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On the general sum-connectivity index of connected graphs with given order and girth

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Abstract

In this paper, we show that in the class of connected graphs G of order $n \geq 3$ having girth at least equal to k, $3 \leq k \leq n$, the unique graph G having minimum general sum-connectivity index $\chi_\alpha(G)$ consists of C_k and $n-k$ pendant vertices adjacent to a unique vertex of C_k, if $-1 \leq \alpha < 0$. This property does not hold for zeroth-order general Randić index ${}^0R_\alpha(G)$.

Keywords:
Girth, pendant vertex, general sum-connectivity index, zeroth-order general Randić index, subadditive function, convex function, Jensen's inequality
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1. Introduction

Let G be a simple graph having vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{G}_n$ denote the set of connected graphs of fixed order n and size $m \geq n$. The girth of a graph $G \in \mathcal{G}_n$ will be denoted $g(G)$. The degree of a vertex $u \in V(G)$ is denoted $d(u)$ and $N(u)$ is the set of vertices adjacent with u. If $d(u) = 1$ then u is called pendant; a pendant edge is an edge containing a pendant vertex. The minimum and maximum degrees of G are denoted $\delta(G)$ and $\Delta(G)$, respectively. For $A \subset E(G)$, $G - A$ denotes the graph deduced from G by deleting the edges of A and the graph obtained by the deletion of an edge $uv \in E(G)$ is denoted $G - uv$. Conversely, if $A \subset E(\overline{G})$, $G + A$ is the graph obtained from G by adding the edges of A. If $x \in V(G)$, $G - x$ denotes the

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subgraph of G obtained by deleting x and its incident edges.
For $n \geq 3$ and $3 \leq k \leq n$, let $C_{k,n-k}$ denote the graph of order n consisting of a cycle C_k and $n-k$ pendant edges attached to a unique vertex of C_k. For other notations in graph theory, we refer [1].

The general sum-connectivity index of graphs was proposed by Zhou and Trinajstić [10]. It is denoted by $\chi_\alpha(G)$ and defined as

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha,$$

where α is a real number. A particular case of the general sum-connectivity index is the harmonic index, denoted by $H(G)$ and defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)} = 2\chi_{-1}(G).$$

The zeroth-order general Randić index, denoted by ${}^0R_\alpha(G)$ is defined as

$${}^0R_\alpha(G) = \sum_{u \in V(G)} d(u)^\alpha,$$

where α is a real number. For $\alpha = 2$ this index is also known as first Zagreb index (see [4]).

For $-1 \leq \alpha < 0$ Du, Zhou and Trinajstić [2] showed that among the set of n-vertex unicyclic graphs with $n \geq 5$, $C_{3,n-3}$ is the unique graph with the minimum general sum-connectivity index and Tomescu and Kanwal [6] showed that in the same set of graphs having girth $k \geq 4$ the unique extremal graph is $C_{k,n-k}$. Zhong [9] proved that in the set of connected graphs of order n and m edges, where $m \geq n$, with girth $g(G) \geq k$ ($3 \leq k \leq n$), minimum harmonic index $H(G)$ is reached only for $C_{k,n-k}$. Other extremal properties of the general sum-connectivity index for trees were proposed in [3, 5].

In this paper, we study the minimum general sum-connectivity index $\chi_\alpha(G)$ in the class of connected graphs G of fixed order $n \geq 3$ and size $m \geq n$ with girth $g(G) \geq k$. Theorem 3.1 extends the above result of Zhong for every $-1 \leq \alpha < 0$ (including the case of the harmonic index, when $\alpha = -1$), Corollary 3.3 those of Du, Zhou and Trinajstić, and Corollary 3.2 the result of Tomescu and Kanwal (which holds for unicyclic graphs, when $m = n$). In section 2 we state some parametric inequalities which will be used in the last section. In section 3 we determine the connected graphs G of order $n \geq 3$ with girth at least k ($3 \leq k \leq n$) having minimum $\chi_\alpha(G)$ for $-1 \leq \alpha < 0$.

2. Some preliminary results

Let $g(n,k) = (n-k)(n-k+3)^\alpha + 2(n-k+4)^\alpha + (k-2)4^\alpha$. Note that $g(n,k) = \chi_\alpha(C_{k,n-k})$.

Lemma 2.1. [8] *The function* $f(n,k) = k(k+3)^\alpha + 2(k+4)^\alpha + (n-k-2)4^\alpha$ *is strictly decreasing in* $k \geq 0$ *for* $-1 \leq \alpha < 0$.

Since $g(n,k) = f(n,n-k)$ we deduce the following property.

Corollary 2.1. *The function $g(n,k)$ is strictly increasing in k, $3 \le k \le n$ for $-1 \le \alpha < 0$.*

Lemma 2.2. [8] *The function*

$$\psi(x) = 2(x+5)^{\alpha} + (x-1)(x+4)^{\alpha} - x(x+3)^{\alpha}$$

defined for $x \ge 0$ and $-1 \le \alpha < 0$ is strictly decreasing.

Lemma 2.3. [7] *Let uv be an edge of a graph G such that $d(u)+d(v)$ is minimum. If $-1 \le \alpha < 0$ then $\chi_{\alpha}(G-uv) < \chi_{\alpha}(G)$.*

Lemma 2.4. [8] a) *Let $x > 0$. If $\alpha < 0$ or $\alpha > 1$ then $(1+x)^{\alpha} > 1 + \alpha x$.*
b) *Let $x > 0$. If $\alpha < 0$ or $1 < \alpha < 2$ then $(1+x)^{\alpha} < 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2$ (for $\alpha = 2$ equality holds).*

Lemma 2.5. *The function $g(n,k)$ is strictly subadditive in n for $-1 \le \alpha < 0$, i.e.,*

$$g(n_1+n_2,k) < g(n_1,k) + g(n_2,k), \tag{1}$$

where $n_1, n_2 \ge k \ge 3$.

Proof. By letting $n_1 + n_2 = n \ge 2k, n_1 = x$ we deduce $n_2 = n - x$ and (1) leads to

$$g(x,k) + g(n-x,k) > g(n,k)$$

for every $k \le x \le n-k$. Using formula for $g(n,k)$ this inequality is equivalent to

$$(x-k)(x-k+3)^{\alpha} + 2(x-k+4)^{\alpha} + (n-x-k)(n-x-k+3)^{\alpha} + 2(n-x-k+4)^{\alpha} + (k-2)4^{\alpha}$$
$$> (n-k)(n-k+3)^{\alpha} + 2(n-k+4)^{\alpha}. \tag{2}$$

Let

$$\eta(x) = (x-k)(x-k+3)^{\alpha} + 2(x-k+4)^{\alpha} + (n-x-k)(n-x-k+3)^{\alpha} + 2(n-x-k+4)^{\alpha}.$$

We have $\eta(x) = \eta(n-x)$; we can write $\eta(x) = \gamma(x) + \gamma(n-x)$, where

$$\gamma(x) = (x-k)(x-k+3)^{\alpha} + 2(x-k+4)^{\alpha}.$$

We get

$$\gamma''(x) = \alpha(\alpha-1)(x-k)(x-k+3)^{\alpha-2} + 2\alpha(x-k+3)^{\alpha-1} + 2\alpha(\alpha-1)(x-k+4)^{\alpha-2}$$
$$< \alpha(\alpha-1)(x-k)(x-k+3)^{\alpha-2} + 2\alpha(x-k+3)^{\alpha-1} + 2\alpha(\alpha-1)(x-k+3)^{\alpha-2}$$
$$= \alpha(x-k+3)^{\alpha-2}((\alpha+1)(x-k+2)+2) < 0.$$

Similarly, $\gamma''(n-x) < 0$, so $\eta''(x) < 0$, hence $\eta(x)$ is a concave function. Because $\eta(x) = \eta(n-x)$ where $k \le x \le n-k$, so the minimum of $\eta(x)$ is reached at $x = k$ and $x = n-k$. Replacing $x = k$ in (2) yields

$$k4^\alpha + (n-2k)(n-2k+3)^\alpha + 2(n-2k+4)^\alpha > (n-k)(n-k+3)^\alpha + 2(n-k+4)^\alpha. (3)$$

In order to prove (3) we shall consider a new variable $x = n \ge 2k$ and the function

$$\varphi(x) = (x-2k)(x-2k+3)^\alpha + 2(x-2k+4)^\alpha - (x-k)(x-k+3)^\alpha - 2(x-k+4)^\alpha$$

defined for $x \ge 2k \ge 6$. We deduce

$$\varphi'(x) = (x-2k+3)^{\alpha-1}((x-2k)(\alpha+1)+3) + 2\alpha(x-2k+4)^{\alpha-1}$$
$$-(x-k+3)^{\alpha-1}((x-k)(\alpha+1)+3) - 2\alpha(x-k+4)^{\alpha-1} > (x-2k+3)^{\alpha-1}(x(\alpha+1)$$
$$-2k(\alpha+1)+3+2\alpha) - (x-k+3)^{\alpha-1}(x(\alpha+1)-k(\alpha+1)+3) - 2\alpha(x-k+4)^{\alpha-1}$$
$$= E(x,k,\alpha)(x-k+4)^{\alpha-1}.$$

We have

$$E(x,k,\alpha) = \left[1 + \frac{k+1}{x-2k+3}\right]^{1-\alpha}[x(\alpha+1) - 2k(\alpha+1) + 3 + 2\alpha]$$
$$-\left[1 + \frac{1}{x-k+3}\right]^{1-\alpha}[x(\alpha+1) - k(\alpha+1) + 3] - 2\alpha.$$

By Lemma 2.5 we get

$$E(x,k,\alpha) > \left[1 + \frac{(1-\alpha)(k+1)}{x-2k+3}\right][x(\alpha+1) - 2k(\alpha+1) + 3 + 2\alpha]$$
$$-\left[1 + \frac{1-\alpha}{x-k+3} + \frac{\alpha(\alpha-1)}{2(x-k+3)^2}\right][x(\alpha+1) - k(\alpha+1) + 3] - 2\alpha$$
$$= -\alpha k(1+\alpha) + \alpha(\alpha-1)F(x,k,\alpha),$$

where

$$F(x,k,\alpha) = \frac{(1+\alpha)(k-x)-3}{2(x-k+3)^2} - \frac{3}{x-k+3} + \frac{k+1}{x-2k+3}.$$

Finally,

$$F(x,k,\alpha) > \frac{k-x-3}{(x-k+3)^2} - \frac{3}{x-k+3} + \frac{k+1}{x-2k+3} = -\frac{4}{x-k+3} + \frac{k+1}{x-2k+3} > 0$$

since $k \ge 3$ implies $\frac{k+1}{x-2k+3} > \frac{4}{x-k+3}$.
Because $\varphi'(x) > 0$ it follows that $\varphi(x)$ is strictly increasing and (3) holds if it holds for $n = 2k$ and $k \ge 3$. Substituting $n = 2k$ in (3) yields

$$(k+2)4^\alpha > k(k+3)^\alpha + 2(k+4)^\alpha,$$

which is true because $k \ge 3$. □

Lemma 2.6. *Let $G \in \mathcal{G}_n$ such that $g(G) \geq k$. We have $\Delta(G) \leq n - k + 2$ and the bound is tight.*

Proof. Let $v \in V(G)$ such that $d(v) = \Delta(G)$. Suppose that v belongs to a cycle in G and denote by C a shortest cycle containing v. It follows that v is adjacent to exactly 2 vertices of C, thus implying $\Delta(G) \leq n - l + 2$, where l denotes the length of C. Since $l \geq g(G)$ we obtain $\Delta(G) \leq n - g(G) + 2 \leq n - k + 2$.
If v does not belong to any cycle in G, it follows that a shortest cycle of G contains at most one vertex in the set $N(v)$ and we deduce $\Delta(G)+1+g(G)-1 \leq n$, or $\Delta(G) \leq n-g(G) < n-k+2$. The bound is reached because $\Delta(C_{k,n-k}) = n - k + 2$. □

3. Main Results

Theorem 3.1. *Let G be a connected graph of order $n \geq 3$ and size $m \geq n$ with girth $g(G) \geq k$ $(3 \leq k \leq n)$. If $-1 \leq \alpha < 0$ then $\chi_\alpha(G) \geq g(n, k) = (n-k)(n-k+3)^\alpha+2(n-k+4)^\alpha+(k-2)4^\alpha$. Equality holds if and only if $G = C_{k,n-k}$.*

Proof. The proof is by induction on $m + n$. For $n = 3$ we have $m = k = 3$, $G = C_3$ and in this case the property holds. Also we can suppose that $n \geq k + 1$, since for $n = k$ there exists a unique graph, namely $C_{n,0} = C_n$. Let $m \geq n \geq 4$. Suppose the property is true for smaller values of $m + n$. Let $G \in \mathcal{G}_n$ having girth $g(G) \geq k$ such that $\chi_\alpha(G)$ is minimum. We shall consider two cases: A. $\delta(G) = 1$ and B. $\delta(G) \geq 2$.

A. In this case there exists a pendant vertex $u \in V(G)$ and let $uv \in E(G)$. We have $d(v) = d \geq 2$ and let $N(v)\backslash\{u\} = \{u_1, \ldots, u_{d-1}\}$. Since G is a connected graph containing at least one cycle, we get that there exists at least one vertex in $\{u_1, \ldots, u_{d-1}\}$ with degree at least 2. Suppose there exists exactly one vertex in this set with degree at least 2, say w. Let $d(w) = s \geq 2$ and let $N(w)\backslash\{v\} = \{v_1, \ldots, v_{s-1}\}$. Define $G_1 = G - \{wv_1, \ldots, wv_{s-1}\} + \{vv_1, \ldots, vv_{s-1}\}$. It follows that $G_1 \in \mathcal{G}_n$ and $g(G_1) = g(G) \geq k$. We deduce

$$\chi_\alpha(G) - \chi_\alpha(G_1) = (d-1)[(d+1)^\alpha - (d+s)^\alpha] + \sum_{i=1}^{s-1}[(d(v_i)+s)^\alpha - (d(v_i)+d+s-1)^\alpha] > 0$$

since $d \geq 2$ and $s \geq 2$. This contradicts the assumption about the minimality of G.
So we deduce that there exist at least two vertices in $\{u_1, \ldots, u_{d-1}\}$ with degree at least 2, thus implying $d \geq 3$. Let $G_2 = G - u$. We have $G_2 \in \mathcal{G}_{n-1}$ and $g(G_2) = g(G) \geq k$.
It follows that

$$\chi_\alpha(G) = \chi_\alpha(G_2) + (d+1)^\alpha + \sum_{i=1}^{d-1}[(d+d(u_i))^\alpha - (d+d(u_i)-1)^\alpha].$$

Since the function $h(x) = (d+x)^\alpha - (d+x-1)^\alpha$ has $h'(x) > 0$ for any $\alpha < 0$, one has

$$\sum_{i=1}^{d-1}[(d+d(u_i))^\alpha - (d+d(u_i)-1)^\alpha] \geq 2[(d+2)^\alpha - (d+1)^\alpha] + (d-3)[(d+1)^\alpha - d^\alpha],$$

equality holds if and only if two degrees of $u_1, \dots, u_{d-1}$ are equal to 2, the remaining ones being 1.

By the induction hypothesis we obtain $\chi_\alpha(G_2) \geq g(n-1,k)$, which yields

$$\chi_\alpha(G) \geq g(n-1,k) + 2(d+2)^\alpha + (d-4)(d+1)^\alpha - (d-3)d^\alpha.$$

Inequality $g(n-1,k) + 2(d+2)^\alpha + (d-4)(d+1)^\alpha - (d-3)d^\alpha \geq g(n,k)$ is equivalent to

$$(n-k-1)(n-k+2)^\alpha + 2(d+2)^\alpha + (d-4)(d+1)^\alpha - (d-3)d^\alpha$$
$$\geq (n-k-2)(n-k+3)^\alpha + 2(n-k+4)^\alpha. \quad (4)$$

Let $\varrho(x) = 2(x+2)^\alpha + (x-4)(x+1)^\alpha - (x-3)x^\alpha$. Since $\varrho(x) = \psi(x-3)$, by Lemma 2.3 it follows that $\varrho(x)$ is strictly decreasing for $x \geq 3$ and $-1 \leq \alpha < 0$. Note that by Lemma 2.7 we have $d \leq \Delta(G) \leq n-k+2$ since $g(G) \geq k$. This leads to the inequality $2(d+2)^\alpha + (d-4)(d+1)^\alpha - (d-3)d^\alpha \geq 2(n-k+4)^\alpha + (n-k-2)(n-k+3)^\alpha - (n-k-1)(n-k+2)^\alpha$ and equality holds only for $d = n-k+2$. In this case (4) becomes an equality. Summarizing, we have $\chi_\alpha(G) = g(n,k)$ only if $G_2 = C_{k,n-1-k}$, $d(v) = n-k+2$ and v is adjacent in G_2 to $k-1$ pendant vertices and to 2 vertices of degree 2. We have $\chi_\alpha(G) \geq g(n,k)$ and equality holds only if $G = C_{k,n-k}$.

B. In this case $\delta(G) \geq 2$. We shall prove that $\chi_\alpha(G) > g(n,k)$. Since $\delta(G) \geq 2$ we may assume that $m \geq n+1$ because $m = n$ implies G is 2-regular, hence $G = C_n = C_{n,0}$ and $\chi_\alpha(C_n) = g(n,n) > g(n,k)$ for every $3 \leq k \leq n-1$ by Corollary 2.2.

Let $e = uv \in E(G)$ such that $d(u) + d(v)$ is minimum. By Lemma 2.4 we have $\chi_\alpha(G - uv) < \chi_\alpha(G)$. Since $m \geq n+1$, $g(G-uv) \geq k$ holds since the cyclomatic number of G is equal to two. We shall consider two subcases B1 and B2, according to e is a cut-edge in G or not, respectively.

B1. e being a cut-edge, $G - e$ has two components, say G_1 and G_2, where $u \in V(G_1)$ and $v \in V(G_2)$. By denoting $|V(G_i)| = n_i$ for $1 \leq i \leq 2$ we get $n = n_1 + n_2$. Because $\delta(G) \geq 2$ and $g(G) \geq k$ we obtain that each G_i has at least one cycle and $g(G_i) \geq g(G) \geq k$, which implies $n_i \geq k$ for $1 \leq i \leq 2$. By induction, since $G_i \in \mathcal{G}_{n_i}$ for each i, we deduce $\chi_\alpha(G) > \chi_\alpha(G-e) = \chi_\alpha(G_1) + \chi_\alpha(G_2) \geq g(n_1,k) + g(n_2,k) > g(n,k)$ by Lemma 2.6.

B2. In this case $G - e$ is a connected graph of order n and size $m-1$, with $m-1 \geq n$ and $g(G-e) \geq k$. By induction $\chi_\alpha(G-e) \geq g(n,k)$, which implies $\chi_\alpha(G) > g(n,k)$ and the proof is complete. □

Since extremal graph $C_{k,n-k}$ has girth equal to k, we deduce the following corollary.

Corollary 3.1. *Let G be a connected graph of order $n \geq 3$ and size $m \geq n$ with girth $g(G) = k$ $(3 \leq k \leq n)$. If $-1 \leq \alpha < 0$ then $\chi_\alpha(G) \geq g(n,k)$. Equality holds if and only if $G = C_{k,n-k}$.*

Since $H(G) = 2\chi_{-1}(G)$, the result also holds for the harmonic index.

If $-1 \leq \alpha < 0$ note that $C_{k,n-k}$ is not extremal for zeroth-order general Randić index ${}^0R_\alpha(G)$. If G_1 denotes the graph consisting of C_{n-2} and two pendant edges incident to two distinct vertices of C_{n-2}, then we get ${}^0R_\alpha(G_1) <^0 R_\alpha(C_{n-2,2})$. This inequality is equivalent to $2 \cdot 3^\alpha < 2^\alpha + 4^\alpha$, which is valid by Jensen's inequality.

Because by Corollary 2.2 the minimum of the function $g(n, k)$ is reached only for $k = 3$, an extremal property deduced by other means for unicyclic graphs in [2] follows:

Corollary 3.2. *If $-1 \leq \alpha < 0$, in the class of connected graphs G of fixed order n and variable size $m \geq n$, $\chi_\alpha(G)$ is minimum if and only if $G = C_{3,n-3}$.*

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