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Modular irregularity strength on some flower graphs

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Abstract

Let $G = (V(G), E(G))$ be a graph with the nonempty vertex set $V(G)$ and the edge set $E(G)$. Let $\mathbb{Z}_n$ be the group of integers modulo n and let k be a positive integer. A modular irregular labeling of a graph G of order n is an edge k-labeling $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$, such that the induced weight function $\sigma : V(G) \rightarrow \mathbb{Z}_n$ defined by $\sigma(v) = \sum_{u \in N(v)} \varphi(uv) \pmod n$ for every vertex $v \in V(G)$ is bijective. The minimum number k such that a graph G has a modular irregular k-labeling is called the modular irregularity strength of a graph G, denoted by $ms(G)$. In this paper, we determine the exact values of the modular irregularity strength of some families of flower graphs, namely rose graphs, daisy graphs and sunflower graphs.

Keywords: modular irregular labeling, modular irregularity strength, daisy graphs, rose graphs, sunflower graphs
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1. Introduction

Let $G = (V(G), E(G))$ be a graph with the nonempty vertex set $V(G)$ and the edge set $E(G)$. In this paper all graphs are simple of order n. The degree of a vertex v is denoted by $\deg(v)$ and we shall denote the maximum degree of G by $\Delta(G)$. A graph is called regular if degree of each vertex is equal, while a graph with the different degree for all vertices is called an irregular graph. A biregular graph is a graph having vertices either of degree d_1 or of degree d_2. In this paper we consider two families of biregular graphs.

A mapping which sends a set of graph elements to a set of numbers is called a labeling. In 1988, Chartrand et al. in [7] defined an *irregular labeling* as an edge k-labeling $\varphi : E(G) \to \{1, 2, \dots, k\}$, where k is a positive integer, such that the vertex weights are different for all vertices. The function $w(v) = \sum_{u \in N(v)} \varphi(uv)$ is called the weight of a vertex $v \in V(G)$ and $N(v)$ stands for the neighborhood of the vertex v. The minimum number k such that a graph G has an irregular k-labeling is called the *irregularity strength* of a graph G and is denoted by $s(G)$. This graph invariant has attracted much attention see [1, 2, 6, 8, 9, 11, 12]. People alsi study the variation on irregular labeling, as an example total irregular labeling where the label is definied for vertices and edges, see [10, 13] as examples.

In 2020, Bača et al. [5] introduced a variation of an irregular labeling which is called a modular irregular labeling. Let G be a graph of order n and let $\mathbb{Z}_n$ be a group of integers modulo n. A *modular irregular labeling* of a graph G is an edge k-labeling $\varphi : E(G) \to \{1, 2, \dots, k\}$ such that the induced function $\sigma : V(G) \to \mathbb{Z}_n$ defined for every vertex $v \in V(G)$ by $\sigma(v) = \sum_{u \in N(v)} \varphi(uv) \pmod n$ is bijective. The sum $\sigma(v) = wt_{\varphi}(v)$ is called the *modular weight* of the vertex v. The minimum number k such that a graph G has modular irregular k-labeling is called the *modular irregularity strength* of a graph G, denoted by $ms(G)$. If there exists no such labeling for the graph G then we put $\mathrm{ms}(G) = \infty$.

For some classes of graphs their modular irregularity strength is known. Bača et al. in [5] obtained the precise values of this graph invariant of paths P_n for $n \geq 3$, stars $K_{1,n}$ for $n \geq 2$, triangular graphs T_n for $n \geq 3$, cycles C_n for $n \geq 3$, and gear graphs G_n for $n \geq 3$. In 2021, Bača et al. [4] characterized the modular irregularity strength of fan graphs F_n for $n \geq 2$. In the same year, Bača et al. in [3] found the modular irregularity strength of wheels W_n. Sugeng et al. in [14] provided the results for the modular irregularity strength of double-stars $S_{k,k}$ and friendship graphs f_n.

2. Known Results

Chartrand et al. in [7] proved the following lemma.

Lemma 2.1. [7] *Let G be a connected graph of order $n \geq 3$ and let G have n_i vertices with degree i, then*

$$s(G) \geq \max_{1 \leq i \leq \Delta(G)} \left\{ \frac{n_i - 1}{i} + 1 \right\}.$$

Bača et al. in [5] stated the following theorem.

Theorem 2.1. [5] *Let G be a graph with no component which has order ≤ 2. Then*

$$s(G) \leq ms(G).$$

Theorem 2.2. [5] *If G is a graph of order n, $n \equiv 2 \pmod 4$, then G has no modular irregular k-labeling, i.e., $ms(G) = \infty$.*

3. New Results

In this section, we determine the modular irregularity strength of three families of flower graphs, which are the rose graphs (also known as the middle graph of cycles), daisy graphs and sunflower graphs.

Rose Graphs

The rose graph is also known as a middle graph of a cycle. The *middle graph* $M(G)$ of a connected graph G is a graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if either they are adjacent edges in G or one is a vertex in G and the other is an edge in G incident with it. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n, $n \geq 3$ and let the n edges of C_n be $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. Thus the *rose graph* $M(C_n)$ can be constructed from the cycle C_n with vertices $v_1, v_2, \ldots, v_n$ and n isolated vertices $w_1, w_2, \ldots, w_n$ and then connecting every two vertices v_i, v_{i+1} with w_i, for $i = 1, 2, \ldots, n$ where $v_{n+1} = v_1$. Thus $M(C_n)$ contains n vertices of degree 2 and n vertices of degree 4. Thus according to Lemma 2.1 and Theorem 2.1 we have that

$$ms(M(C_n)) \geq s(M(C_n)) \geq \max\left\{\tfrac{n-1}{2} + 1, \tfrac{n-1}{4} + 1\right\} = \tfrac{n+1}{2}.$$

As the irregularity strength is an integer we get

$$ms(M(C_n)) \geq s(M(C_n)) \geq \left\lceil \tfrac{n+1}{2} \right\rceil. \tag{1}$$

Theorem 3.1. *Let $M(C_n)$ be a rose graph with $n \geq 3$, then*

$$ms(M(C_n)) = \begin{cases} \frac{n}{2} + 1, & \text{when } n \text{ is even,} \\ \infty, & \text{when } n \text{ is odd.} \end{cases}$$

Proof. We consider two cases according to the parity of n.

Case 1. When n is odd then $|V(M(C_n))| = 2n \equiv 2 \pmod 4$. Thus, following Theorem 2.2, the graph does not have a modular irregular labeling.

Case 2. When n is even we label the edges of $M(C_n)$ as follows.

$$\begin{aligned} \varphi_1(v_iv_{i+1}) &= \tfrac{n}{2}, && \text{for } i = 1, 2, \ldots, n-1, \\ \varphi_1(v_1v_n) &= \tfrac{n}{2}, \\ \varphi_1(v_iw_i) &= \begin{cases} i, & \text{for } i = 1, 2, \ldots, \frac{n}{2} + 1, \\ n + 2 - i, & \text{for } i = \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n, \end{cases} \end{aligned}$$

$$\varphi_1(v_{i+1}w_i) = \begin{cases} i, & \text{for } i = 1, 2, \dots, \frac{n}{2}, \\ n+1-i, & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2, \dots, n-1, \end{cases}$$
$$\varphi(v_1w_n) = 1.$$

It is easy to see that the maximal edge label is $\frac{n}{2}+1$. Thus φ_1 is an $(\frac{n}{2}+1)$-labeling of $M(C_n)$.

Now we check the induced modular weights of the vertices in $M(C_n)$. For the vertices v_i, $i = 1, 2, \dots, n$ we have the following:

$$wt_{\varphi_1}(v_1) = \varphi_1(v_1v_2) + \varphi_1(v_1v_n) + \varphi_1(v_1w_1) + \varphi_1(v_1w_n) = \tfrac{n}{2} + \tfrac{n}{2} + 1 + 1 = n+2,$$

for $i = 2, 3, \dots, \frac{n}{2}+1$

$$wt_{\varphi_1}(v_i) = \varphi_1(v_iv_{i+1}) + \varphi_1(v_{i-1}v_i) + \varphi_1(v_iw_i) + \varphi_1(v_iw_{i-1}) = \tfrac{n}{2} + \tfrac{n}{2} + i + (i-1)$$
$$= n + 2i - 1,$$

i.e., the corresponding modular weights are $1, n+3, n+5, \dots, 2n-1$,

$$wt_{\varphi_1}(v_{\frac{n}{2}+2}) = \varphi_1(v_{\frac{n}{2}+2}v_{\frac{n}{2}+3}) + \varphi_1(v_{\frac{n}{2}+1}v_{\frac{n}{2}+2}) + \varphi_1(v_{\frac{n}{2}+2}w_{\frac{n}{2}+2}) + \varphi_1(v_{\frac{n}{2}+2}w_{\frac{n}{2}+1})$$
$$= \tfrac{n}{2} + \tfrac{n}{2} + \tfrac{n}{2} + \tfrac{n}{2} = 2n \equiv 0 \pmod{2n},$$

for $i = \frac{n}{2}+3, \frac{n}{2}+4, \dots, n-1$

$$wt_{\varphi_1}(v_i) = \varphi_1(v_iv_{i+1}) + \varphi_1(v_{i-1}v_i) + \varphi_1(v_iw_i) + \varphi_1(v_iw_{i-1}) = \tfrac{n}{2} + \tfrac{n}{2} + (n+2-i)$$
$$+ (n+2-i) = 3n + 4 - 2i,$$

i.e., the corresponding modular weights are $n+6, n+8, \dots, 2n-2$,

$$wt_{\varphi_1}(v_n) = \varphi_1(v_1v_n) + \varphi_1(v_{n-1}v_n) + \varphi_1(v_nw_n) + \varphi_1(v_nw_{n-1}) = \tfrac{n}{2} + \tfrac{n}{2} + 2 + 2 = n+4.$$

Now we check the weights of the vertices w_i, $i = 1, 2, \dots, n$.

For $i = 1, 2, \dots, \frac{n}{2}$

$$wt_{\varphi_1}(w_i) = \varphi_1(v_iw_i) + \varphi_1(v_{i+1}w_i) = i + i = 2i,$$

i.e., the corresponding modular weights are $2, 4, \dots, n$,

$$wt_{\varphi_1}(w_{\frac{n}{2}+1}) = \varphi_1(v_{\frac{n}{2}+1}w_{\frac{n}{2}+1}) + \varphi_1(v_{\frac{n}{2}+2}w_{\frac{n}{2}+1}) = (\tfrac{n}{2}+1) + \tfrac{n}{2} = n+1,$$

for $i = \frac{n}{2}+2, \frac{n}{2}+3, \dots, n-1$,

$$wt_{\varphi_1}(w_i) = \varphi_1(v_iw_i) + \varphi_1(v_{i+1}w_i) = (n+2-i) + (n+1-i) = 2n+3-2i,$$

i.e., the corresponding modular weights are $5, 7, \dots, n-1$,

$$wt_{\varphi_1}(w_n) = \varphi_1(v_nw_n) + \varphi_1(v_1w_n) = 2 + 1 = 3.$$

Combining the previous we get that the modular weights of all vertices constitute the set $\{0, 1, \dots, 2n-1\}$. Thus according to (1) we have $ms(M(C_n)) = \frac{n}{2}+1$ for n even, $n \geq 4$. □

Figure 1 illustrates a modular irregular 5-labeling of $M(C_8)$.

Immediately from the previous theorem we get the result for the irregularity strength of the rose graph $M(C_n)$ for even n.

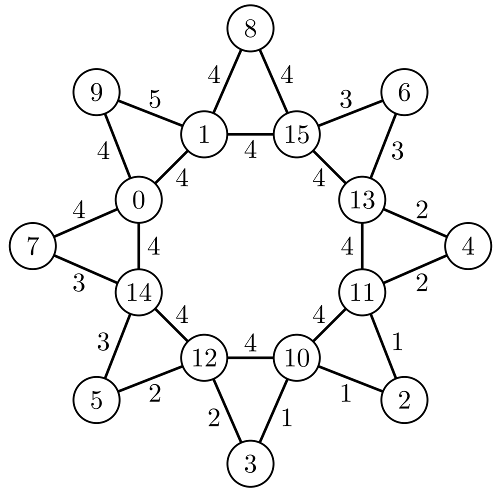

Figure 1. A modular irregular 5-labeling of $M(C_8)$.

Corollary 3.1. *Let $M(C_n)$ be a rose graph with n even, $n \geq 4$, then*

$$s(M(C_n)) = \tfrac{n}{2} + 1.$$

Daisy Graphs

The *daisy graph* DK_n is constructed from the complete graph K_n with vertices $v_1, v_2, \dots, v_n$ and n isolated vertices $w_1, w_2, \dots, w_n$ such that the vertices v_i, v_{i+1} are connected to w_i for $i = 1, 2, \dots, n$ where $v_{n+1} = v_1$. A daisy graph DK_n has n vertex of degree 2 and n vertices of degree $n+1$. Based on Lemma 2.1 and Theorem 2.1 we have that

$$ms(DK_n) \geq s(DK_n) \geq \max\left\{\tfrac{n-1}{2} + 1, \tfrac{n-1}{n+1} + 1\right\} = \tfrac{n+1}{2}.$$

Since $s(DK_n)$ is an integer so

$$ms(DK_n) \geq s(DK_n) \geq \left\lceil \tfrac{n+1}{2} \right\rceil. \quad (2)$$

Theorem 3.2. *Let DK_n be a daisy graph with $n \geq 3$, then*

$$ms(DK_n) = \begin{cases} \frac{n}{2} + 1, & \text{when } n \text{ is even,} \\ \infty, & \text{when } n \text{ is odd.} \end{cases}$$

Proof. We again distinguish two cases, when n is odd and when n is even.

Case 1. When n is odd then $|V(DK_n)| = 2n \equiv 2 \pmod 4$. By Theorem 2.2 we obtain that there does not exist a modular irregular labeling of DK_n in this case.

Case 2. When n is even we define a suitable edge labeling φ_2 of DK_n in the following way.

$$\varphi_2(v_i v_{i+1}) = \begin{cases} 1, & \text{for } i = 1, 3, \dots, n-1, \\ 2, & \text{for } i = 2, 4, \dots, n-2, \end{cases}$$

$$\begin{aligned}
\varphi_2(v_1v_n) &= 2, \\
\varphi_2(v_1v_j) &= 1, && \text{for } j = 3, 4, \dots, n-1, \\
\varphi_2(v_iv_j) &= 1, && \text{for } i = 2, 3, \dots, n-2 \text{ and } i+2 \le j \le n, \\
\varphi_2(v_iw_i) &= \begin{cases} i, & \text{for } i = 1, 2, \dots, \frac{n}{2}+1, \\ n+2-i, & \text{for } i = \frac{n}{2}+2, \frac{n}{2}+3, \dots, n, \end{cases} \\
\varphi_2(v_{i+1}w_i) &= \begin{cases} i, & \text{for } i = 1, 2, \dots, \frac{n}{2}, \\ n+1-i, & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2, \dots, n-1, \end{cases} \\
\varphi_2(v_1w_n) &= 1.
\end{aligned}$$

Evidently the labeling φ_2 is an $(\frac{n}{2}+1)$-labeling.

Now we evaluate the modular vertex weights under the labeling φ_2. We use the fact every vertex v_i, $i = 1, 2, \dots, n$, is in the subgraph K_n of DK_n incident with one edge labeled by 2 and with $n-2$ edges labeled by 1. Thus

$$wt_{\varphi_2}(v_1) = \sum_{j\neq 1} \varphi_2(v_1v_j) + \varphi_2(v_1w_1) + \varphi_2(v_1w_n) = n+1+1 = n+2,$$

for $i = 2, 3, \dots, \frac{n}{2}+1$

$$wt_{\varphi_2}(v_i) = \sum_{j\neq i} \varphi_2(v_iv_j) + \varphi_2(v_iw_i) + \varphi_2(v_iw_{i-1}) = n+i+(i-1) = n+2i-1,$$

i.e., the corresponding modular weights are $1, n+3, n+5, \dots, 2n-1$,

$$\begin{aligned}
wt_{\varphi_2}(v_{\frac{n}{2}+2}) &= \sum_{j\neq \frac{n}{2}+2} \varphi_2(v_{\frac{n}{2}+2}v_j) + \varphi_2(v_{\frac{n}{2}+2}w_{\frac{n}{2}+2}) + \varphi_2(v_{\frac{n}{2}+2}w_{\frac{n}{2}+1}) = n + \tfrac{n}{2} + \tfrac{n}{2} = 2n \\
&\equiv 0 \pmod{2n},
\end{aligned}$$

for $i = \frac{n}{2}+3, \frac{n}{2}+4, \dots, n-1$

$$\begin{aligned}
wt_{\varphi_2}(v_i) &= \sum_{j\neq i} \varphi_2(v_iv_j) + \varphi_2(v_iw_i) + \varphi_2(v_iw_{i-1}) = n + (n+2-i) + (n+2-i) \\
&= 3n+4-2i,
\end{aligned}$$

i.e., the corresponding modular weights are $n+6, n+8, \dots, 2n-2$,

$$wt_{\varphi_2}(v_n) = \sum_{j\neq n} \varphi_2(v_nv_j) + \varphi_2(v_nw_n) + \varphi_2(v_nw_{n-1}) = n+2+2 = n+4,$$

for $i = 1, 2, \dots, \frac{n}{2}$

$$wt_{\varphi_2}(w_i) = \varphi_2(v_iw_i) + \varphi_2(v_{i+1}w_i) = i + i = 2i,$$

i.e., the corresponding modular weights are $2, 4, \dots, n$,

$$wt_{\varphi_2}(w_{\frac{n}{2}+1}) = \varphi_2(v_{\frac{n}{2}+1}w_{\frac{n}{2}+1}) + \varphi_2(v_{\frac{n}{2}+2}w_{\frac{n}{2}+1}) = (\tfrac{n}{2}+1) + \tfrac{n}{2} = n+1,$$

for $i = \frac{n}{2}+2, \frac{n}{2}+3, \dots, n-1$

$$wt_{\varphi_2}(w_i) = \varphi_2(v_iw_i) + \varphi_2(v_{i+1}w_i) = (n+2-i) + (n+1-i) = 2n+3-2i,$$

i.e., the corresponding modular weights are $5, 7, \dots, n-1$,

$wt_{\varphi_2}(w_n) = \varphi_2(v_n w_n) + \varphi_2(v_1 w_n) = 2 + 1 = 3.$

Therefore, the modular weights of vertices constitute the set $\{0, 1, \dots, 2n-1\}$. By combining this result with (2) we conclude that $ms(DK_n) = \frac{n}{2} + 1$ for even $n \geq 4$. □

Figure 2 gives an example of a modular irregular 4-labeling of DK_6.

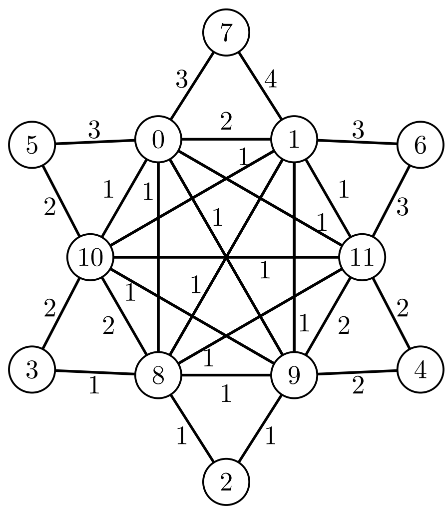

Figure 2. A modular irregular 4-labeling of DK_6.

From Theorem 3.2 we have also the following result for the irregularity strength of DK_n.

Corollary 3.2. *Let DK_n be a daisy graph with n even, $n \geq 4$, then*

$$s(DK_n) = \frac{n}{2} + 1.$$

Sunflower Graphs

A *wheel* W_n, $n \geq 3$, is a graph obtained by joining all vertices of a cycle on n vertices to a further vertex c. Let us denote the vertices of degree 3 in W_n by $v_1, v_2, \dots, v_n$ such that the edges of W_n are $cv_1, cv_2, \dots, cv_n, v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ and v_1v_n. A *sunflower graph* Sf_n is a graph constructed from a wheel W_n and n additional vertices $w_1, w_2, \dots, w_n$ where w_i is adjacent to v_i and v_{i+1}, $i = 1, 2, \dots, n$ with $v_{n+1} = v_1$. A sunflower graph Sf_n has one vertex of degree n, n vertices of degree 5 and n vertices of degree 2. According to Lemma 2.1 and Theorem 2.1 we obtain the following lower bound

$$ms(Sf_n) \geq s(Sf_n) \geq \left\lceil \frac{n+1}{2} \right\rceil. \tag{3}$$

Theorem 3.3. *Let Sf_n be a sunflower graph, $n \geq 3$, then*

$$ms(Sf_n) = \begin{cases} 3, & \textit{when } n = 3, \\ 4, & \textit{when } n = 5, \\ \left\lceil \frac{n+1}{2} \right\rceil, & \textit{otherwise}. \end{cases}$$

Proof. The proof consists of five cases.

Case 1. By (3) we get that $ms(Sf_3) \geq 2$. By a contradiction we prove that $ms(Sf_3) \neq 2$. Assume that Sf_3 admits a modular irregular 2-labeling. Then as Sf_3 contains three vertices of degree 2 their weights must be 2, 3 and 4. On the other side the maximal vertex weight is at most 10 and can be obtained only on a vertex of degree 5. Because Sf_3 has 7 vertices we get that the set of the vertex weights must consists of numbers $2, 3, \ldots, 8$, i.e., the modular vertex weights are $0, 1, \ldots, 6$. As the sum of the weights $2+3+\cdots+8 = 35$ we get a contradiction because the sum of all vertex weights must be even as every edge label contributes twice to the vertex weights.

Case 2. According to (3) we have that $ms(Sf_5) \geq 3$. Assume that Sf_5 has a modular irregular 3-labeling. Under this labeling the five vertices of degree 2 must have weights $2, 3, 4, 5$ and 6. The other vertices have degree 5 and thus the maximal vertex weight is at most 15 and can be obtained as the sum of five labels 3. The graph Sf_5 is of order 11. Thus the set of the vertex weights must be $\{2, 3, \ldots, 12\}$, i.e., the modular weights are $0, 1, \ldots, 10$. A contradiction as the sum of the vertex weights is 77.

Case 3. When $n = 7$ we consider the modular irregular 4-labeling of Sf_7 illustrated on Figure 3.

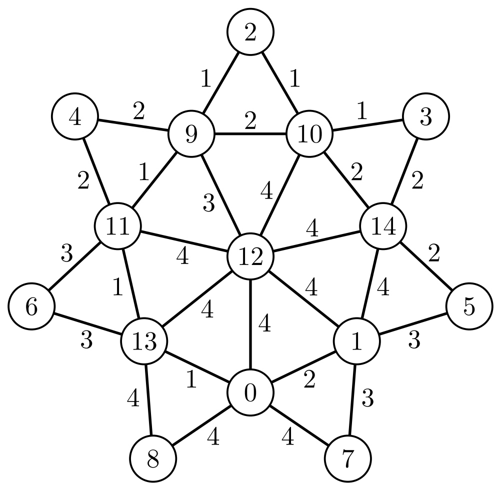

Figure 3. A modular irregular 4-labeling of Sf_7.

Case 4. When n is even, $n \geq 4$, we define an edge labeling φ_3 of Sf_n such that

$$\varphi_3(cv_i) = 2, \qquad \text{for } i = 1, 2, \ldots, n,$$

$$\varphi_3(v_i v_{i+1}) = \begin{cases} \frac{n}{2}, & \text{for } i = 1, \\ \frac{n}{2} - 1, & \text{for } i = 2, 3, \ldots, n, \end{cases}$$

$$\varphi_3(v_{i+1} w_i) = \begin{cases} \frac{n}{2} + 1, & \text{for } i = 1, \\ \frac{n}{2} + 2 - i, & \text{for } i = 2, 3, \ldots, \frac{n}{2} + 1, \\ i - \frac{n}{2}, & \text{for } i = \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n, \end{cases}$$

$$\varphi_3(v_iw_i) = \begin{cases} \frac{n}{2}, & \text{for } i = 1, \\ \frac{n}{2} + 2 - i, & \text{for } i = 2, 3, \dots, \frac{n}{2} + 1, \\ i - \frac{n}{2} - 1, & \text{for } i = \frac{n}{2} + 2, \frac{n}{2} + 3, \dots, n. \end{cases}$$

As the maximal edge label is $\frac{n}{2} + 1$ we get that φ_3 is an $(\frac{n}{2} + 1)$-labeling. The corresponding modular edge weights are the following.

$$\begin{aligned} wt_{\varphi_3}(v_1) &= \varphi_3(cv_1) + \varphi_3(v_nv_1) + \varphi_3(v_1v_2) + \varphi_3(w_nv_1) + \varphi_3(w_1v_1) = 2 + (\tfrac{n}{2} - 1) + \tfrac{n}{2} + \tfrac{n}{2} \\ &\quad + \tfrac{n}{2} = 2n + 1 \equiv 0 \pmod{(2n+1)}, \\ wt_{\varphi_3}(v_2) &= \varphi_3(cv_2) + \varphi_3(v_1v_2) + \varphi_3(v_2v_3) + \varphi_3(w_1v_2) + \varphi_3(w_2v_2) = 2 + \tfrac{n}{2} + (\tfrac{n}{2} - 1) \\ &\quad + (\tfrac{n}{2} + 1) + \tfrac{n}{2} = 2n + 2 \equiv 1 \pmod{(2n+1)}, \end{aligned}$$

for $i = 3, 4, \dots, \frac{n}{2} + 1$

$$\begin{aligned} wt_{\varphi_3}(v_i) &= \varphi_3(cv_i) + \varphi_3(v_{i-1}v_i) + \varphi_3(v_iv_{i+1}) + \varphi_3(w_{i-1}v_i) + \varphi_3(w_iv_i) = 2 + (\tfrac{n}{2} - 1) \\ &\quad + (\tfrac{n}{2} - 1) + (\tfrac{n}{2} + 3 - i) + (\tfrac{n}{2} + 2 - i) = 2n + 5 - 2i, \end{aligned}$$

i.e., the corresponding modular weights are $n + 3, n + 5, \dots, 2n - 1$,

$$\begin{aligned} wt_{\varphi_2}(v_{\frac{n}{2}+2}) &= \varphi_3(cv_{\frac{n}{2}+2}) + \varphi_3(v_{\frac{n}{2}+1}v_{\frac{n}{2}+2}) + \varphi_3(v_{\frac{n}{2}+2}v_{\frac{n}{2}+3}) + \varphi_3(w_{\frac{n}{2}+1}v_{\frac{n}{2}+2}) \\ &\quad + \varphi_3(w_{\frac{n}{2}+2}v_{\frac{n}{2}+2}) = 2 + (\tfrac{n}{2} - 1) + (\tfrac{n}{2} - 1) + 1 + 1 = n + 2, \end{aligned}$$

for $i = \frac{n}{2} + 3, \frac{n}{2} + 4, \dots, n$

$$\begin{aligned} wt_{\varphi_3}(v_i) &= \varphi_3(cv_i) + \varphi_3(v_{i-1}v_i) + \varphi_3(v_iv_{i+1}) + \varphi_3(w_{i-1}v_i) + \varphi_3(w_iv_i) = 2 + (\tfrac{n}{2} - 1) \\ &\quad + (\tfrac{n}{2} - 1) + (i - 1 - \tfrac{n}{2}) + (i - \tfrac{n}{2} - 1) = 2i - 2, \end{aligned}$$

i.e., the corresponding modular weights are $n + 4, n + 6, \dots, 2n - 2$,

$$wt_{\varphi_3}(w_1) = \varphi_3(v_2w_1) + \varphi_3(v_1w_1) = (\tfrac{n}{2} + 1) + \tfrac{n}{2} = n + 1,$$

for $i = 2, 3, \dots, \frac{n}{2} + 1$

$$wt_{\varphi_3}(w_i) = \varphi_3(v_{i+1}w_i) + \varphi_3(v_iw_i) = (\tfrac{n}{2} + 2 - i) + (\tfrac{n}{2} + 2 - i) = n + 4 - 2i,$$

i.e., the corresponding modular weights are $2, 4, \dots, n$,

for $i = \frac{n}{2} + 2, \frac{n}{2} + 3, \dots, n$

$$wt_{\varphi_3}(w_i) = \varphi_3(v_{i+1}w_i) + \varphi_3(v_iw_i) = (i - \tfrac{n}{2}) + (i - \tfrac{n}{2} - 1) = 2i - n - 1,$$

i.e., the corresponding modular weights are $3, 5, \dots, n - 1$,

$$wt_{\varphi_3}(c) = \sum_{i=1}^{n} \varphi_3(cv_i) = 2n.$$

Thus the modular vertex weights are $0, 1, \dots, 2n$ which means that for even $n \geq 4$ the labeling φ_3 is a modular irregular $(\frac{n}{2} + 1)$-labeling of Sf_n.

Case 5. When n is odd, $n \geq 9$, we consider the following labeling φ_4 of Sf_n

$$\varphi_4(cv_i) = \begin{cases} 5, & \text{for } i = 1, 2, \frac{n+1}{2}, \\ 4, & \text{otherwise}, \end{cases}$$

$$\varphi_4(v_iv_{i+1}) = \begin{cases} \frac{n-5}{2}, & \text{for } i = 1 \text{ and } i = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n, \\ \frac{n-3}{2}, & \text{for } i = 2, 3, \dots, \frac{n-1}{2}, \end{cases}$$

$$\varphi_4(v_{i+1}w_i) = \begin{cases} \frac{n+1}{2} - i, & \text{for } i = 1, 2, \dots, \frac{n-1}{2}, \\ i - \frac{n-1}{2}, & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n, \end{cases}$$

$$\varphi_4(v_iw_i) = \begin{cases} \frac{n+3}{2} - i, & \text{for } i = 1, 2, \dots, \frac{n-1}{2}, \\ i - \frac{n-1}{2}, & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n. \end{cases}$$

Evidently φ_4 is an $\frac{n+1}{2}$-labeling. Now we evaluate the modular edge weights induced by the labeling φ_4.

$$\begin{aligned} wt_{\varphi_4}(v_1) &= \varphi_4(cv_1) + \varphi_4(v_nv_1) + \varphi_4(v_1v_2) + \varphi_4(w_nv_1) + \varphi_4(w_1v_1) = 5 + \tfrac{n-5}{2} + \tfrac{n-5}{2} + \tfrac{n+1}{2} \\ &\quad + \tfrac{n+1}{2} = 2n + 1 \equiv 0 \pmod{(2n+1)}, \\ wt_{\varphi_4}(v_2) &= \varphi_4(cv_2) + \varphi_4(v_1v_2) + \varphi_4(v_2v_3) + \varphi_4(w_1v_2) + \varphi_4(w_2v_2) = 5 + \tfrac{n-5}{2} + \tfrac{n-3}{2} \\ &\quad + \tfrac{n-1}{2} + \tfrac{n-1}{2} = 2n, \end{aligned}$$

for $i = 3, 4, \dots, \frac{n-1}{2}$

$$\begin{aligned} wt_{\varphi_4}(v_i) &= \varphi_4(cv_i) + \varphi_4(v_{i-1}v_i) + \varphi_4(v_iv_{i+1}) + \varphi_4(w_{i-1}v_i) + \varphi_4(w_iv_i) = 4 + \tfrac{n-3}{2} \\ &\quad + \tfrac{n-3}{2} + (\tfrac{n+3}{2} - i) + (\tfrac{n+3}{2} - i) = 2n + 4 - 2i, \end{aligned}$$

i.e., the corresponding modular weights are $n + 5, n + 7, \dots, 2n - 2$,

$$\begin{aligned} wt_{\varphi_4}(v_{\frac{n+1}{2}}) &= \varphi_4(cv_{\frac{n+1}{2}}) + \varphi_4(v_{\frac{n-1}{2}}v_{\frac{n+1}{2}}) + \varphi_4(v_{\frac{n+1}{2}}v_{\frac{n+3}{2}}) + \varphi_4(w_{\frac{n-1}{2}}v_{\frac{n+1}{2}}) \\ &\quad + \varphi_4(w_{\frac{n+1}{2}}v_{\frac{n+1}{2}}) = 5 + \tfrac{n-3}{2} + \tfrac{n-5}{2} + 1 + 1 = n + 3, \end{aligned}$$

for $i = \frac{n+3}{2}, \frac{n+5}{2}, \dots, n$

$$\begin{aligned} wt_{\varphi_4}(v_i) &= \varphi_4(cv_i) + \varphi_4(v_{i-1}v_i) + \varphi_4(v_iv_{i+1}) + \varphi_4(w_{i-1}v_i) + \varphi_4(w_iv_i) = 4 + \tfrac{n-5}{2} \\ &\quad + \tfrac{n-5}{2} + (i - \tfrac{n+1}{2}) + (i - \tfrac{n-1}{2}) = 2i - 1, \end{aligned}$$

i.e., the corresponding modular weights are $n + 2, n + 4, \dots, 2n - 1$,

for $i = 1, 2, \dots, \frac{n-1}{2}$

$$wt_{\varphi_4}(w_i) = \varphi_4(v_{i+1}w_i) + \varphi_4(v_iw_i) = (\tfrac{n+1}{2} - i) + (\tfrac{n+3}{2} - i) = n + 2 - 2i,$$

i.e., the corresponding modular weights are $3, 5, \dots, n$,

$$wt_{\varphi_4}(w_{\frac{n+1}{2}}) = \varphi_3(v_{\frac{n+3}{2}}w_{\frac{n+1}{2}}) + \varphi_3(v_{\frac{n+1}{2}}w_{\frac{n+1}{2}}) = 1 + 1 = 2,$$

for $i = \frac{n+3}{2}, \frac{n+5}{2}, \dots, n$

$$wt_{\varphi_4}(w_i) = \varphi_4(v_{i+1}w_i) + \varphi_4(v_iw_i) = (i - \tfrac{n-1}{2}) + (i - \tfrac{n-1}{2}) = 2i - n + 1,$$

i.e., the corresponding modular weights are $4, 6, \dots, n + 1$,

$$wt_{\varphi_4}(c) = \sum_{i=1}^{n} \varphi_4(cv_i) = 5 \cdot 3 + 4 \cdot (n - 3) = 4n + 3 \equiv 1 \pmod{(2n+1)}.$$

According to the previous we get that the modular vertex weights are $0, 1, \dots, 2n$. Thus for odd $n \geq 9$ the labeling φ_4 is a modular irregular $\frac{n+1}{2}$-labeling of Sf_n. This concludes the proof. □

We conclude the results with the following corollary.

Corollary 3.3. *Let Sf_n be a sunflower graph, $n = 4$ or $n \geq 6$, then*

$$s(Sf_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

4. Conclusion

In this paper, we determined the modular irregularity strength for three families of flower graphs which are rose graphs, daisy graphs and sunflower graphs. The first two of these families of flower graphs are biregular. For the future research we propose to find the modular irregularity strengths of other biregular graphs.

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