging to $G[V(K_{j\times s}) \setminus W]$.

Case 1: $u \in U$, $v \in V$ and $w \in W'$ where U, V and W' are distinct partite sets of $K_{j\times s}$.

Sub-case 1.1: $j = 3$
Since $j = 3$, without loss of generality, select a red edge (a, b) of the red $(n-1)K_2$ such that $a \in U$ and $b \in V$. a cannot be adjacent to both $\{v, w\}$ in blue as it would force a blue C_4. If (a, v) is a red edge, clearly by Lemma 2.1 we get that (b, u) and (b, w) are blue. Then we get a blue C_4, a contradiction. Therefore, we may assume that (a, w) is a red edge. Next applying the Lemma 2.1, we can conclude that (b, u) is a blue edge. To avoid a blue C_4, (b, w) has to be red. Also there exists $c \in W'$ and without loss of generality, there is a vertex $d \in V$ such that $(c, d) \in (n-1)K_2$. Further, both (u, c), and (v, c) cannot be simultaneously blue. If (v, c) is a red edge, applying Lemma 2.1, we get (d, u) and (d, w) are blue. But then (d, u, v, w, d) will be a blue C_4, a contradiction. If (v, c) is a blue edge then (u, c) is a red edge. Then by Lemma 2.1, (d, w) has to be blue edge. Next, in order to avoid a blue C_4, (d, u) has to be a red edge. However, in this situation (b, c) has to be a blue edge in order to avoid a red nK_2 consisting of the red edges (a, w), (d, u) and (b, c). But then (b, c, v, u, b) will be a blue C_4, a contradiction.

Sub-case 1.2: $j \geq 4$
Since $j \geq 4$, select a red edge (a, b) of the red $(n-1)K_2$ such that $b \notin U \cup V \cup W'$ and either $a \in U$ or $a \notin U \cup V \cup W'$. a cannot be adjacent to both $\{v, w\}$ in blue as it would force a blue C_4. Therefore, without loss of generality we may assume that (a, v) is a red edge. Next applying the Lemma 2.1 we can conclude that (b, w) is a blue edge. By a similar argument we would get that (b, u) is also a blue edge. But then (u, b, w, v, u) will be a blue C_4, a contradiction.

Case 2: $u, v, w \in U$ where U is a partite set of $K_{j\times s}$.
Since $j \geq 3$, select an red edge (a, b) of the red $(n-1)K_2$ such that $a, b \notin U$. If both a and b are adjacent to exactly one vertex of $S = \{u, v, w\}$ in red then by Lemma 2.1 they will have to be adjacent to the same vertex of S(say u). This would force a and b to be adjacent to all the other vertices v and w in blue, resulting in a blue C_4. Therefore without loss of generality, b is adjacent to no vertices of S or b is adjacent to at least two vertices of S in red. In the first possibility, b will be forced to be adjacent to all the vertices u, v and w in blue and in the later possibility a will be forced to be adjacent to all the vertices u, v and w in blue. Using the same argument for another red edge (c, d) of the red $(n-1)K_2$ such that $c, d \notin U$, we will get u, v and w are adjacent to c or d in blue. Therefore, the vertices u, v will be adjacent to one vertex of $\{a, b\}$ and one vertex of $\{c, d\}$ in blue. This will yield a blue C_4 consisting of u, v, one vertex of $\{a, b\}$ and one vertex of $\{c, d\}$, a contradiction.

Case 3: $u, w \in U$ and $v \in V$ where U and V are two partite sets of $K_{j\times s}$. Since $j \geq 3$, select a red edge (a, b) of the red $(n-1)K_2$ such that $a \in V$ and $b \notin U \cup V$. If a is adjacent to a vertex of $\{u, w\}$ in red remove (a, b) from the red $(n-1)K_2$ and adding this edge will get a new red $(n-1)K_2$. Then the remaining three vertices along with the new red $(n-1)K_2$ will satisfy the conditions of Sub-case 1.1. Therefore, a is adjacent to the vertices u and w in blue. But then, (w, a, u, v, w) will be a blue C_4, a contradiction.

□

Theorem 4.1. $m_2(nK_2, C_4) = n+1$ *and if* $j \geq 3$*, then*

$$m_j(nK_2, C_4) = \begin{cases} 2 & \textit{if } n = 1 \textit{ and } j = 3 \\ \left\lceil \dfrac{2n+1}{j} \right\rceil & \textit{otherwise} \end{cases}$$

Proof. $m_2(nK_2, C_4) = n+1$ was proved in Lemma 2.5. Henceforth assume $j \geq 3$. We know that the theorem is true for $n \in \{1, 2\}$ since,

$$m_j(nK_2, C_4) = \begin{cases} 2 & \text{if } j = 3 \text{ and } n \in \{1,2\} \text{ or } j = 4, n = 2 \\ 1 & \text{otherwise} \end{cases}$$

Henceforth we will assume that $n \geq 3$.

To show that $m_j(nK_2, C_4) \geq \left\lceil \frac{2n+1}{j} \right\rceil$. Assume that $\left\lceil \frac{2n+1}{j} \right\rceil > 1$, and consider the coloring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{2n+1}{j} \right\rceil - 1$, generated by coloring all edges leaving a singleton vertex (say v) of $K_{j\times s}$ by blue and all the other edges by red. Then, the graph has no blue C_4, and $sj - 1 = \left(\left\lceil \frac{2n+1}{j} \right\rceil - 1 \right) j - 1 < 2n$. Also the vertex v is not adjacent to any vertices in red. Therefore, the graph contains no red nK_2. Hence, $m_j(nK_2, C_4) \geq \left\lceil \frac{2n+1}{j} \right\rceil$.

Assume $j \geq 3$ and $n \geq 3$. Now we are left to show $m_j(nK_2, C_4) \leq \left\lceil \frac{2n+1}{j} \right\rceil$. The theorem is true for $n = 2$.
Thus we would get $m_j(nK_2, C_4) \leq \left\lceil \frac{2n+1}{j} \right\rceil$ by applying induction on n, Lemma 4.1 along with $m_j(nK_2, C_4) \geq \left\lceil \frac{2n+1}{j} \right\rceil$. We conclude the proof.

□

5. Size Ramsey numbers related to stripes versus five cycles

Lemma 5.1. *Suppose $j \geq 4$ and $n \geq 3$. Given that $m_j((n-1)K_2, C_5) = \left\lceil \frac{2n-2}{j-1} \right\rceil$ it follows that $m_j(nK_2, C_5) \leq \left\lceil \frac{2n}{j-1} \right\rceil$.*

Proof. Let $j \geq 4$. Assume that $m_j((n-1)K_2, C_5) = \left\lceil \frac{2n-2}{j-1} \right\rceil$ is true. To prove $m_j(nK_2, C_5) \leq \left\lceil \frac{2n}{j-1} \right\rceil$, consider any red and blue, red nK_2 free coloring given by $K_{j \times s} = H_R \oplus H_B$ where $s = \left\lceil \frac{2n}{j-1} \right\rceil$. Assume $K_{j \times s}$ doesn't contain a blue C_5. Then the subgraph $K_{j \times s_0}$ where $s_0 = \left\lceil \frac{2n-2}{j-1} \right\rceil$ has no blue C_5, so it has a red $(n-1)K_2$. Let $W = V((n-1)K_2)$. Let W_i be the subgraph consisting of the disjoint edges of the matching $(n-1)K_2$ that belong to $G[V(K_{j \times s}) \setminus V_i]$; $1 \leq i \leq j$.

Since

$$s(j-1) - 2(n-1) \geq \left(\frac{2n}{j-1} \right)(j-1) - 2(n-1) \geq 2,$$

the graph contains two vertices in any $V(K_{(j-1) \times s}) \setminus V(W_i)$. Therefore, by the repeated use of previous statement we would get one of the following cases.

Case 1: $u, v \in U$, $w, y \in V$ where U and V are distinct partite sets of $K_{j \times s}$ and u, v, w, y in $G[V(K_{j \times s}) \setminus W]$.

Since $j \geq 4$, select an red edge (a, b) of the red $(n-1)K_2$ such that $a, b \notin U \cup V$. As H_B is C_5 free a is incident to at least two vertices of $\{u, v, w, y\}$ in red. By applying the Lemma 2.1 to H_B we get b is incident to all vertices of $\{u, v, w, y\}$. But then (b, y, u, w, v, b) will be a blue C_5, a contradiction.

Case 2: $u \in U$, $v \in V$ and $w \in W'$ where U, V and W' are distinct partite sets of $K_{j \times s}$ and u, v, w in $G[V(K_{j \times s}) \setminus W]$.

Claim 1: There exists a vertex s such that $\{u, v, w, s\}$ induces a blue B_2.
Proof of Claim 1: Since $j \geq 4$, select a red edge (a, b) of the red $(n-1)K_2$ such that $a \notin U \cup V \cup W'$ and $b \in U$. Suppose the claim is false. In order for $\{u, v, w, a\}$ not to induce a blue B_2, a must be adjacent in red to at least two vertices of $\{u, v, w\}$. In order for $\{u, v, w, b\}$ not to induce a blue B_2, b must be adjacent in red to all vertices of $\{v, w\}$, a contradiction by Lemma 2.1.

By the claim we can select a red edge (a, b) of the red $(n-1)K_2$ where $a \in U$, $b \notin W'$ such that without loss of generality $\{u, v, w, s, a, b\}$ will fall in to one of the possibilities as illustrated in the following diagram.

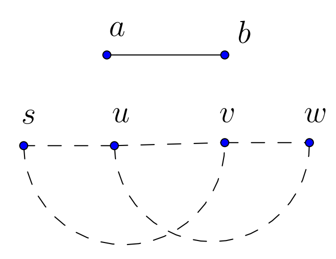

Figure 1. (Option 1 of Case 2)

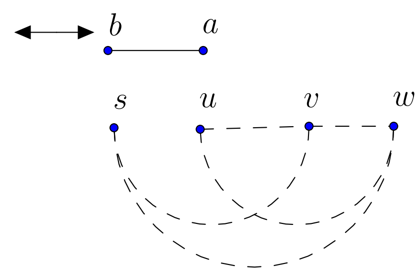

Figure 2. (Option 2 of Case 2)

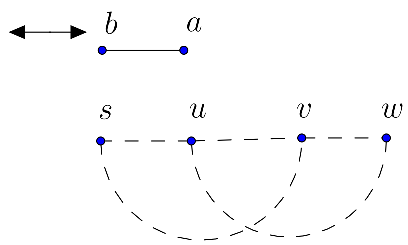

Figure 3. (Option 3 of Case 2)

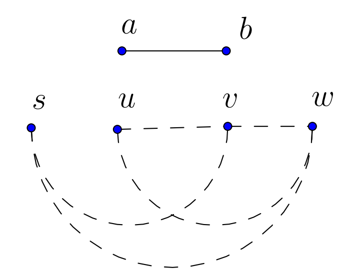

Figure 4. (Option 4 of Case 2)

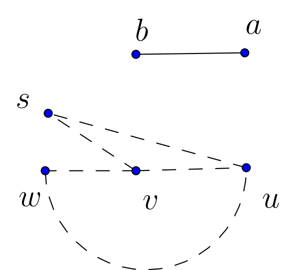

Figure 5. (Option 5 of Case 2)

But in all these five possibilities by Lemma 2.1 we would obtain a blue C_5 or a red nK_2. The details are left to the reader. $\square$

Theorem 5.1. *If $j \geq 2$, then*

$$m_j(nK_2, C_5) = \begin{cases} \infty & \text{if } j = 2 \\ 2 & \text{if } n = 1, j = 3 \text{ or } n = 1, j = 4 \text{ or } n = 2, j = 5 \\ \left\lceil \dfrac{2n}{j-1} \right\rceil & \text{otherwise} \end{cases}$$

Proof. Since a bipartite graph contains no odd cycle we get $m_2(nK_2, C_5) = \infty$. By Lemma 2.6, $m_3(nK_2, C_5) = n$ for $n \geq 2$ and $m_3(K_2, C_5) = 2$. Henceforth, assume that $j \geq 4$. Inspection shows us that the theorem is true for $n \in \{1, 2, 3\}$.

To show that $m_j(nK_2, C_5) \geq \left\lceil \dfrac{2n}{j-1} \right\rceil$, assume that $\left\lceil \dfrac{2n}{j-1} \right\rceil > 1$. Consider the coloring given by $K_{j \times s} = H_R \oplus H_B$, where $s = \left\lceil \dfrac{2n}{j-1} \right\rceil - 1$, generated by coloring all edges adjacent to one partite set of $K_{j \times s}$ by blue and all the other edges by red. Then, the graph has no blue C_5, and $s(j-1) = \left(\left\lceil \dfrac{2n}{j-1} \right\rceil - 1 \right)(j-1) < 2n$. Therefore, the graph also contains no red nK_2. Hence, $m_j(nK_2, C_5) \geq \left\lceil \dfrac{2n}{j-1} \right\rceil$.

Assume $j \geq 4$ and $n \geq 4$. We are left to show $m_j(nK_2, C_5) \leq \left\lceil \dfrac{2n}{j-1} \right\rceil$. The theorem is true for $n \in \{1, 2, 3\}$. Therefore, applying induction on n, using Lemma 5.1 along with $m_j(nK_2, C_5) \geq \left\lceil \dfrac{2n}{j-1} \right\rceil$, we get $m_j(nK_2, C_5) \leq \left\lceil \dfrac{2n}{j-1} \right\rceil$. Hence it completes the proof. $\square$

6. Size Ramsey numbers related to stripes versus six cycles

Lemma 6.1. *Suppose $j \geq 3$ and $n \geq 4$. Given that $m_j((n-1)K_2, C_6) = \left\lceil \dfrac{2n}{j} \right\rceil$ it follows that $m_j(nK_2, C_6) \leq \left\lceil \dfrac{2n+2}{j} \right\rceil$.*

Proof. Let $j \geq 3$ and $n \geq 4$. Assume that $m_j((n-1)K_2, C_6) = \left\lceil \dfrac{2n}{j} \right\rceil$ is true. To prove $m_j(nK_2, C_6) \leq \left\lceil \dfrac{2n+2}{j} \right\rceil$, consider any red-blue, red nK_2 free coloring given by $K_{j \times s} = H_R \oplus H_B$ where $s = \left\lceil \dfrac{2n+2}{j} \right\rceil$. Assume that the coloring is blue C_6 free. Then the subgraph $K_{j \times s_0}$

where $s_0 = \left\lceil \frac{2n}{j} \right\rceil$ has no blue C_6, so it has a red $(n-1)K_2$. Let $W = V((n-1)K_2)$. Since $sj - 2(n-1) \geq \left(\frac{2n+2}{j}\right) j - 2(n-1) \geq 4$, the graph contains four vertices u, v, w and x in $V(K_{j\times s}) \setminus W$. Let $S = \{u, v, w, x\}$.

Case 1: $u \in U$, $v \in V$, $w \in W'$ and $x \in X$ where U, V, W' and X are distinct partite sets of $K_{j\times s}$. Let $Y = U \cup V \cup W' \cup X$. Note that then all vertices of S are adjacent to each other in blue. Let (a, b) and (a_1, b_1) be edges in $(n-1)K_2$.

Claim 2: Both a and b must be adjacent to at least one vertex of S in red.
Proof of Claim 2: Suppose that a or b is not adjacent in red to any vertex of S. That is, that b or a is adjacent in blue to at least 3 vertices of S. But then by Lemma 2.1, as either a_1 or b_1 is adjacent to two vertices of S in blue; so we will obtain a blue C_6.

By the above Claim 2 and Lemma 2.1, a and b must be adjacent in red to the same vertex say w in S. Since H_R has no red nK_2 a and b cannot be adjacent in red to any vertex of S other than w. But this will force a blue C_6, a contradiction.

Case 2: $u, v, w \in U$ and $x \in V$ where U and V are partite sets of $K_{j\times s}$.
Since $j \geq 3$, select two red edges (a_1, b_1) and (a_2, b_2) of the red $(n-1)K_2$ such that $a_1, a_2 \notin U \cup V$ and $b_1, b_2 \notin U$.

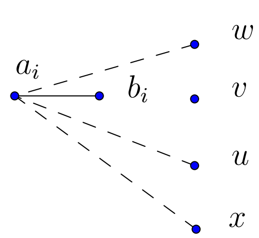

Figure 6. (Option 1 of Case 2)

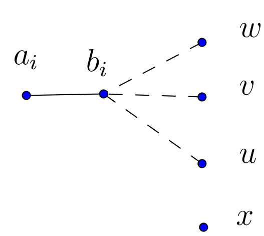

Figure 7. (Option 2 of Case 2)

If b_i; $i \in \{1, 2\}$ is adjacent in red to at least one vertex of S (say including v), then by Lemma 2.1, a_i is adjacent in blue to three vertices of S (see figure 6). If b_i; $i \in \{1, 2\}$ is adjacent in red to no vertices of S, then obviously b_i is adjacent in blue to three vertices of S (see figure 7). Therefore, for $i \in \{1, 2\}$, either one of a_i or b_i will be adjacent in blue to three vertices of S. Inspection will show that this will force a blue C_6 except when b_1 and b_2 are in V and if they are adjacent to the same vertex of $\{u, v, w\}$ (say u) in red. But even in this situation will force a blue C_6, a contradiction.

Case 3: $u, x \in U$, $v \in V$ and $w \in W'$ where U, V and W' are three partite sets of $K_{j\times s}$.
Since $j \geq 3$, by a counting argument we see that there are two red edges (a_1, b_1) and (a_2, b_2) of the red $(n-1)K_2$ such that $a_1, a_2, b_1, b_2 \notin U$.

Then by Lemma 2.1, for $i \in \{1, 2\}$, either one of a_i or b_i is adjacent in blue to three vertices of S. This results in a blue C_6, a contradiction.

Case 4: $u, w \in U$ and $v, x \in V$ where U and V are two partite sets of $K_{j\times s}$. First assume that $j = 3$, then we can select a red edges (a, b) and (a_1, b_1) of the red $(n-1)K_2$ such that $b, b_1 \notin U \cup V$, $a \notin U$ and $a_1 \notin V$. If either one of (a, u), (a, w), (a_1, v) or (a_1, x) is a red edge then by Lemma 2.1 we would obtain a blue C_6, a contradiction. However, in the remaining possibility, if all the the edges (a, u), (a, w), (a_1, v) or (a_1, x) are blue edge then we would get that $(a_1, v, w, a, u, x, a_1)$ is a blue C_6.

If $j \geq 4$, we can select a red edge (a, b) of the red $(n - 1)K_2$ such that $a, b \notin U \cup V$. Without loss of generality we may assume that (a, u) is a red edge. But then by Lemma 2.1, (b, v), (b, w) and (b, x) will be forced to be blue edges. Next applying a similar argument, to another red edge (a_1, b_1) of the red $(n - 1)K_2$, we get that either a_1 or b_1 must be adjacent to three vertices of $\{u, v, w, x\}$ in blue, forcing a blue C_6. □

Theorem 6.1. *$m_2(nK_2, C_6) = n + 2$ and if $j \geq 3$, then*

$$m_j(nK_2, C_6) = \begin{cases} 2 & \textit{if } n = 1, j = 4 \textit{ if } n = 1, j = 5 \textit{ if } n = 2, j = 6 \\ \left\lceil \dfrac{2n+2}{j} \right\rceil & \textit{otherwise} \end{cases}$$

Proof. $m_2(nK_2, C_6) = n + 2$ was proved in Lemma 2.8. Henceforth assume $j \geq 3$. Clearly

$$m_j(K_2, C_6) = \begin{cases} 2 & \text{if } j \in \{3, 4, 5\} \\ 1 & \text{otherwise} \end{cases}$$

and

$$m_j(2K_2, C_6) = \begin{cases} 2 & \text{if } j \in \{3, 4, 5, 6\} \\ 1 & \text{otherwise} \end{cases}$$

and

$$m_j(3K_2, C_6) = \begin{cases} 2 & \text{if } j \in \{3, 4, 5, 6, 7\} \\ 1 & \text{otherwise} \end{cases}$$

Henceforth, assume that $n \geq 4$.

First to show that $m_j(nK_2, C_6) \geq \left\lceil \frac{2n+2}{j} \right\rceil$ assume $\left\lceil \frac{2n+2}{j} \right\rceil > 1$. Consider the coloring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{2n+2}{j} \right\rceil - 1$, generated by coloring all edges incident to the two vertices (say u, v) of $K_{j\times s}$ by blue and all the other edges by red. Then, the graph has no blue C_6, and $sj - 2 = \left(\left\lceil \frac{2n+2}{j} \right\rceil - 1 \right) j - 2 < 2n$. Also the vertices u and v are not adjacent to any vertices in red. Therefore, the graph contains no red nK_2.

We are left to show that $m_j(nK_2, C_6) \leq \left\lceil \frac{2n+2}{j} \right\rceil$ for $n \geq 4$. This result is true for $n = 3$. Therefore, we would get the result by applying induction on n (for $n \geq 4$) and using Lemma 6.1 along with $m_j(nK_2, C_6) \geq \left\lceil \frac{2n+2}{j} \right\rceil$. Hence the theorem follows.

□

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