Electronic Journal of Graph Theory and Applications **11** (1) (2023), 155–164

Electronic Journal of
Graph Theory and Applications

The Alon-Tarsi number of two kinds of planar graphs

Zhiguo Li*, Qing Ye, Zeling Shao

Department of Mathematics, Hebei University of Technology, China

zhiguolee@hebut.edu.cn, hebutyeqing@163.com, zelingshao@163.com

Abstract

The *Alon-Tarsi number* $AT(G)$ of a graph G is the least k for which there is an orientation D of G with max outdegree $k-1$ such that the number of spanning Eulerian subgraphs of G with an even number of edges differs from the number of spanning Eulerian subgraphs with an odd number of edges. In this paper, the exact value of the Alon-Tarsi number of two kinds of planar graphs is obtained.

Keywords: Alon-Tarsi number, choice number, chromatic number, Combinatorial Nullstellensatz, planar graph
Mathematics Subject Classification : 05C15
DOI: 10.5614/ejgta.2023.11.1.13

1. Introduction

All graphs considered in this article are finite and simple. One of the most popular topics in graph theory is graph coloring. In addition to classical coloring, list coloring is also a hot topic, it is a well-established generalization of graph coloring and has been widely studied. The study of list coloring problems was obtained in the 1970s by Vizing [1] and independently by Erdős, Rubin, and Taylor [2].

A *k-list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of k permissible colors. Given a k-list assignment L of G, an *L-coloring* of G is a mapping ϕ which assigns to each vertex v a color $\phi(v) \in L(v)$ such that $\phi(u) \neq \phi(v)$ for every edge uv of G. A graph G is *k-choosable* if G has an L-coloring for every k-list assignment L. The *choice number* of a graph G is the least positive integer k such that G is k-choosable, denoted by $ch(G)$.

Received: 11 August 2022, Revised: 20 March 2023, Accepted: 25 March 2023.

In the classic article [3], an upper bound for the choice number and for some related parameters of graphs is obtained by applying algebraic techniques, which was later called the Alon-Tarsi number of G, and denoted by $AT(G)$ (see e.g. Jensen and Toft (1995) [4]).

The *Alon-Tarsi number* of G, $AT(G)$, is the smallest k for which there is an orientation D of G with max outdegree $k-1$ such that the number of odd spanning Eulerian subgraphs of G is not the same as the number of even spanning Eulerian subgraphs of G. For convenience, all Eulerian subgraphs in this paper represent spanning Eulerian subgraphs. Furthermore, there is an equivalent definition of the Alon-Tarsi number by Alon-Tarsi polynomial method (see Definition 2.3).

Let $\chi_p(G)$ be the paint number of G. Schauz [5] has pointed out that $ch(G) \leq \chi_p(G) \leq AT(G)$ for any graph G and the equalities are not held in general. For more details about the paint number, the reader is referred to [6]. A graph G is *chromatic-choosable*, if $\chi(G) = ch(G)$. In [7], Thomassen proves with a very elegant argument that every planar graph is 5-choosable, with proof that can be translated into a simple linear algorithm for finding a list coloring. Voigt [8] has shown that not every planar graph is 4-choosable. Recently, Zhu [9] showed that every planar graph G has $AT(G) \leq 5$ which generalizes Thomassen's result. In [10], the authors get the Alon-Tarsi number of Halin graphs which have upper bound 4.

In this paper, we are interested in the Alon-Tarsi number of two kinds of planar graphs. One kind of graph is a class of 4-regular planar graphs [11], which is defined by $R_n = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E = \{v_iv_{i+1}, u_iu_{i+1}, u_iv_i, u_iv_{i+1} | i = 1, 2, \ldots, n\}$, where $v_{n+1} = v_1, u_{n+1} = u_1$ (see Figure 1). Obviously R_n contains $v_1v_2\cdots v_nv_1$ and $u_1u_2\cdots u_nu_1$ n-cycles as subgraphs. Another kind of graph is the biwheel B_n [12], there exists two vertices y_1 and y_2 adjacent to every vertices on cycles C_n. A biwheel B_n has $2n$ triangle faces, $3n$ edges, and $2+n$ vertices (see Figure 2 for $n = 6$).

In this article, we study the Alon-Tarsi number of R_n and B_n respectively and obtain two results as follows:

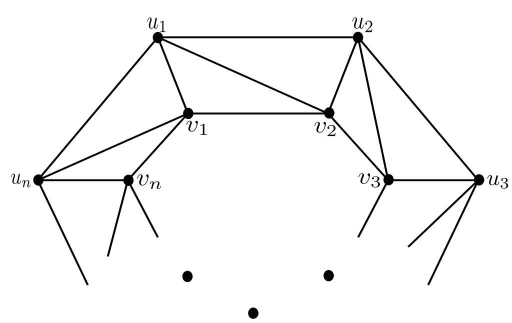

Figure 1. 4-regular planar graph $R_n (n \geq 3)$.

Theorem 1.1. *For a* 4*-regular planar graph* R_n*,*

$$\chi(R_n) = AT(R_n) = \begin{cases} 3, & n \equiv 0 \pmod 3; \\ 4, & \text{otherwise.} \end{cases}$$

Consequently, R_n *is chromatic-choosable.*

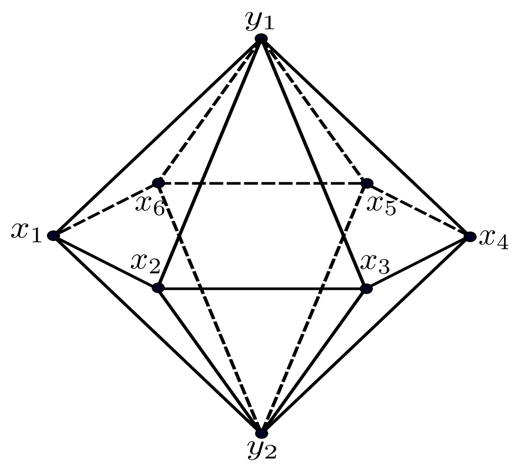

Figure 2. A biwheel B_6.

Theorem 1.2. *For a biwheel B_n ($n \geq 3$),*

$$AT(B_n) = \begin{cases} 3, & n = 4, \\ 4, & \text{otherwise.} \end{cases}$$

2. Preliminaries

Definition 2.1. [3] *A subdigraph H of a directed graph D is called Eulerian if $V(H) = V(G)$ and the indegree $d_H^-(v)$ of every vertex v of H in H is equal to its outdegree $d_H^+(v)$. Note that H might not be connected. For a digraph D, we denote by $\mathcal{E}(D)$ the family of Eulerian subdigraphs of D. H is even if it has an even number of edges, otherwise, it is odd. Let $\mathcal{E}_e(D)$ and $\mathcal{E}_o(D)$ denote the family of even and odd Eulerian subgraphs of D, respectively. Let* $\text{diff}(D) = |\mathcal{E}_e(D)| - |\mathcal{E}_o(D)|$. *We say that D is Alon-Tarsi if* $\text{diff}(D) \neq 0$. *If an orientation D of G yields an Alon-Tarsi digraph, then we say D is an Alon-Tarsi orientation (or an AT-orientation, for short) of G.*

Definition 2.2. [13] *Assume that G is an undirected simple graph whose vertices are linearly ordered and $\mathbb{F}$ is a field. Associate to each vertex v of G a variable x_v. The graph polynomial $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ of G is defined as*

$$f_G(\mathbf{x}) = \prod_{uv \in E(G),\ u<v} (x_u - x_v).$$

It is clear that the graph polynomial encodes information about its proper colorings. Indeed, a graph G is k-colorable if and only if there exists an n-tuple $(a_1, a_2, \ldots, a_n) \in \{0, 1, \ldots, k-1\}^n$ such that $f_G(a_1, a_2, \ldots, a_n) \neq 0$. Similarly, G is k-choosable if and only if for an arbitrary field $\mathbb{F}$ and for every family of sets $S_i \subset \mathbb{F} : 1 \leq i \leq n$, each of size at least k, there exists an n-tuple $(a_1, a_2, \ldots, a_n) \in S_1 \times S_2 \times \ldots \times S_n$ such that $f_G(a_1, a_2, \ldots, a_n) \neq 0$.

The following theorem gives a sufficient condition for the existence of such an n-tuple.

Theorem 2.1 (Combinatorial Nullstellensatz)**.** [14] *Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^{n} t_i$, where* *each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, there are $s_1 \in S_1$, $s_2 \in S_2$, $\ldots$, $s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.*

Combinatorial Nullstellensatz is a landmark theorem in algebraic combinatorics, which is now a widely used tool in tackling many (not necessarily coloring related) combinatorial problems in diverse areas of mathematics.

Finally we give another equivalent definition of the Alon-Tarsi number.

Definition 2.3. [13] *Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We say that G is Alon-Tarsi k-choosable, if there exist a monomial $c\prod_{i=1}^{n} x_i^{t_i}$ in the expansion of f_G such that $c \neq 0$ and $t_i \leq k-1$ for every $1 \leq i \leq n$. The smallest integer k for which G is Alon-Tarsi k-choosable, denote by $AT(G)$, is called the Alon-Tarsi number of G.*

3. Proof of the Theorem 1.1

Lemma 3.1. [15] *If G is a connected graph, and is neither a complete graph nor an odd cycle, then $AT(G) \leq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.*

By Lemma 3.1 and R_n contains odd cycles, we have

Lemma 3.2. $3 \leq \chi(R_n) \leq AT(R_n) \leq 4$ *for each R_n.*

Lemma 3.3. *For a 4-regular planar graph R_n,*

$$\chi(R_n) = \begin{cases} 3, & n \equiv 0 \pmod 3, \\ 4, & \text{otherwise}. \end{cases}$$

Proof. Assume $V(R_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$.

Case 1. $n \equiv 0$ (mod 3).

It is easy to check that there is a proper 3-coloring π: $V(R_n) \to \{0, 1, 2\}$ as follows:

If $i \equiv 1$ (mod 3), then $\pi(u_i) = 0$ and $\pi(v_i) = 1$;

If $i \equiv 2$ (mod 3), then $\pi(u_i) = 1$ and $\pi(v_i) = 2$;

If $i \equiv 0$ (mod 3), then $\pi(u_i) = 2$ and $\pi(v_i) = 0$.

It follows by Lemma 3.2 that $\chi(R_n) = 3$ (see Figure 3 for $n = 6$).

Case 2. $n \equiv 1$ (mod 3) or $n \equiv 2$ (mod 3).

We shall prove that R_n is not 3-colorable. Assume toward the contrary that there is a proper 3-coloring ϕ: $V(R_n) \to \{0, 1, 2\}$. Without loss of generality, let $\phi(u_1) = 0$ and $\phi(v_1) = 1$. $v_2, u_2, v_3, u_3, \ldots, v_{n-1}, u_{n-1}$ can be colored by a unique way.

If $n \equiv 1$ (mod 3), v_n is adjacent to v_1, v_{n-1} and u_{n-1}, but $\phi(v_1) = 1$, $\phi(v_{n-1}) = 0$ and $\phi(u_{n-1}) = 2$, there is no available color for v_n, a contradiction.

If $n \equiv 2$ (mod 3), $\phi(v_1) = \phi(v_{n-1}) = 1$ and $\phi(u_{n-1}) = 0$, so $\phi(v_n) = 2$. However, there is no possible color for u_n, a contradiction.

By Brook's theorem, $\chi(R_n) \leq 4$. Hence $\chi(R_n) = 4$ if $n \equiv 1$ (mod 3) or $n \equiv 2$ (mod 3). □

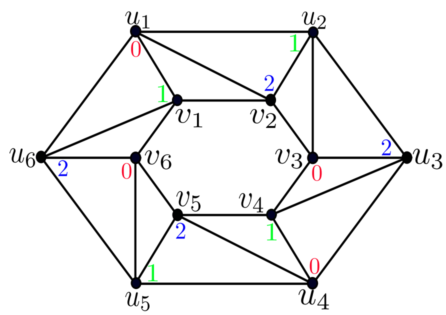

Figure 3. A 3-coloring of R_6.

Lemma 3.4. [16] *Assume that $f(\mathbf{x}) \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is a polynomial over $\mathbb{F}$, and $d_i \geq 0$ are integers such that $\deg f \leq \sum_{i=1}^n d_i$. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$. Then the coefficient of the monomial $\prod_{i=1}^n x_i^{d_i}$ in the expansion of f is*

$$c_{f,\mathbf{d}} = (\prod_{i=1}^n d_i!)^{-1} \sum_{a_1=0}^{d_1} \cdots \sum_{a_n=0}^{d_n} (-1)^{d_1+a_1} \binom{d_1}{a_1} \cdots (-1)^{d_n+a_n} \binom{d_n}{a_n} f(a_1, \ldots, a_n).$$

In particular, if $d_i = d$ for all i, then the coefficient of the monomial $\prod_{i=1}^n x_i^d$ in the expansion of f is

$$c_{f,\mathbf{d}} = (d!)^{-n} \sum_{\sigma} \left(\prod_{i=1}^n (-1)^{d+\sigma(x_i)} \binom{d}{\sigma(x_i)} \right) f(\sigma),$$

where the summation is over all mappings $\sigma : \{x_1, x_2, \ldots, x_n\} \to \{0, 1, \ldots, d\}$ and $f(\sigma)$ is the evaluation of f at $x_i = \sigma(x_i)$ for $i = 1, 2, \ldots, n$.

Lemma 3.5. *For a 4-regular graph R_n,*

$$AT(R_n) = \begin{cases} 3, & n \equiv 0 \pmod 3, \\ 4, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.2 and Lemma 3.3, it follows that $AT(R_n) = 4$ when $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. It remains to show that $AT(R_n) = 3$ if $n \equiv 0 \pmod 3$.

The graph polynomial of R_n is

$$f(\mathbf{x}) = \prod_{1 \leq i \leq n} (x_{v_{i+1}} - x_{v_i})(x_{u_{i+1}} - x_{u_i})(x_{v_i} - x_{u_i})(x_{v_{i+1}} - x_{u_i}),$$

where $u_{n+1} = u_1$, $v_{n+1} = v_1$.

In order to prove that $AT(R_n) \leq 3$, by Definition 2.3, it suffices to show that the monomial $\prod_{i,j=1}^n x_{v_i}^2 x_{u_j}^2$ in the expansion of $f(\mathbf{x})$ is non-vanishing. By Lemma 3.4, it is equivalent to prove that

$$c_{f,2} = (2!)^{-2n} \sum_{\sigma} \left[\prod_{i=1}^n (-1)^{2+\sigma(v_i)} \binom{2}{\sigma(v_i)} (-1)^{2+\sigma(u_i)} \binom{2}{\sigma(u_i)} \right] f(\sigma) \neq 0,$$

where the summation is over all mappings $\sigma : \{v_1, \ldots, v_n, u_1, \ldots, u_n\} \to \{0, 1, 2\}$.

Let $\Phi = \{\sigma \mid f(\sigma) \neq 0\}$ and $\Psi = \{\sigma \mid f(\sigma) = 0\}$. Obviously Ψ contribute nothing to the coefficient $c_{f,2}$. It is clear that $f(\sigma) \neq 0$ if and only if the mapping σ is a 3-proper coloring of R_n. Since R_n is 3-colorable when $n \equiv 0 \pmod 3$, $\Phi \neq \emptyset$. By Lemma 3.3, each color class contains $\frac{2n}{3}$ vertices. Hence

$$\prod_{i=1}^{n}(-1)^{2+\sigma(v_i)}\binom{2}{\sigma(v_i)}(-1)^{2+\sigma(u_i)}\binom{2}{\sigma(u_i)} = \prod_{j=0}^{2}\left[(-1)^{2+j}\binom{2}{j}\right]^{\frac{2n}{3}} = \left[(-1)^{2+1}\binom{2}{1}\right]^{\frac{2n}{3}} = 2^{\frac{2n}{3}},$$

and

$$c_{f,2} = 2^{-2n}2^{\frac{2n}{3}}\sum_{\sigma\in\Phi} f(\sigma) = 2^{-\frac{4n}{3}}\sum_{\sigma\in\Phi} f(\sigma).$$

For each mapping $\sigma \in \Phi$,

$$f(\sigma) = \prod_{1\leq i\leq n}(\sigma(v_{i+1}) - \sigma(v_i))(\sigma(u_{i+1}) - \sigma(u_i))(\sigma(v_i) - \sigma(u_i))(\sigma(v_{i+1}) - \sigma(u_i)). \qquad (*)$$

According to Lemma 3.3, if the coloring of any two adjacent vertices are determined, then other vertices can be colored in a unique way. Furthermore, it is quite clear that $\sigma(v_i) = \sigma(u_{i+1})$ and $\sigma(u_i) = \sigma(v_{i+2})$ for each $1 \leq i \leq n$. On the right hand side of $(*)$, replace $\sigma(u_{i+1}) - \sigma(u_i)$ and $\sigma(v_{i+1}) - \sigma(u_i)$ with $\sigma(v_i) - \sigma(u_i)$ and $\sigma(v_{i+1}) - \sigma(v_{i+2})$ respectively. Then we get

$$f(\sigma) = (-1)^n \prod_{1\leq i\leq n}(\sigma(v_{i+1}) - \sigma(v_i))^2(\sigma(v_i) - \sigma(u_i))^2 = (-1)^n 2^{\frac{4n}{3}}.$$

Hence

$$c_{f,2} = (-1)^n 2^{-\frac{4n}{3}} 2^{\frac{4n}{3}}|\Phi| = (-1)^n|\Phi| \neq 0.$$

Therefore $\prod_{i,j=1}^{n} x_{v_i}^2 x_{u_j}^2$ is a non-vanishing monomial of f_{R_n} and $AT(R_n) \leq 3$. In addition, $AT(R_n) \geq \chi(R_n) = 3$, and this completes the proof. □

It follows from the inequality $\chi(G) \leq ch(G) \leq \chi_p(G) \leq AT(G)$ that

Corollary 3.1.

$$ch(R_n) = \chi_p(R_n) = \begin{cases} 3, & n \equiv 0 \pmod 3, \\ 4, & \text{otherwise.} \end{cases}$$

4. Proof of the Theorem 1.2

The proof will be completed by a sequence of lemmas. By the structure of the biwheel, it is easy to show that

Lemma 4.1. *For a biwheel B_n,*

$$\chi(B_n) = \begin{cases} 3, & n \text{ is even,} \\ 4, & n \text{ is odd.} \end{cases}$$

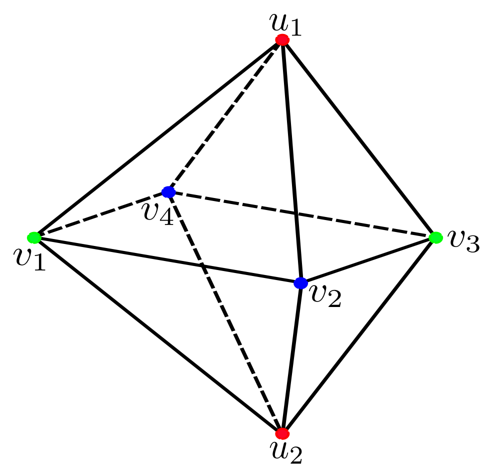

Figure 4. A 3-proper coloring of B_4.

Lemma 4.2. $AT(B_4) = 3$.

Proof. According to Lemma 4.1, $\chi(B_4) = 3$ (see Figure 4). Thus $AT(B_4) \geq 3$. What is left is to show that $AT(B_4) \leq 3$. The graph polynomial of B_4 is

$$f(\mathbf{x}) = \prod_{1\leq i\leq 4} (x_{v_{i+1}} - x_{v_i})(x_{u_1} - x_{v_i})(x_{u_2} - x_{v_i}),$$

where $v_5 = v_1$.

In order to show that $AT(B_4) \leq 3$, it suffices to prove that the coefficient

$$c_{f,2} = (2!)^{-6} \sum_{\sigma} \left[\prod_{i=1}^{4} (-1)^{2+\sigma(v_i)} \binom{2}{\sigma(v_i)} \prod_{j=1}^{2} (-1)^{2+\sigma(u_j)} \binom{2}{\sigma(u_j)} \right] f(\sigma)$$

of the term $x_{v_1}^2 x_{v_2}^2 x_{v_3}^2 x_{v_4}^2 x_{u_1}^2 x_{u_2}^2$ in $f(\mathbf{x})$ is not zero, where the summation is over all mappings $\sigma : \{v_1, \ldots, v_4, u_1, u_2\} \to \{0, 1, 2\}$. Note that if σ is not a proper coloring of B_4, then $f(\sigma) = 0$. Therefore, we can restrict the summation to proper colorings σ of B_4 with color set $\{0, 1, 2\}$.

It is easy to check that every proper coloring σ of B_4 is of the form $\sigma(v_1) = \sigma(v_3) = a$, $\sigma(v_2) = \sigma(v_4) = b$ and $\sigma(u_1) = \sigma(u_2) = c$, where (a, b, c) is a permutation of the color set $\{0, 1, 2\}$. So each color class contains 2 vertices. It follows that

$$\prod_{i=1}^{4} (-1)^{2+\sigma(v_i)} \binom{2}{\sigma(v_i)} \prod_{j=1}^{2} (-1)^{2+\sigma(u_j)} \binom{2}{\sigma(u_j)} = \prod_{k=0}^{2} \left[(-1)^{2+k} \binom{2}{k} \right]^2 = [(-1)^{2+1} \tbinom{2}{1}]^2 = 2^2,$$

and

$$\begin{aligned} f(\sigma) &= \prod_{1\leq i\leq 4} (\sigma(v_{i+1}) - \sigma(v_i))(\sigma(u_1) - \sigma(v_i))(\sigma(u_2) - \sigma(v_i)) \\ &= [(\sigma(v_2) - \sigma(v_1))(\sigma(u_1) - \sigma(v_1))(\sigma(u_1) - \sigma(v_2))]^4 > 0. \end{aligned}$$

Therefore

$$c_{f,2} = 2^{-4} \sum_{\sigma} f(\sigma) \neq 0.$$

$\square$

Lemma 4.3. *If n is even and $n > 4$, then $AT(B_n) = 4$.*

Proof. Assume $V(B_n) = \{v_1, v_2, \ldots, v_{2k}, u_1, u_2\}$, D is an arbitrary orientation of B_n. Since $\sum_{x \in V(D)} d_D^+(x) = |A(D)| = 6k$ and $|V(D)| = 2k+2$, there are some vertices that have outdegree at least 3, so $AT(B_{2k}) \geq 4$. It remains to show that $AT(B_{2k}) \leq 4$.

Let D_1 be an orientation of B_{2k} in which the edges of B_{2k} are oriented in such a way by orientating the cycle C_{2k} in clockwise and orientating edge $v_i u_j$ as $(v_i, u_j), i = 1, 2, \ldots, 2k$, $j = 1, 2$ (see Figure $5(a)$). Since D_1 has no odd directed cycle, it follows that D_1 is an AT-orientation with maximum outdegree 3. Therefore $AT(B_{2k}) \leq 4$. $\square$

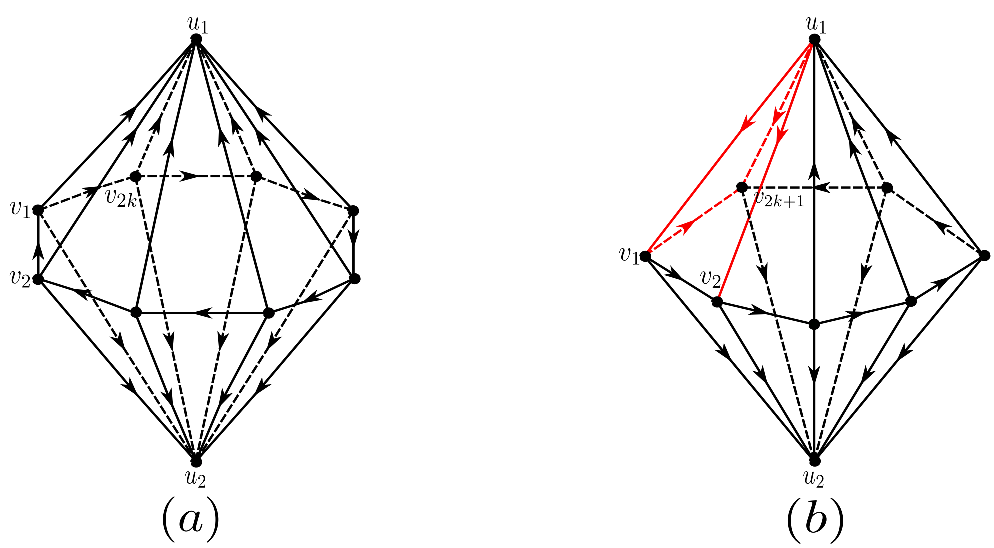

Figure 5. (a). The AT-orientation D_1 of B_{2k} ($k > 2$). (b). The AT-orientation D_2 of B_{2k+1} ($k \geq 1$).

Lemma 4.4. *If n is odd, then $AT(B_n) = 4$.*

Proof. Assume $V(B_n) = \{v_1, v_2, \ldots, v_{2k+1}, u_1, u_2\}$ and $k \geq 1$. $v_1 v_2 \cdots v_{2k+1} v_1$ is an odd cycle, denoted by C_{2k+1}. Let $G' = B_{2k+1} - u_2$. G' has an orientation D' as the following way: orient the edge $v_i v_{i+1}$ as (v_i, v_{i+1}) for each $1 \leq i \leq 2k$, the edge $v_{2k+1} v_1$ as (v_1, v_{2k+1}), the edge $v_j u_1$ as (v_j, u_1) for $j = 3, 4, \ldots, 2k$, the unoriented edges between u_1 and $V(C_{2k+1})$ are oriented from u_1 to $V(C_{2k+1})$. It is easily seen that $u_1 v_1 v_2 \cdots v_i u_1$ is an odd directed cycle and $u_1 v_2 v_3 \cdots v_i u_1$ is an even directed cycle when i is even, $u_1 v_1 v_2 \cdots v_i u_1$ is an even directed cycle and $u_1 v_2 v_3 \cdots v_i u_1$ is an odd directed cycle when i is odd, where $3 \leq i \leq 2k$. Therefore, D' contains $2(2k-2)$ directed cycles.

Specifically D' contains $(2k-2)$ odd directed cycles and $(2k-2)$ even directed cycles. It is clear that the arc (v_2, v_3) is contained in all directed cycles. Since Eulerian subdigraph is the arc disjoint union of directed cycles and empty subdigraph is an even Eulerian subdigraph, D' has $(2k-2)$ odd Eulerian subdigraphs and $(2k-1)$ even Eulerian subdigraphs. Therefore $\text{diff}(D') = |\mathcal{E}_e(D')| - |\mathcal{E}_o(D')| = 1 \neq 0$, D' is an AT-orientation of G'.

Let D_2 be obtained from D' by adding arcs (v_i, u_2), $i = 1, 2, \ldots, 2k+1$ (see Figure $5(b)$). Observe that no arc incident to u_2 is contained in a directed cycle of D_2, so none of these arcs is contained in an Eulerian subdigraph of D_2, $\mathcal{E}(D_2) = \mathcal{E}(D')$. Additionally, D_2 is an orientation of B_{2k+1} in which each vertex has outdegree at most 3. Therefore $AT(B_{2k+1}) \leq 4$.

Since $\chi(B_{2k+1}) = 4$, it follows that $AT(B_{2k+1}) \geq 4$. The result is established. □

Remark 4.1. In Section 3, we conclude that $\chi(R_n) = ch(R_n) = AT(R_n)$. However, when n is even and $n > 4$, by Lemma 4.1 and Lemma 4.3, $\chi(B_n) = 3$ and $AT(B_n) = 4$. Furthermore, we can prove B_n is not chromatic-choosable for all n.

In fact, let L be the list assignment of B_{6k} $(k \geq 2)$ defined as $L(u_1) = \{1,2,3\}$, $L(u_2) = \{4,5,6\}$, $L(v_1) = \cdots = L(v_k) = \{1,4,5\}$, $L(v_{k+1}) = \cdots = L(v_{2k}) = \{1,4,6\}$, $L(v_{2k+1}) = \cdots = L(v_{3k}) = \{2,4,5\}$, $L(v_{3k+1}) = \cdots = L(v_{4k}) = \{2,4,6\}$, $L(v_{4k+1}) = \cdots = L(v_{5k}) = \{3,4,5\}$, $L(v_{5k+1}) = \cdots = L(v_{6k}) = \{3,4,6\}$.

Now we can show that B_{6k} is not L-colorable. Assume, for the sake of contradiction, that φ is a proper L-coloring of B_{6k}. Without loss of generality, let $\varphi(u_1) = 1$, then the vertices $v_1, \ldots, v_{2k}$ will use up colors 4, 5 and 6 (see Figure 6). Hence there is no available color for u_2, a contradiction.

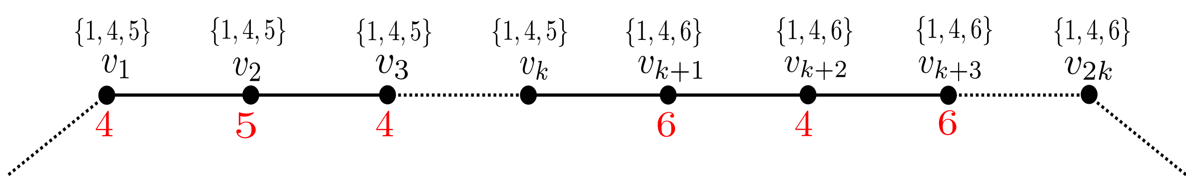

Figure 6. The case of $\varphi(u_1) = 1$.

5. Conclusions

In this paper, we have obtained the exact value of the Alon-Tarsi number of two kinds of planar graphs R_n and B_n mainly by the AT-orientation method and polynomial method. As is well known, for a simple graph G, $\chi(G) \leq ch(G) \leq \chi_p(G) \leq AT(G)$. Therefore, as byproducts, we also get that R_n is chromatic-choosable while B_n is not for all n.

Acknowledgement

This work was partially funded by Science and Technology Project of Hebei Education Department, China (No. ZD2020130) and the Natural Science Foundation of Hebei Province, China (No. A2021202013).

References

[1] V.G. Vizing, Vertex colorings with given colors, *Metody Diskret. Anal.* **29** (1976), 3–10.

[2] P. Erdős, A.L. Rubin, and H. Taylor, Choosability in graphs, *Congr. Numer.* **26** (1979), 126–157.

[3] N. Alon and M. Tarsi, Colorings and orientations of graphs, *Combinatorica* **12**(2) (1992), 125–134.

[4] T.R. Jensen and B. Toft, Graph coloring problems, *Wiley, New York*, 1995.

[5] U. Schauz, A paintability version of the Combinatorial Nullstellensatz and list colorings of k-partite k-uniform hypergraphs, *Electron. J. Combin.* **17**(1) (2010), 3940–3946.

[6] X.D. Zhu, On-line list coloring of graphs, *Electron. J. Combin.* **16**(1) (2009), R127.

[7] C. Thomassen, Every planar graph is 5-choosable, *J. Combin. Theory Ser. B* **62**(1) (1994), 180–181.

[8] M. Voigt and B. Wirth, On 3-colorable non-4-choosable planar graphs, *J. Graph Theory* **24**(3) (1997), 233–235.

[9] X.D. Zhu, The Alon-Tarsi number of planar graphs, *J. Combin. Theory Ser. B* **134** (2019), 354–358.

[10] Z.G. Li, Q. Ye, and Z.L. Shao, The Alon-Tarsi number of Halin graphs, *arXiv preprint arXiv:2110.11617*.

[11] L. Zhang, H.Y. Chen, and X.Y. Yuan, The adjacent vertex distinguishing incidence coloring numbers of a class of 4-regular planar graphs, *Math. Pract. Theory* **42** (19) (2012), 197–201.

[12] M. Pankov and A. Tyc, On two types of Z-monodromy in triangulations of surfaces, *Discrete Math.* **342** (9) (2019), 2549–2558.

[13] D. Hefetz, On two generalizations of the Alon-Tarsi polynomial method, *J. Combin. Theory Ser. B* **101** (6) (2011), 403–414.

[14] N. Alon, Combinatorial Nullstellensatz, *Combin. Probab. Comput.* **8**(1-2) (1999), 7–29.

[15] J. Hladký, D. Král, and U. Schauz, Brooks' theorem via the Alon-Tarsi theorem, *Discrete Math.* **310** (23) (2010), 3426–3428.

[16] U. Schauz, Algebraically solvable problems: describing polynomials as equivalent to explicit solutions, *Electron. J. Combin.* **15**(1) (2008), 273–295.

