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Forbidden family of P_h-magic graphs

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Abstract

Let G be a simple, finite, and undirected graph and H be a subgraph of G. The graph G admits an H-covering if every edge in G belongs to a subgraph isomorphic to H. A bijection $f : V(G) \cup E(G) \to [1, n]$ is a magic total labeling if for every subgraphs H' isomorphic to H, the sum of labels of all vertices and edges in H' is constant. If there exists such f, we say G is H-magic. A graph F is said to be a forbidden subgraph of H-magic graphs if $F \subseteq G$ implies G is not an H-magic graph. A set that contains all forbidden subgraph of H-magic is called forbidden family of H-magic graphs, denoted by $\mathcal{F}(H)$. In this paper, we consider $\mathcal{F}(P_h)$, where P_h is a path of order h. We present some sufficient conditions of a graph being a member of $\mathcal{F}(P_h)$. Besides that, we show the uniqueness of a minimal tree which belongs to $\mathcal{F}(P_3)$ and characterize P_3-(super)magic trees.

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1. Introduction

Let G and H be finite, simple, undirected graphs. We write G admits an H-covering if every edge in the graph belongs to a subgraph H' which is isomorphic to H. The graph G is called H-magic if G admits H-covering and there exists total labeling $f : V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ such that there exists positive integer k which $w(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) =$

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k, for each subgraph $H' \cong H$ of G. Furthermore, if f also have extra property $f(V(G)) = [1, |V(G)|]$, then G is H-supermagic. A special case of K_2-supermagic graphs is called edge-supermagic graphs. Some results concering H-(super)magic graphs can be seen in [1], [5], [9]. For more information about (super)magic labeling and its variations, readers may consult to [3].

A graph F is called a *forbidden subgraph* of H-magic if $F \subseteq G$ implies G is not H-magic. Let $\mathcal{F}(H)$ be a set containing every graph admitting H-covering which is not allowed to be a subgraph of any H-magic graph. We call such set as forbidden family $\mathcal{F}(H)$. Known studies about forbidden subgraph in magic labeling may be seen in [4], [6], [7], [8]. We adopt these results in our notation.

Theorem 1.1. *[4] Let $h \geq 3$ be positive integer. We have $C_h \in \mathcal{F}(P_h)$.*

A (n, k)-tadpole is a graph constructed by identifying an end vertex of P_k with a vertex of C_n. Maryati et al. [7] write $C_n^{+1} \cong (n, 1)$-tadpole.

Theorem 1.2. *[7] Let $h \geq 4$ be positive integer. We have $\{C_{h-1}^{+1}, C_{h+1}^{+1}\} \subseteq \mathcal{F}(P_h)$.*

Moreover, Maryati et al. [6, 7] defined H_n graph with a vertex and edge set

$$V(H_n) = \{v_{1,i}, v_{2,i} \mid i \in [1, 2n+1]\},$$
$$E(H_n) = \{v_{1,i}v_{1,i+1}, v_{2,i}v_{2,i+1} \mid i \in [1, 2n]\} \cup \{v_{1,n+1}v_{2,n+1}\}.$$

They determined that this graph is also a forbidden subgraph of P_h.

Theorem 1.3. *[6, 7] Let $h \geq 3$ be positive integer. We have $H_h \in \mathcal{F}(P_h)$.*

This paper is written as follows. In Section 2 and 3 we investigate sufficient conditions for a graph which belongs to $\mathcal{F}(P_h)$. Section 2 mainly deals with tree graphs, while Section 3 deals with unicyclic graphs. Furthermore, we found that there is no tree other than H_1 which belongs to $\mathcal{F}(P_3)$ of small order in Section 4.

2. Trees in $\mathcal{F}(P_h)$

We define $Dt(v, u)$ as a set of every length of possible paths formed with endpoints of v, u. Clearly, $d(v, u) \in Dt(v, u)$ and for u, v vertices in trees we have $Dt(v, u) = \{d(v, u)\}$. To start, two supplementary lemmas are provided which arose from the implications of graphs being P_h-magic. The first lemma tells us that some parts in every paths having length more than h in a graph will induce constant sums.

Lemma 2.1. *Let $n \geq 3, m \in [1, \lfloor \frac{n-1}{2} \rfloor]$ be integers. Let G be a graph that has f as a P_h-magic labeling of G. If there exists $u, v \in V(G)$ with $n \in Dt(u, v)$, then there exists consecutive vertices $x_0, x_1, ..., x_m = u$ and $y_0, y_1, ..., y_m = v$ such that*

$$\sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{m} f(x_{i-1}x_i) = \sum_{i=1}^{m} f(y_i) + \sum_{i=1}^{m} f(y_{i-1}y_i).$$

Proof. Since $n \in Dt(u,v)$, then there exists consecutive vertices $u = z_1, z_2, z_3, ..., z_{n+1} = v$. By taking weights of two subgraphs from consecutive vertices $z_1, z_2, ..., z_{n-m+1}$ and $z_{m+1}, z_{m+2}, ..., z_{n+1}$ we have

$$\sum_{i=1}^{n-m+1} f(z_i) + \sum_{i=2}^{n-m+1} f(z_{i-1}z_i) = \sum_{i=m+1}^{n+1} f(z_i) + \sum_{i=m+2}^{n+1} f(z_{i-1}z_i),$$

which implies

$$\sum_{i=1}^{m} f(z_i) + \sum_{i=2}^{m+1} f(z_{i-1}z_i) = \sum_{i=n-m+2}^{n+1} f(z_i) + \sum_{i=n-m+2}^{n+1} f(z_{i-1}z_i).$$

substituting $x_i = z_{m-i+1}$ and $y_i = z_{n-m+i+1}$ we got the result as desired. □

Next, constant sums may also appear in parts of a subgraph isomorphic to a certain tree with three pendants.

Lemma 2.2. *Let $n \geq 3, m \in [\lfloor \frac{n+1}{2} \rfloor, n-1]$ be integers. Let G be a graph that has f as a P_h-magic labeling of G. If there exists four vertices x_1, w, y, z such that*

1. *there exists m satisfying $m \in Dt(w,y)$ and $m \in Dt(w,z)$,*
2. *there exists n satisfying so that $m + n \in Dt(x_1, y)$,*

then there exists a consecutive vertices $x_1, x_2, ..., x_n = w, v_1, v_2, ..., v_m = y$ and $x_1, x_2, ..., x_n = w, u_1, u_2, ..., u_m = z$ such that

$$\sum_{i=1}^{m} f(v_i) + \sum_{i=1}^{m} f(v_{i-1}v_i) = \sum_{i=1}^{m} f(u_i) + \sum_{i=1}^{m} f(u_{i-1}u_i)$$

with $x_n = v_0 = u_0$.

Proof. By taking two subgraph of consecutive vertices $x_1, ..., x_n, v_1, ..., v_m$ and $x_1, ..., x_n, u_1, ..., u_m$ we got

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n-1} f(x_i x_{i+1}) + \sum_{i=1}^{m} f(v_i) + \sum_{i=1}^{m} f(v_{i-1}v_i) \\ =&\sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n-1} f(x_i x_{i+1}) + \sum_{i=1}^{m} f(u_i) + \sum_{i=1}^{m} f(u_{i-1}u_i). \end{aligned}$$

This implies

$$\sum_{i=1}^{m} f(v_i) + \sum_{i=1}^{m} f(v_{i-1}v_i) = \sum_{i=1}^{m} f(u_i) + \sum_{i=1}^{m} f(u_{i-1}u_i),$$

therefore the lemma holds. □

One kind of a graph belonging to $\mathcal{F}(P_h)$ is a new class of graph namely *Tiara graphs*. We define a *Tiara graph* $G = Ti_n(p, q, r)$ as follows

$$\begin{aligned} V(G) &= \{v_i \mid i \in [1,(n-1)(q+1)+1]\} \cup \{x_{b,j} \mid b \in \{1,(n-1)(q+1)+1\}, j \in [1,r]\} \\ &\quad \cup \{w_{(q+1)k+1,l} \mid k \in [0,n-1], l \in [1,p]\}, \\ E(G) &= \{v_i v_{i+1} \mid i \in [1,(n-1)(q+1)]\} \\ &\quad \cup \{v_b x_{b,1}, x_{b,j}x_{b,j+1} \mid b \in \{1,(n-1)(q+1)+1\}, j \in [1,r-1]\} \\ &\quad \cup \{v_{(q+1)k+1}w_{(q+1)k+1,1}, w_{(q+1)k+1,l}w_{(q+1)k+1,l+1} \mid k \in [0,n-1], l \in [1,p-1]\}. \end{aligned}$$

An example of $Ti_4(1,1,3)$ is depicted in Figure 1. Theorem 2.1 and Theorem 2.2 deals with tiara graphs which belongs to $\mathcal{F}(P_h)$.

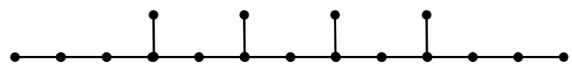

Figure 1. Tiara $Ti_4(1,1,3)$.

Theorem 2.1. *Let h, s be positive integers with $s \geq 2$. For every s being a solution of $h(s)$, the following statements are true.*

a) If $h = 2s+1$, then $Ti_2(s, s-1, s) \in \mathcal{F}(P_h)$.

b) If $h = 2s$, then $Ti_2(s-1, s-1, s) \in \mathcal{F}(P_h)$.

Proof. Let h be fixed. To prove part a) and b) simultaneously we set $G \cong Ti_2(h-s-1, s-1, s)$. Suppose G is P_h-magic with f as a P_h-magic labeling for G. In this proof, define $w_{1,0} = v_1$ and $w_{s+1,0} = v_{s+1}$. Consider $x_{1,s}, v_1, v_{h-s}, w_{1,h-s-1}$. Notice that $h-s-1 \in Dt(v_1, v_{h-s})$, $h-s-1 \in Dt(v_1, w_{1,h-s-1})$ and $(h-s-1)+s = h-1 \in Dt(x_{1,s}, v_{h-s})$. Therefore, by Lemma 2.2 we have

$$\sum_{i=2}^{h-s} f(v_i) + \sum_{i=2}^{h-s} f(v_{i-1}v_i) = \sum_{i=1}^{h-s-1} f(w_{1,i}) + \sum_{i=1}^{h-s-1} f(w_{1,i-1}w_{1,i}). \tag{1}$$

Next, consider $w_{1,h-s-1}$ and $w_{s+1,h-s-1}$. Since $2h-s-2 \in Dt(w_{1,h-s-1}, w_{s+1,h-s-1})$, by Lemma 2.1 (setting $m = h-s-1$) we have

$$\sum_{i=1}^{h-s-1} f(w_{1,i}) + \sum_{i=1}^{h-s-1} f(w_{1,i-1}w_{1,i}) = \sum_{i=1}^{h-s-1} f(w_{s+1,i}) + \sum_{i=1}^{h-s-1} f(w_{s+1,i-1}w_{s+1,i}). \tag{2}$$

Then, consider $x_{s+1,s}, v_{s+1}v_{2s+2-h}, w_{s+1,h-s-1}$. Notice that $h-s \in Dt(v_{s+1}, v_{2s+2-h})$, $h-s \in Dt(v_{s+1}, w_{s+1,h-s-1})$ and $(h-s)+s = h \in Dt(x_{s+1,s}, v_{s+1})$. Therefore, by Lemma 2.2 we have

$$\sum_{i=1}^{h-s-1} f(w_{s+1,i}) + \sum_{i=1}^{h-s-1} f(w_{s+1,i-1}w_{s+1,i}) = \sum_{i=2s+2-h}^{s} f(v_i) + \sum_{i=2s+s-h+1}^{s+1} f(v_{i-1}v_i). \tag{3}$$

From (1), (2) and (3), we have

$$\sum_{i=2}^{h-s} f(v_i) + \sum_{i=2}^{h-s} f(v_{i-1}v_i) = \sum_{i=2s+2-h}^{s} f(v_i) + \sum_{i=2s+2-h+1}^{s+1} f(v_{i-1}v_i).$$

If $h = 2s+1$, this would imply $f(v_1) = f(v_{s+1})$. On the other hand, $h = 2s$ implies $f(v_1v_2) = f(v_sv_{s+1})$. The contradictions of injectivity of f in both cases are implying $Ti_2(h-s-1, s-1, s) \in \mathcal{F}(P_h)$. □

Theorem 2.2. *Let h, s, t be positive integers. For every pair s, t being a solution of $h = s(t+3)+1$, then*

$$Ti_{(t+3)}(s, s-1, s(t+2)) \in \mathcal{F}(P_h).$$

Proof. Let h be fixed. Suppose $G \cong Ti_{(t+3)}(s, s-1, s(t+2))$ is P_h-magic with a magic labeling f. In this proof, define $x_{k,0} = v_k = w_{k,0}$ for every $k \in [1, h-s]$ (note that $h-s = (t+2)s+1$). First, consider $v_{h-s}, v_1, x_{1,s}, w_{1,s}$. Notice that $s \in Dt(v_1, x_{1,s}), s \in Dt(v_1, w_{1,s})$ and $s + s(t+2) = s(t+3) \in Dt(v_{h-s}, x_{1,s})$. Hence, by Lemma 2.2, we have

$$\sum_{i=1}^{s} f(x_{1,i}) + \sum_{i=1}^{s} f(x_{1,i-1}x_{1,i}) = \sum_{i=1}^{s} f(w_{1,i}) + \sum_{i=1}^{s} f(w_{1,i-1}w_{1,i}). \quad (4)$$

Then, considering $w_{1,s}$ and $w_{h-s,s}$ with $s(t+4) \in Dt(w_{1,s}, w_{h-s,s})$ by Lemma 2.1 (setting $m = s$) we have

$$\sum_{i=1}^{s} f(w_{1,i}) + \sum_{i=1}^{s} f(w_{1,i-1}w_{1,i}) = \sum_{i=1}^{s} f(w_{h-s,i}) + \sum_{i=1}^{s} f(w_{h-s,i-1}w_{h-s,i}). \quad (5)$$

Next, consider $v_1, v_{h-s}, x_{h-s,s}, w_{h-s,s}$. We can see that $s \in Dt(v_{h-s}, x_{h-s,s}), s \in Dt(v_{h-s}, w_{h-s,s})$ and $s + s(t+2) = s(t+3) \in Dt(v_1, v_{h-s,s})$. Therefore, by Lemma 2.2 implies

$$\sum_{i=1}^{s} f(w_{h-s,i}) + \sum_{i=1}^{s} f(w_{h-s,i-1}w_{h-s,i}) = \sum_{i=1}^{s} f(x_{h-s,i}) + \sum_{i=1}^{s} f(x_{h-s,i-1}x_{h-s,i}). \quad (6)$$

Combining (4),(5) and (6), we got

$$\sum_{i=1}^{s} f(x_{1,i}) + \sum_{i=1}^{s} f(x_{1,i-1}x_{1,i}) = \sum_{i=1}^{s} f(x_{h-s,i}) + \sum_{i=1}^{s} f(x_{h-s,i-1}x_{h-s,i}). \quad (7)$$

Let $j \in [1, t+1]$. Considering $x_{1,s(t+2-j)}$ and $w_{sj+1,s}$ with $s(t+4) \in Dt(x_{1,s(t+2-j)}, w_{sj+1,s})$, by Lemma 2.1 (setting $m = s$) we have

$$\sum_{i=s(t+1-j)+1}^{s(t+2-j)} f(x_{1,i}) + \sum_{i=s(t+1-j)+1}^{s(t+2-j)} f(x_{1,i-1}x_{1,i}) = \sum_{i=1}^{s} f(w_{sj+1,i}) + \sum_{i=1}^{s} f(w_{sj+1,i-1}w_{sj+1,i}). \quad (8)$$

Similarly, considering $x_{h-s,sj+1}$ and $w_{sj+1,s}$ with $s(t+4) \in Dt(x_{h-s,sj+1}, w_{sj+1,s})$, by Lemma 2.1 (setting $m = s$) we got

$$\sum_{i=1}^{s} f(w_{sj+1,i}) + \sum_{i=1}^{s} f(w_{sj+1,i-1}w_{sj+1,i}) = \sum_{i=sj+1}^{s(j+1)} f(x_{h-s,i}) + \sum_{i=sj+1}^{s(j+1)} f(x_{h-s,i-1}x_{h-s,i}). \tag{9}$$

Combining (8) and (9) for every j yields

$$\sum_{i=s(t+1-j)+1}^{s(t+2-j)} f(x_{1,i}) + \sum_{i=s(t+1-j)+1}^{s(t+2-j)} f(x_{1,i-1}x_{1,i}) = \sum_{i=sj+1}^{s(j+1)} f(x_{h-s,i}) + \sum_{i=sj+1}^{s(j+1)} f(x_{h-s,i-1}x_{h-s,i}). \tag{10}$$

Finally, consider two paths of length h with the consecutive vertices $x_{1,h-s-1}, ..., x_{1,1}, v_1, w_{1,1}, ..., w_{1,s}$ and $x_{h-s,h-s-1}, ..., x_{h-s,1}, v_{h-s}, w_{h-s,1}, ..., w_{h-s,s}$. Since G is P_h-magic, we have

$$\sum_{i=0}^{h-s} f(x_{1,i}) + \sum_{i=1}^{h-s} f(x_{1,i-1}x_{1,i}) + \sum_{i=1}^{s} f(w_{1,i}) + \sum_{i=1}^{s} f(w_{1,i-1}w_{1,i}) \tag{11}$$

$$= \sum_{i=0}^{h-s} f(x_{h-s,i}) + \sum_{i=1}^{h-s} f(x_{h-s,i-1}x_{h-s,i}) + \sum_{i=1}^{s} f(w_{h-s,i}) + \sum_{i=1}^{s} f(w_{h-s,i=1}w_{h-s,i}). \tag{12}$$

Applying (10) for every j in (11), proceeded by (5) and (7), we have

$$f(x_{1,0}) = f(x_{h-s,0})$$

which is a contradiction of f being a P_h-magic labeling. Therefore, $G \in \mathcal{F}(P_h)$. □

Another class of graphs belonging to $\mathcal{F}(P_h)$ are *bandana graphs*. Here, we define *bandana graphs* $G = Bd(p, q, r, n)$ as follows

$$\begin{aligned} V(G) &= \{v_i \mid i \in [1, 2q+1]\} \cup \{x_{b,j}, w_{b,l} \mid b \in \{1, 2q+1\}, j \in [1, r], l \in [1, p]\} \\ &\quad \cup \{y_k \mid k \in [1, n]\}, \\ E(G) &= \{v_iv_{i+1} \mid i \in [1, 2q]\} \cup \{v_bx_{b,1}, x_{b,j}x_{b,j+1} \mid b \in \{1, 2q+1\}, j \in [1, r-1]\} \\ &\quad \cup \{v_bw_{b,1}, w_{b,l}w_{b,l+1} \mid b \in \{1, 2q+1\}, l \in [1, p-1]\} \cup \{v_{q+1}y_1, y_ky_{k+1} \mid k \in [1, n-1]\}. \end{aligned}$$

An example of bandana graph is illustrated in Figure 2. The proceeding theorem are some bandana graphs which belongs to $\mathcal{F}(P_h)$.

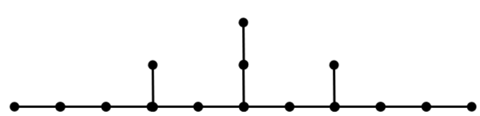

Figure 2. Bandana $Bd(1, 1, 3, 2)$.

Theorem 2.3. *Let h, s, t be positive integers. For every pair s, t being a solution of $h = 4s + t$, then*

$$Bd(2s-1, s, 2s+t, 3s-1) \in \mathcal{F}(P_h)$$

Proof. Let h be fixed. Suppose $G \cong Bd(2s-1, s, 2s+t, 3s-1)$ is P_h-magic with a magic labeling f. In this proof, define $x_{1,0} = v_1 = w_{1,0}$ and $x_{2q+1,0} = v_{2q+1} = w_{2q+1,0}$. First, consider $x_{1,2s+t}, v_1, w_{1,2s-1}, v_{2s}$. We can see that $2s-1 \in Dt(v_1, w_{1,2s-1}), 2s-1 \in Dt(v_1, v_{2s})$ and $(2s+t)+(2s-1) = 4s+t-1 \in Dt(x_{1,2s+t}, w_{1,2s-1})$. Therefore, using Lemma 2.2 yields

$$\sum_{i=2}^{2s} f(v_i) + \sum_{i=2}^{2s} f(v_{i-1}v_i) = \sum_{i=1}^{2s-1} f(w_{1,i}) + \sum_{i=1}^{2s-1} f(w_{1,i-1}w_{1,i}). \tag{13}$$

Then, considering $w_{1,2s-1}$ and $x_{2q+1,2s+t-1}$ with $6s+t-2 \in Dt(w_{1,2s-1}, x_{2q+1,2s+t-1})$, by Lemma 2.1 (and setting $m = 2s-1$) we have

$$\sum_{i=1}^{2s-1} f(w_{1,i}) + \sum_{i=1}^{2s-1} f(w_{1,i-1}w_{1,i}) = \sum_{i=t+1}^{2s+t-1} f(x_{2q+1,i}) + \sum_{i=t+1}^{2s+t-1} f(x_{2q+1,i-1}x_{2q+1,i}). \tag{14}$$

Next, consider $x_{2q+1,2s+t-1}$ and y_{3s-1}. Since $6s+t-2 \in Dt(x_{2q+1,2s+t-1}, y_{3s-1})$, by Lemma 2.1 (and setting $m = 2s-1$) we got

$$\sum_{i=t+1}^{2s+t-1} f(x_{2q+1,i}) + \sum_{i=t+1}^{2s+t-1} f(x_{2q+1,i-1}x_{2q+1,i}) = \sum_{i=s+1}^{3s-1} f(y_i) + \sum_{i=s+1}^{3s-1} f(y_{i-1}y_i). \tag{15}$$

Similarly, considering y_{3s-1} and $x_{1,2s+t-1}$ with $6s+t-2 \in Dt(y_{3s-1}, x_{1,2s+t-1})$, by Lemma 2.1 (and setting $m = 2s-1$) we have

$$\sum_{i=s+1}^{3s-1} f(y_i) + \sum_{i=s+1}^{3s-1} f(y_{i-1}y_i) = \sum_{i=t+1}^{2s+t-1} f(x_{1,i}) + \sum_{i=t+1}^{2s+t-1} f(x_{1,i-1}x_{1,i}). \tag{16}$$

Again, consider $x_{1,2s-t+1}$ and $w_{2q+1,2s-1}$ with $6s+t-2 \in Dt(x_{1,2s-t+1}, w_{2q+1,2s-1})$, by Lemma 2.1 (setting $m = 2s-1$) we got

$$\sum_{i=t+1}^{2s+t-1} f(x_{1,i}) + \sum_{i=t+1}^{2s+t-1} f(x_{1,i-1}x_{1,i}) = \sum_{i=1}^{2s-1} f(w_{2q+1,i}) + \sum_{i=1}^{2s-1} f(w_{2q+1,i-1}w_{2q+1,i}). \tag{17}$$

Finally, consider $x_{2q+1,2s+t}, v_{2q+1}, w_{2q+1,2s-1}, v_2$. Notice that $2s-1 \in Dt(v_{2q+1}, w_{2q+1,2s-1}), 2s-1 \in Dt(v_{2q+1}, v_2)$ and $(2s+t)+(2s-1) = 4s+t-1 \in Dt(x_{2q+1,2s+t}, w_{2q+1,2s-1})$. Hence, using Lemma 2.2 yields

$$\sum_{i=1}^{2s-1} f(w_{2q+1,i}) + \sum_{i=1}^{2s-1} f(w_{2q+1,i-1}w_{2q+1,i}) = \sum_{i=2}^{2s} f(v_i) + \sum_{i=3}^{2s+1} f(v_{i-1}v_i). \tag{18}$$

Solving (13) to (18) we have

$$\sum_{i=2}^{2s} f(v_i) + \sum_{i=2}^{2s} f(v_{i-1}v_i) = \sum_{i=2}^{2s} f(v_i) + \sum_{i=3}^{2s+1} f(v_{i-1}v_i)$$

which implies $f(v_1v_2) = f(v_{2s}v_{2s+1})$. This contradiction of injectivity of f implies $G \in \mathcal{F}(P_h)$. $\square$

3. Unicyclic graphs in $\mathcal{F}(P_h)$

A result of [7] which states that $(n, 1)$-tadpole $\in \mathcal{F}(P_{n+1})$ may be generalized into the following theorem.

Theorem 3.1. *Let $n \geq 3$, $p \geq 1$, and n, p be an integer, and $m = \left\lfloor \frac{n+1}{2} \right\rfloor$.*

a) (n,p)-*tadpole* $\in \mathcal{F}(P_{n+p})$,

b) (n,p)-*tadpole* $\in \mathcal{F}(P_{m+p})$.

Proof. For $n \geq 3, p \geq 1$, let $G \cong (n,p)$-tadpole be a graph that has a vertex set

$$V(G) = \{v_i, w_j \mid i \in [1,n], j \in [1,p]\},$$

and an edge set

$$E(G) = \{w_{j-1}w_j, v_{i-1}v_i \mid i \in [1,p], j \in [1,n]\}$$

with $w_0 = v_1$ and $v_0 = v_n$.

First, we want to prove (n,p)-tadpole $\in \mathcal{F}(P_{n+p})$. Suppose G is a P_{n+p}-magic graph and f is a P_{n+p}-magic labeling of G. By taking P_{n+p} subgraph of G with consecutive vertices $w_p, w_{p-1}, ..., w_1, v_1, v_2, ..., v_n$ and $w_p, w_{p-1}, ..., w_1, v_1, v_n, v_{n-1}, ..., v_2$, we have

$$\sum_{i=1}^{p} f(w_i) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{p-1} f(w_iw_{i+1}) + f(w_1v_1) + \sum_{i=1}^{n-1} f(v_iv_{i+1})$$
$$= \sum_{i=1}^{p} f(w_i) + \sum_{i=1}^{n} f(v_i) + \sum_{i=1}^{p-1} f(w_iw_{i+1}) + f(w_1v_1) + \sum_{i=2}^{n-1} f(v_iv_{i+1}) + f(v_1v_n)$$

this implies $f(v_1v_2) = f(v_1v_n)$ which is a contradiction from a fact that f is injective.

Next, we will show $G \cong (n,p)$-tadpole $\in \mathcal{F}(P_{m+p})$. Suppose G is a P_{m+p}-magic graph. Consider w_p and v_{m+1} with $m+p-1 \in Dt(w_p, v_{m+1})$. Using Lemma 2.1, we have

$$f(w_p) + f(w_{p-1}w_p) = f(v_{m+1}) + f(v_mv_{m+1}). \tag{19}$$

Similarly, considering w_p and v_m with $m+p-1 \in Dt(w_p, v_m)$, applying Lemma 2.1 yields

$$f(w_p) + f(w_{p-1}w_p) = f(v_{n-m+1}) + f(v_{n-m+1}v_{n-m+2}). \tag{20}$$

Therefore, (19) and (20) yields

$$f(v_{m+1}) + f(v_m v_{m+1}) = f(v_{n-m+1}) + f(v_{n-m+1} v_{n-m+2}). \tag{21}$$

Now, divide the problem into cases based on parity of n.

Case 1. n is even
If n is even, let $n = 2i$, then $m = \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{2i+1}{2} \rfloor = i$ implying

$$n - m = m.$$

Plugging this into (21) yields

$$f(v_{m+1}) + f(v_m v_{m+1}) = f(v_{m+1}) + f(v_{m+1} v_{m+2})$$

which implies $f(v_m v_{m+1}) = f(v_{m+1} v_{m+2})$ and this leads to a contradiction.

Case 2. n is odd
If n is odd, let $n = 2i+1$, then $m = \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{2i+2}{2} \rfloor = i+1$ which means

$$n - m + 1 = m.$$

Plugging this into (21) giving us

$$f(v_{m+1}) + f(v_m v_{m+1}) = f(v_m) + f(v_m v_{m+1})$$

implying $f(v_{m+1}) = f(v_m)$ and this also leads to a contradiction. □

In general, most graphs containing cycles belongs to $\mathcal{F}(P_h)$. The proceeding theorem provide some sufficient conditions to determine whether a given graph belongs to $\mathcal{F}(P_h)$.

Theorem 3.2. *Let $h \geq 3, n \geq 2$ and $v_i, i \in [1, n]$ denotes leaves in a given graph G. If these conditions are satisfied for graph G:*

a) $h \in Dt(v_i, v_{i+1})$ *for every* $i \in [1, n]$,

b) $2h - 1 \in Dt(v_1, v_n)$ *or* $2h \in Dt(v_1, v_n)$,

then $G \in \mathcal{F}(P_h)$.

Proof. Suppose G is P_h-magic and has properties as stated in the theorem. For convience, denote e_v as an edge which is incident to a leaf v. For every $i \in [1.n]$, since $h \in Dt(v_i, v_{i+1})$ then there exists a vertex sequence $v_i = x_1, x_2, ..., x_{n+1} = v_{i+1}$ in the graph. Using Lemma 2.1 (setting $m = 1$), we have

$$f(x_1) + f(x_1 x_2) = f(x_{n+1}) + f(x_n x_{n+1})$$
$$f(v_i) + f(e_{v_i}) = f(v_{i+1}) + f(e_{v_{i+1}})$$

for all i. Consequently, iterating i from 1 to $n-1$ yields

$$f(v_1) + f(e_{v_1}) = f(v_n) + f(e_{v_n}). \tag{22}$$

Let $r \in \{2h-1, 2h\}$ such that $r \in Dt(v_1, v_n)$. Then, there exists a vertex sequence $v_1 = y_1, y_2, ..., y_{r+1} = v_n$. Take the subsequence $y_1, y_2, ..., y_{h+1}$ and apply Lemma 2.1 (setting $m = 1$). We have

$$f(y_1) + f(y_1y_2) = f(y_{h+1}) + f(y_hy_{h+1}). \tag{23}$$

Similarly, taking the subsequence $y_{r-h+1}, y_{r-h+2}, ..., y_{r+1}$ and applying Lemma 2.1 (setting $m = 1$ yields

$$f(y_{r-h+1}) + f(y_{r-h+1}y_{r-h+2}) = f(y_{r+1}) + f(y_ry_{r+1}). \tag{24}$$

From (22), (23) and (24), we have

$$\begin{aligned} f(y_{h+1}) + f(y_hy_{h+1}) &= f(y_1) + f(y_1y_2) \\ &= f(v_1) + f(e_{v_1}) \\ &= f(v_n) + f(e_{v_n}) \\ &= f(y_{r+1}) + f(y_ry_{r+1}) \\ &= f(y_{r-h+1}) + f(y_{r-h+1}y_{r-h+2}). \end{aligned}$$

If $r = 2h-1$, then we got

$$f(y_{h+1}) = f(y_h)$$

which will contradicts the injectivity of f. Similarly, if $r = 2h$ we have

$$f(y_hy_{h+1}) = f(y_{r-h+1}y_{r-h+2})$$

which also contradicts the injectivity of f. We conclude that $G \in \mathcal{F}(P_h)$. □

In Figure 3, we give an example of a graph satisfying conditions in Theorem 3.2.

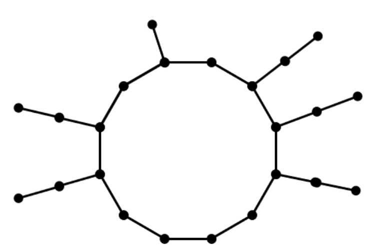

Figure 3. A graph G satisfying condition in Theorem 3.2 for $h = 5$. Hence $G \in \mathcal{F}(P_5)$.

4. Uniqueness of minimal tree in $\mathcal{F}(P_3)$

Let G be H-magic with its H-magic labeling f. Recall that K_2-supermagic graphs is also called edge-supermagic graphs. Enomoto et al. [2] suggests that there exists a supermagic labeling for every given trees.

Conjecture 1. *[2] All trees are edge-supermagic.*

The implication of this conjecture is written as follows.

Remark 4.1. If Conjecture 1 is true, then there does not exist trees in $\mathcal{F}(K_2)$.

Therefore, we want to do similar approach for trying to find trees in $\mathcal{F}(P_3)$. According to Theorem 1.3, $H_1 \in \mathcal{F}(P_3)$. Our goal is to find whether there exists other trees $T \in \mathcal{F}(P_3)$ which does not contain H_1 while also characterizing trees which are P_3-supermagic.
To characterize these trees, we need some theorems that have been established before to be used in our proof. A sufficient condition for trees to have P_h-supermagic has been presented by Maryati et al. [6] with following theorem.

Theorem 4.1. *[6] Let G be a tree that admits P_h-covering for some certain integer $h \geq 2$. If for every subgraph P_h in G contains a fixed vertex c, then G is P_h-supermagic.*

For one class of the tree graph, which is a path, Gutiérrez and Lladó [4] showed a sufficient condition for paths P_n to have P_h-magic with a theorem as follows.

Theorem 4.2. *[4] Let $n \geq 1$ be an integer, then a path P_n is P_h-supermagic for every integer $h \in [2, n]$.*

Next, we start to characterize trees of order $n \in [3, 9]$ which are P_3-supermagic. Some labelings are obtained by using the provided theorems.

Theorem 4.3. *Every tree of order $n \in [3, 9]$ is P_3-supermagic if and only if the tree is H_1-free.*

Proof. The forward direction is just a result from Theorem 1.3 by taking n to be small. To prove the backward direction, we enumerate all trees of order $n \in [3, 9]$ which is H_1-free is P_3-supermagic. All graphs which satisfies the condition is shown to be P_3-supermagic by Figure 4. Hence, the theorem holds.

□

Considering the theorems and results for P_3-(super)magic labeling in these trees, we establish a conjecture and its implication as a closure in this section.

Conjecture 2. *Every H_1-free tree is P_3-(super)magic.*

Remark 4.2. If Conjecture 2 is true, then $T \in \mathcal{F}(P_3)$ implies $H_1 \subseteq T$.

5. Concluding Remarks

For future investigation, there are some problems which we found to be interesting.

Problem 1. *Can Remark 4.2 be shown without using Conjecture 2?*

Problem 2. *What are forbidden subgraphs in $\mathcal{F}(H)$ for other kind of H?*

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References

[1] Darmaji, S. Wahyudi, Rinurwati, and S.W. Saputro, On the construction of super edge-magic total graphs, *Electron. J. Graph Theory Appl.* **10**(1) (2022) 301–309.

[2] H. Enomoto, A. Llado, T. Nakamigawa, and G. Ringel, Super edge magic graphs, *SUT Journal of Mathematics* **34**(2) (1998) 105–109.

[3] J.A. Gallian, A dynamic survey of graph labelings, *Electron. J. Combin.* **14** (2021) #DS6.

[4] A. Gutiérrez and A. Lladó, Magic covering, *J. Combin. Math. Combin. Comput.* **55** (2005) 43–56.

[5] R. Ichishima, F.A. Muntaner-Batle, and A. Oshima, The consecutively super edge-magic deficiency of graphs and related concepts, *Electron. J. Graph Theory Appl.* **8**(1) (2020) 71–92.

[6] T.K. Maryati, E.T. Baskoro, and A.N.M. Salman, P_h-supermagic labelings of some trees, *J. Combin. Math. Combin. Comput.* **65** (2008) 197–204.

[7] T.K. Maryati, E.T. Baskoro, A.N.M. Salman, and Irawati, On the path-(super)magicness of a cycle with some pendants, *Util. Math.* **96** (2015) 319–330.

[8] Y.F. Ashari, A.N.M. Salman, and R. Simanjuntak, On forbidden subgraphs of (K_2, H)-sim-(super)magic graphs, *Symmetry* **13**(8) (2021) 1346.

[9] S. P. Subbiah and J. Pandimadevi, H-E-Super magic decomposition of graphs, *Electron. J. Graph Theory Appl.* **2**(2) (2014) 115–128.

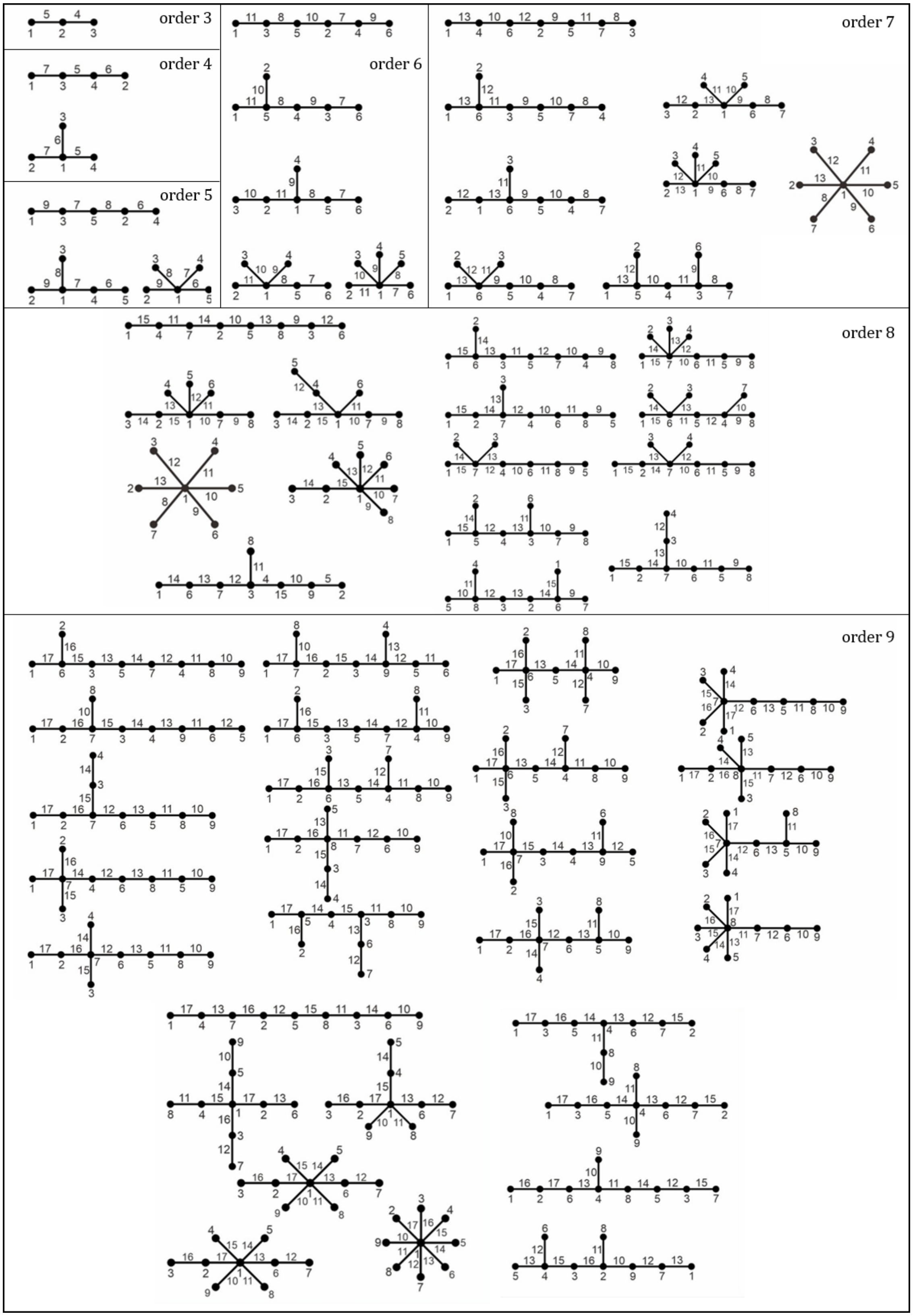

Figure 4. P_3-supermagic trees.