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On the inverse graph of a finite group and its rainbow connection number

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Abstract

A rainbow path in an edge-colored graph G is a path that every two edges have different colors. The minimum number of colors needed to color the edges of G such that every two distinct vertices are connected by a rainbow path is called the rainbow connection number of G. Let $(\Gamma, *)$ be a finite group with $T_\Gamma = \{t \in \Gamma | t \neq t^{-1}\}$. The inverse graph of Γ, denoted by $IG(\Gamma)$, is a graph whose vertex set is Γ and two distinct vertices, u and v, are adjacent if $u * v \in T_\Gamma$ or $v * u \in T_\Gamma$. In this paper, we determine the necessary and sufficient conditions for the inverse graph of a finite group to be connected. We show that the inverse graph of a finite group is connected if and only if the group has a set of generators whose all elements are non-self-invertible. We also determine the rainbow connection numbers of the inverse graphs of finite groups.

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1. Introduction

The concept of the rainbow connection number of a graph was introduced by Chartrand et al. in 2008 [9]. This concept has attracted the attention of many researchers to study. For a graph G, the set of vertices of G is denoted by $V(G)$, and the set of edges of G is denoted by $E(G)$. An *edge coloring* of a graph G is a mapping from $E(G)$ to a set of a finite number of colors. In this paper, we use edge colorings that allow adjacent edges to have the same color. A path of an edge-colored graph is called a *rainbow path* if every two edges of the path have different colors. If every two distinct vertices in $V(G)$ are connected by a rainbow path, then G is called a *rainbow-connected graph*, and its edge coloring is called *rainbow coloring*. If a rainbow coloring uses k colors, it is called the *rainbow k-coloring*. The minimum number of colors needed to color the edges of G such that G is rainbow-connected is called the *rainbow connection number* of G, denoted by $rc(G)$. A rainbow $rc(G)$-coloring of a graph G is called a *minimum rainbow coloring* of G.

Some properties of the rainbow connection number of a graph have been determined in [9]. For a connected graph G, $rc(G) = 1$ if and only if G is a complete graph. If $diam(G)$ is the diameter of G and p is the number of edges of G, then $diam(G) \leq rc(G) \leq p$. If H is a connected spanning subgraph of G, then $rc(G) \leq rc(H)$.

Several researchers have investigated the rainbow connection numbers of some graphs. Chartrand et al. [9] studied the rainbow connection number of complete graphs, trees, wheel graphs, bipartite graphs, and multipartite graphs. Li et al. [16] studied the rainbow connection numbers of line graphs. Fitriani et al. investigated the rainbow connection number of amalgamation of some graphs [13] and comb products of graphs [14].

The rainbow connection number of a graph can be used to measure the security of a communication or computer network modeled by the graph [19]. Among various types of graphs used to model networks, there are graphs of finite groups. The graphs of finite groups have attracted the attention of some researchers in the last decades. Besides modeling networks, through graphs of finite groups, we can study the algebraic structures of groups by using combinatorial properties of graphs. Some of the graphs of finite groups are Cayley graphs [7], commuting graphs [5], non-commuting graphs [2], power graphs of finite groups [8], and enhanced power graphs [1].

One of the graphs of finite groups used to model networks is Cayley graph. Akers et al. [3] used Cayley graphs for designing and analyzing symmetric interconnection networks. They also showed that some symmetry graphs which have been used as processor/communication interconnection networks, such as the ring, n-dimensional Boolean hypercube, and cube-connected cycles, can be represented by Cayley graph models. These facts motivated some researchers to study the rainbow connection numbers of Cayley graphs to measure the security of the networks modeled by the graphs. Several studies related to the rainbow connection numbers of Cayley graphs have been conducted by Li et al. [15], Lu et al. [17], Ma et al. [19], and Bau et al. [6]. Motivated by these studies, some researchers investigated the rainbow connection numbers of the other graphs of finite groups, such as the rainbow connection number of the power graph of a finite group [18], the rainbow connectivity of the non-commuting graph of a finite group [20], and the rainbow connection number of the enhanced power graph [12].

In 2017, Alfuraidan and Zakariya [4] introduced a new graph of a finite group called the inverse graph of a finite group. Given a finite group $(\Gamma, *)$ with $T_\Gamma = \{t \in \Gamma | t \neq t^{-1}\}$. The *inverse graph* of $(\Gamma, *)$, denoted by $IG(\Gamma)$, is a graph whose vertices are the elements of Γ such that two distinct vertices u and v are adjacent if and only if $u * v \in T_\Gamma$ or $v * u \in T_\Gamma$. Alfuraidan and Zakariya have studied some properties of $IG(\Gamma)$ [4]. If $(\Gamma, *)$ is a finite group with $T_\Gamma \neq \emptyset$, then every element of T_Γ is adjacent to the identity element of Γ. The set $E(IG(\Gamma))$ is an empty set if and only if T_Γ is an empty set. They also proved that there is no inverse graph that is complete for any nontrivial finite group.

In the context of networks, one of the interconnection networks, which is not a complete graph, is a partial mesh network, an interconnection network whose not all nodes are connected directly to each other. Since a connected inverse graph of a nontrivial finite group is not a complete graph, we can construct a partial mesh network based on the graph. Hence, determining the rainbow connection number of a connected inverse graph of a finite group is the same as determining the rainbow connection number of the partial mesh network formed based on the graph. Therefore, research on the rainbow connection number of the inverse graphs of finite groups should be meaningful. Figure 1 shows an example of a partial mesh network, which is also the inverse graph of group $\mathbb{Z}_6$.

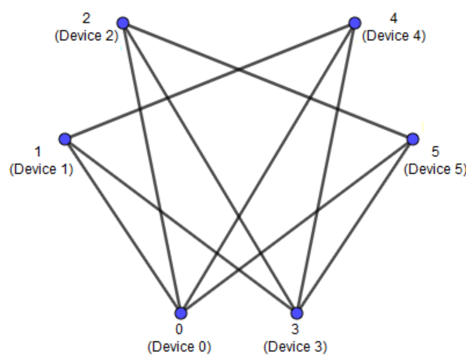

Figure 1. An example of a partial mesh network

Our research continues the research conducted by Alfuraidan and Zakariya. Alfuraidan and Zakariya [4] proved that if the group is abelian, then the inverse graph of the group is connected. In this research, we determine the necessary and sufficient conditions for the inverse graph of a finite group to be connected. We also obtain that if the inverse graph of a finite group is connected, then the group has a minimal set of generators whose all members are non-self-invertible. We also investigate the rainbow connection numbers of the inverse graph of finite groups.

This paper is organized as follows. In Section 2, we present some definitions and properties in Graph Theory and Group Theory that will be used to obtain the main results. In Section 3, we discuss the necessary and sufficient conditions for the inverse graph of a finite group to be connected. We also discuss a property of a minimal set of generators of a group whose inverse graph is connected and some families of groups whose inverse graphs are connected. In section 4, we discuss the rainbow connection numbers of the inverse graphs of finite groups. In the last section, we give some conclusions.

2. Preliminaries

This section presents some definitions and properties in Group Theory and Graph Theory that will be useful in the next sections.

2.1. Some definitions and properties in group theory

Throughout the remaining sections, we refer to [11] for definitions and notations in Group Theory that are not described here. Let $(\Gamma, *)$ be a finite group, where Γ is the set of all elements of the group, and $*$ is the binary operation of the group. The *order* of the group $(\Gamma, *)$, denoted by $|\Gamma|$, is the number of elements of Γ. If $|\Gamma|$ is finite, then $(\Gamma, *)$ is called a *finite group*. For $x \in \Gamma$, the product $x * x * \cdots * x$ (n terms) is denoted by x^n. The *order* of $x \in \Gamma$, denoted by $|x|$, is the smallest positive integer n such that $x^n = e$. According to Lagrange's Theorem, if $(\Gamma, *)$ is a finite group and $(H, *)$ is a subgroup of $(\Gamma, *)$, then the order of $(H, *)$ divides the order of $(\Gamma, *)$. The following theorem is a consequence of Lagrange's Theorem.

Theorem 2.1. *[11] In a finite group, the order of a group element divides the order of its group.*

A subset A of elements of a group $(\Gamma, *)$ is called a *set of generators* of $(\Gamma, *)$ if every element of Γ can be expressed as a (finite) product of some elements of A and their inverses. If A does not contain any other set of generators of $(\Gamma, *)$, then A is called a *minimal set of generators* of $(\Gamma, *)$.

For a group $(\Gamma, *)$, we define $S_\Gamma = \{s \in \Gamma | s = s^{-1}\}$ and $T_\Gamma = \{t \in \Gamma | t \neq t^{-1}\}$. It is clear that $S_\Gamma \cap T_\Gamma = \emptyset$ and $S_\Gamma \cup T_\Gamma = \Gamma$. According to the definition of T_Γ, if an element t is in T_Γ, then t^{-1} is also in T_Γ. Thus, $|T_\Gamma|$ is even. In the remaining sections, the group notation $(\Gamma, *)$ will be written as Γ for simplicity.

2.2. Some definitions and properties in graph theory

All graphs discussed in this paper are simple, undirected, and nontrivial. Throughout the remaining sections, we refer to [10] for definitions and notations in Graph Theory that are not described here. A *path* is a non-empty graph having the set of all vertices $V = \{x_0, x_1, \ldots, x_k\}$ and the set of all edges $E = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$, where $\epsilon_i = x_{i-1}x_i$ and the x_i are all distinct. The *length* of a path is the number of edges in the path. A graph G is called a *connected graph* if any two of its vertices are linked by a path in G. The *distance* of two vertices x and y in a graph G, denoted by $d_G(x, y)$, is the length of the shortest path in G that connects x and y. The greatest distance between any two vertices in G is called the *diameter* of G, denoted by $diam(G)$.

3. Connected Inverse Graph of a Finite Group

In this section, we discuss finite groups whose inverse graphs are connected. We begin with a finite group Γ of odd order with $T_\Gamma \neq \emptyset$.

Theorem 3.1. *Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. If the order of Γ is odd, then $IG(\Gamma)$ is a connected graph.*

Proof. Let Γ be a finite group of odd order. Since the order of a group element divides the order of the group, Γ does not have any non-identity element of order 2. Hence, the only member of S_Γ is e, the identity element of Γ. For every $t \in T_\Gamma$, we get $e * t = t$. Hence, e is adjacent to every $t \in T_\Gamma$ in $IG(\Gamma)$. Thus, $IG(\Gamma)$ is a connected graph. □

It is known that not all finite groups have an odd order. Therefore, we need to determine the necessary and sufficient conditions for an inverse graph of a finite group to be connected. For a finite group Γ with $T_\Gamma \neq \emptyset$, according to the definition of $IG(\Gamma)$, two elements $g_1, g_2 \in \Gamma$ are adjacent in $IG(\Gamma)$ if and only if $g_1 * g_2 \in T_\Gamma$ or $g_2 * g_1 \in T_\Gamma$. In the case of $g_1 * g_2 \in T_\Gamma$, there exists an element $t_1 \in T_\Gamma$ such that $g_1 * g_2 = t_1$. Let r_1 be the inverse of t_1. Then, $g_1 * g_2 = t_1$ if and only if $r_1 * g_1 = g_2^{-1}$. In the case of $g_2 * g_1 \in T_\Gamma$, there exists an element $t_2 \in T_\Gamma$ such that $g_2 * g_1 = t_2$. Let r_2 be the inverse of t_2. Then, $g_2 * g_1 = t_2$ if and only if $g_1 * r_2 = g_2^{-1}$. Thus, two elements $g_1, g_2 \in \Gamma$ are adjacent in $IG(\Gamma)$ if and only if $r_1 * g_1 = g_2^{-1}$ or $g_1 * r_2 = g_2^{-1}$ for $r_1, r_2 \in T_\Gamma$. We use this fact to determine the necessary and sufficient conditions for an inverse graph of a finite group to be connected.

Theorem 3.2. *Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. The inverse graph $IG(\Gamma)$ is connected if and only if T_Γ is a set of generators of Γ.*

Proof. Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. Recall that every t in T_Γ is adjacent to the identity element e. For any s in S_Γ, we get $e * s = s * e = s$. Hence, each s in S_Γ is not adjacent to e.

Let the inverse graph $IG(\Gamma)$ be a connected graph. In order to prove that T_Γ is a set of generators of Γ, it is enough to show that each $s \in S_\Gamma$ can be expressed as a product of some elements of T_Γ. The identity element e can be expressed as $e = t * t^{-1}$, with $t \in \Gamma$. Since t^{-1} is also in T_Γ, then e is a product of elements of T_Γ. Since $IG(\Gamma)$ is a connected graph, there is a path connecting e to every $s \in S_\Gamma \setminus \{e\}$. Choose any $s \in S_\Gamma \setminus \{e\}$. Let the path from e to s is $eg_1g_2 \dots g_{m-1}g_ms$, with $m \geq 1$. Because e is adjacent to g_1, we get $e * g_1 \in T_\Gamma$ or $g_1 * e \in T_\Gamma$. Since $e * g_1 = g_1 * e = g_1$, we get $g_1 \in T_\Gamma$. Next, we use induction to show that if g_i is a product of some elements of T_Γ and g_{i+1} is adjacent to g_i, then g_{i+1} is also a product of some elements of T_Γ for every $i \in \{1, 2, \dots, m-1\}$. Let g_i be a product of some elements of T_Γ and g_{i+1} be adjacent to g_i. Since g_i is adjacent to g_{i+1}, $g_i * g_{i+1} \in T_\Gamma$ or $g_{i+1} * g_i \in T_\Gamma$. In the case of $g_i * g_{i+1} \in T_\Gamma$, there exists an element $r_i \in T_\Gamma$ such that $g_i * g_{i+1} = r_i$. Hence, $g_{i+1} = g_i^{-1} * r_i$. In the case of $g_{i+1} * g_i \in T_\Gamma$, there exists an element $t_i \in T_\Gamma$ such that $g_{i+1} * g_i = t_i$. Hence, $g_{i+1} = t_i * g_i^{-1}$. Because g_i is a product of some elements of T_Γ, g_i^{-1} is also a product of some elements of T_Γ. Therefore, we get that g_{i+1} is a product of some elements of T_Γ. We conclude that g_i is a product of some elements of T_Γ for every $i \in \{1, 2, \dots, m\}$. Because g_m is adjacent to s, $g_m * s \in T_\Gamma$ or $s * g_m \in T_\Gamma$. In the case of $g_m * s \in T_\Gamma$, there exists an element $r_m \in T_\Gamma$ such that $g_m * s = r_m$. Hence, $s = g_m^{-1} * r_m$. In the case of $s * g_m \in T_\Gamma$, there exists an element $t_m \in T_\Gamma$ such that $s * g_m = t_m$. Hence, $s = t_m * g_m^{-1}$. Since g_m is a product of some elements of T_Γ, g_m^{-1} is also a product of some elements of T_Γ. Thus, s is a product of some elements of T_Γ. In conclusion, T_Γ is a set of generators of Γ.

Now let T_Γ be a set of generators of Γ. We have to prove that $IG(\Gamma)$ is a connected graph. Since each $t \in T_\Gamma$ is adjacent to e, it is sufficient to prove that every $s \in S_\Gamma \setminus \{e\}$ is connected to e. Choose any $s \in S_\Gamma \setminus \{e\}$. Since T_Γ is a set of generators of Γ, s can be expressed as

$s = \prod_{i=1}^{k} t_i = t_1 * t_2 * \cdots * t_k$, where $t_i \in T_\Gamma$ for every $i \in \{1, \ldots, k\}$ and $2 \leq k \leq |T_\Gamma|$. Recall that $a, b \in \Gamma$ are adjacent in $IG(\Gamma)$ if and only if $r_1 * a = b^{-1}$ for an element $r_1 \in T_\Gamma$ or $a * r_2 = b^{-1}$ for an element $r_2 \in T_\Gamma$. Since $s = t_1 * t_2 * \cdots * t_{k-1} * t_k$, we get $s^{-1} = t_k^{-1} * t_{k-1}^{-1} * \cdots * t_1^{-1}$. Now write $t_{k-1}^{-1} * t_{k-2}^{-1} * \cdots * t_1^{-1} = g_1$. Hence, we get $s^{-1} = t_k^{-1} * g_1$. Thus, s is adjacent to g_1. If $k = 2$, then $g_1 = t_1^{-1} \in T_\Gamma$. Therefore, g_1 is adjacent to e and s is connected to e. If $k \geq 3$, we use the iterative steps in Algorithm 1 below to construct g_i for $i \in \{2, \ldots, k-1\}$.

Algorithm 1 Constructing g_i for $i \in \{2, \ldots, k-1\}$

```
Input: t_1, ..., t_{k-1}
Output: g_2, ..., g_{k-1}
Set l_1 = 1, l_2 = 2, m_1 = k - 1, m_2 = k - 1
for each integer i in {2, ..., k-1} do
    if i is odd then
        g_i = ∏_{j=l_1}^{m_1} t_j^{-1}
        l_1 = l_1 + 1
        m_1 = m_1 - 1
    else
        g_i = ∏_{j=l_2}^{m_2} t_j
        l_2 = l_2 + 1
        m_2 = m_2 - 1
```

At the end of the iterative steps, we get $g_{k-1} = t_{k/2}^{-1}$ if k is even or $g_{k-1} = t_{\lceil k/2 \rceil}$ if k is odd. Hence, g_{k-1} is adjacent to e. We also get $g_i^{-1} = t * g_{i+1}$ or $g_i^{-1} = t^{-1} * g_{i+1}$, where t is an element of T_Γ, for $i \in \{1, \ldots, k-2\}$. Hence, g_i is adjacent to g_{i+1} for $i \in \{1, \ldots, k-2\}$. Thus, the path $eg_{k-1} \cdots g_1 s$ connects s to e. We conclude that $IG(\Gamma)$ is a connected graph. □

It has been mentioned that if Γ is a group of odd order, then $S_\Gamma = \{e\}$, where e is the identity element of Γ. The identity e can be expressed as $e = t * t^{-1}$ for any $t \in T_\Gamma$. Therefore, T_Γ is a set of generators of Γ. Hence, a group of odd order, whose inverse graph is connected, satisfies Theorem 3.2. Theorem 3.2 also gives us the following corollary.

Corollary 3.1. *Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. The inverse graph $IG(\Gamma)$ is connected if and only if Γ has a minimal set of generators whose all members are non-self-invertible.*

Proof. Let Γ be a finite group with a connected inverse graph $IG(\Gamma)$. According to Theorem 3.2, T_Γ is a set of generators of Γ. Thus, Γ has a subset of T_Γ as its minimal set of generators. Therefore, all elements of the minimal set of generators are non-self-invertible in Γ. Conversely, let Γ has a minimal set of generators whose all elements are non-self-invertible. Clearly, the minimal set of generators is a subset of T_Γ. Hence, T_Γ is also a set of generators of Γ. According to Theorem 3.2, $IG(\Gamma)$ is connected. □

Theorem 3.2 and Corollary 3.1 give us an interesting fact. If a finite group has a connected inverse graph, then we can ensure that the group has at least one minimal set of generators whose all members are non-self-invertible elements. If the inverse graph of a finite group is not connected, then the group cannot have a minimal set of generators whose all members are non-self-invertible.

The following examples show some applications of Theorem 3.2 and Corollary 3.1.

Example 1. Consider the symmetric group $Sym(4)$, the group of all permutations of 4 objects. This group has 24 elements. The subset $S_{Sym(4)}$ ={(1), (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23)}, where (1) is the identity permutation, is the set of all self-invertible elements of $Sym(4)$. The set $T_{Sym(4)} = Sym(4) \setminus S_{Sym(4)}$ is the set of all non-self-invertible elements of $Sym(4)$. In a group of permutations, the binary operation is the composition of two permutations. If we compose each non-identity permutation in $S_{Sym(4)}$ with some elements of $T_{Sym(4)}$, the results are also an element of $T_{Sym(4)}$. For example, $(234)(12) = (1342)$, $(234)(13) = (1423)$, $(234)(14) = (1234)$, $(341)(23) = (1324)$, $(341)(24) = (1342)$, $(412)(34) = (1243)$, $(412)((12)(34)) = (143)$, $(412)((13)(24)) = (132)$, and $(412)((14)(23)) = (234)$. Therefore, each non-identity element in $S_{Sym(4)}$ is adjacent to at least one element in $T_{Sym(4)}$. Since every element in $T_{Sym(4)}$ is adjacent to the identity, the inverse graph $IG(Sym(4))$ is connected. Since $IG(Sym(4))$ is connected, according to Theorem 3.2 and Corollary 3.1, $T_{Sym(4)}$ is a set of generators of $Sym(4)$, and we can ensure that $Sym(4)$ has a minimal set of generators whose all elements are non-self-invertible.

Example 2. The dihedral group $D_{2n} = \langle r, s : r^n = s^2 = e, srs = r^{-1}\rangle$, with $n \geq 3$, is an example of a group with no set of generators whose all elements are non-self-invertible. All non-self-invertible elements of the group have the form r^i, where $i \in \{1, 2, \ldots, n\}$ and $i \neq n/2$ if n is even. The element $s \in D_{2n}$ cannot be expressed as a product of some finite r^i or their inverses. Therefore, the inverse graph of D_{2n} is not connected according to Theorem 3.2. Figure 2 shows the inverse graph of group D_6.

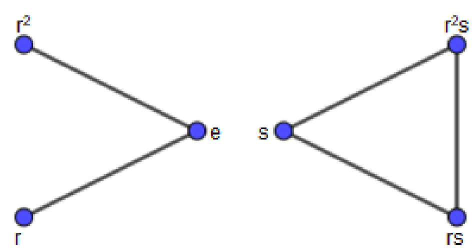

Figure 2. The inverse graph of group D_6

Theorem 3.2 can help us determine which families of groups have connected inverse graphs. Some of them are presented in the following corollaries.

Corollary 3.2. *Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. If for every $s \in S_\Gamma$ there exist some $t \in T_\Gamma$ such that $s * t \neq t^{-1} * s$, then $IG(\Gamma)$ is a connected graph.*

Proof. Let Γ be a finite group with $T_\Gamma \neq \emptyset$ and for every $s \in S_\Gamma$ there exist some $t \in T_\Gamma$ such that $s * t \neq t^{-1} * s$. Choose any $s \in S_\Gamma$. Since $t^{-1} * s = (s * t)^{-1}$, we get $s * t \neq (s * t)^{-1}$ for some $t \in T_\Gamma$. Hence, $s * t$ is a member of T_Γ for some $t \in T_\Gamma$. Therefore, there exists a $t' \in T_\Gamma$ such that $s * t = t'$. As a consequence, $s = t' * t^{-1}$, which is a product of two members of T_Γ. So, we get that T_Γ is a set of generators of Γ. According to Theorem 3.2, the inverse graph $IG(\Gamma)$ is connected. □

Corollary 3.3. *Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. If for every $s \in S_\Gamma$ there exist some $t \in T_\Gamma$ such that $s * t = t * s$, then $IG(\Gamma)$ is a connected graph.*

Proof. Let Γ be a group of finite order with $T_\Gamma \neq \emptyset$. If for every $s \in S_\Gamma$ there exist some $t \in T_\Gamma$ such that $s * t = t * s$, then for every $s \in S_\Gamma$, $s * t \neq t^{-1} * s$ for some $t \in T_\Gamma$. Thus, according to Corollary 3.2, $IG(\Gamma)$ is a connected graph. □

It has been proven in [4] that the inverse graph of an abelian group is connected. This result conforms to Corollary 3.3 since an abelian group satisfies $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. The following example shows a group that satisfies Corollary 3.2.

Example 3. Consider the group $A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$, which is the alternating group whose all members are even permutations of four objects. Note that $S_{A_4} = \{(1), (12)(34), (13)(24), (14)(23)\}$ and $T_{A_4} = A_4 \setminus S_{A_4}$. We get that $|S_{A_4}| = 4$ and $|T_{A_4}| = 8$. This group satisfies $s * t \neq t^{-1} * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. Hence, every $s \in S_\Gamma$ is adjacent to every $t \in T_\Gamma$. The inverse graph $IG(A_4)$ can be seen in Figure 3.

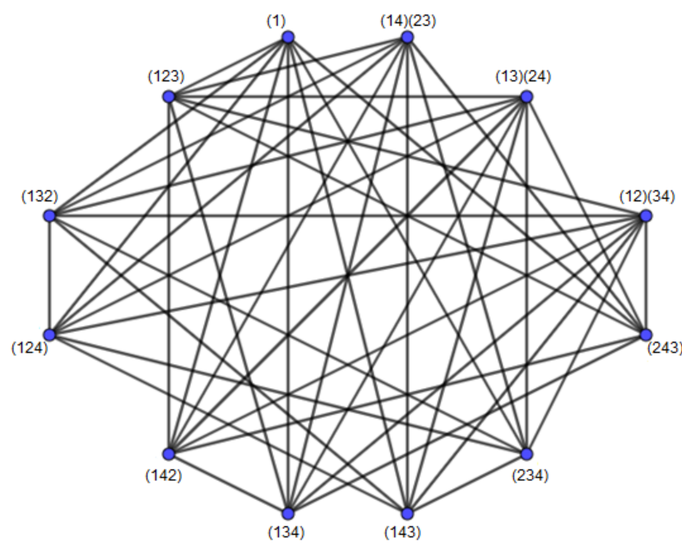

Figure 3. The inverse graph of group A_4

In the next section, we discuss the rainbow connection numbers of the inverse graphs of finite groups. Before determining the rainbow connection number of an inverse graph of a finite group, we have to ensure that the graph is connected since the rainbow connection number is defined only for a connected graph.

4. Rainbow Connection Numbers of the Inverse Graphs of Finite Groups

Now we discuss the rainbow connection numbers of the inverse graphs of finite groups. All groups discussed in this section have some non-self-invertible elements. We begin with the inverse graph of a group of odd order.

Theorem 4.1. *If Γ is a group of odd order with $T_\Gamma \neq \emptyset$, then $rc(IG(\Gamma)) = 2$.*

Proof. Let Γ be a group of odd order with $T_\Gamma \neq \emptyset$. Clearly, Γ is a nontrivial group. According to Theorem 3.1, $IG(\Gamma)$ is connected and $S_\Gamma = \{e\}$, where e is the identity of Γ. Hence, for every $t \in T_\Gamma$, t^{-1} is the only element of Γ which is not adjacent to t in $IG(\Gamma)$. Recall that if Γ is a nontrivial finite group, then $IG(\Gamma)$ is not a complete graph. Therefore, we get that $rc(IG(\Gamma)) \geq 2$. Now write $T_\Gamma = \{t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_{|T_\Gamma|/2}, t_{|T_\Gamma|/2}^{-1}\}$. Next, we color the edge et_i with color 1 and the edge et_i^{-1} with color 2 for every $i \in \{1, 2, \ldots, (|T_\Gamma|)/2\}$, and the other edges of $IG(\Gamma)$ are colored with color 2. Under this coloring, every two distinct vertices of $IG(\Gamma)$ are connected by a rainbow path. Hence, this coloring is a rainbow coloring with two colors, and we get that $rc(IG(\Gamma)) \leq 2$. Since $rc(IG(\Gamma)) \geq 2$, we conclude that $rc(IG(\Gamma)) = 2$. □

Theorem 4.1 gives an exact value for $rc(IG(\Gamma))$ if Γ is any group of odd order with $T_\Gamma \neq \emptyset$. As we already know, for any group Γ of odd order with identity element e, $S_\Gamma = \{e\}$ and $e*t = t*e = t$ for every $t \in T_\Gamma$. For a group Γ of even order which satisfies $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, we obtain the following result.

Theorem 4.2. *Let Γ be a group of even order with $T_\Gamma \neq \emptyset$. If $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$.*

Proof. Let Γ be a finite group of even order with $T_\Gamma \neq \emptyset$, and $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. Since $\Gamma = S_\Gamma \cup T_\Gamma$ and $|T_\Gamma|$ is even, $|S_\Gamma|$ is also even, and hence $|S_\Gamma| > 1$. From Corollary 3.3, we get that $IG(\Gamma)$ is connected. Since $t * s \neq t^{-1} * s = (s * t)^{-1}$, we get $s*t \neq (s*t)^{-1}$, and hence $s*t \in T_\Gamma$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. It means that every $s \in S_\Gamma$ is adjacent to every $t \in T_\Gamma$. Consequently, there exists an element $t' \in T_\Gamma$ for each $s \in S_\Gamma$ and each $t \in T_\Gamma$ such that $s * t = t'$. Suppose that $|S_\Gamma| > |T_\Gamma|$. Then there exist two distinct elements s_1 and s_2 in S_Γ such that $s_1 * t = s_2 * t$ for an element $t \in T_\Gamma$. According to the cancellation law, we get $s_1 = s_2$, which is a contradiction. Thus, $|S_\Gamma|$ cannot be greater than $|T_\Gamma|$.

Choose any $s \in S_\Gamma$ and any $t \in T_\Gamma$. Since $s * t = t' = t * s$ with t' is an element in T_Γ, we get $t * (t')^{-1} = s$ and $(t')^{-1} * t = s$. Write $(t')^{-1} = t''$. Hence, there exists an element $t'' \in T_\Gamma$ which satisfies $t * t'' = t'' * t = s$.

Let the edges of $IG(\Gamma)$ be colored using two different colors in $W = \{1, 2\}$. For every pair of vertices $t_1, t_2 \in T_\Gamma$ and $s_1, s_2 \in S_\Gamma$ which are adjacent in $IG(\Gamma)$, the edges t_1t_2 and s_1s_2 is colored by one of the two colors in W. For every $s_i \in S_\Gamma$, $i \in \{1, 2, \ldots, |S_\Gamma|\}$, we associate an ordered $|T_\Gamma|$-tuple code $C(s_i) = (c_{i1}, c_{i2}, \ldots, c_{i|T_\Gamma|})$ called the color code of s_i, where c_{ij} is the color of the edge s_it_j with $t_j \in T_\Gamma$ for every $j \in \{1, 2, \ldots, |T_\Gamma|\}$. As mentioned before, for each $s_i \in S_\Gamma$ and each $t_j \in T_\Gamma$, there exists an element $t''_j \in T_\Gamma$ such that $t_j * t''_j = t''_j * t_j = s_i$. Hence, we get $s_i * t_j = (t''_j)^{-1} \in T_\Gamma$ and $t''_j * s_i = t_j^{-1} \in T_\Gamma$. Thus, s_i is adjacent to t_j and t''_j. If $t_j \neq t''_j$, then t_j is not adjacent to t''_j. In the case of $t_j \neq t''_j$, if the edge s_it_j is colored by color 1, then the edge $s_it''_j$

must be colored by color 2, and vice versa. In the case of $t_j = t_j''$, if the edge $s_i t_j$ is colored by color 1, then the edge $s_i t_j^{-1}$ is colored by color 2, and vice versa. Therefore, the number of distinct color codes of the elements of S_Γ is at most $2^{|T_\Gamma|/2}$. Since $|S_\Gamma|$ and $|T_\Gamma|$ are even, if $|S_\Gamma| \leq |T_\Gamma|$, then $|S_\Gamma| \leq 2^{|T_\Gamma|/2}$. Therefore, each $s_i \in S_\Gamma$ may have a unique color code. Consequently, every two distinct elements of S_Γ may have different color codes. Under this coloring, the rainbow paths between two distinct vertices of $IG(\Gamma)$ are as follows:

1. for any $s \in S_\Gamma$ and any $t \in T_\Gamma$, the rainbow path between s and t is the edge st,
2. for any two distinct elements $t_i, t_j \in T_\Gamma$ that are not adjacent in $IG(\Gamma)$, the rainbow path between t_i and t_j is $t_i s t_j$, where s is an element of S_Γ such that $t_i * t_j = s$,
3. for any two distinct elements $s_i, s_k \in S_\Gamma$ that are not adjacent in $IG(\Gamma)$, the rainbow path between s_i and s_k is $s_i t_j s_k$, where t_j is an element of T_Γ such that $c_{ij} \neq c_{kj}$,
4. for any two distinct elements $t_i, t_j \in T_\Gamma$ that are adjacent in $IG(\Gamma)$, the rainbow path between t_i and t_j is the edge $t_i t_j$,
5. for any two distinct elements $s_i, s_j \in S_\Gamma$ that are adjacent in $IG(\Gamma)$, the rainbow path that connects s_i and s_j is the edge $s_i s_j$.

Thus, we get that every pair of distinct vertices of $IG(\Gamma)$ is connected by a rainbow path and $rc(IG(\Gamma)) \leq 2$. Since $rc(IG(\Gamma) \geq 2$, we conclude that $rc(IG(\Gamma)) = 2$. □

Based on Theorem 4.2, we can determine the rainbow connection number of the inverse graph of an abelian group of even order.

Corollary 4.1. *If Γ is an abelian group of even order with $T_\Gamma \neq \emptyset$, then $rc(IG(\Gamma)) = 2$.*

Proof. Let Γ be an abelian group of even order with $T_\Gamma \neq \emptyset$. Since Γ is abelian, we get $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. By using Theorem 4.2, we get $rc(IG(\Gamma)) = 2$. □

Theorem 4.2 holds for a group Γ of even order that satisfies $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$. It is known that not all groups meet this condition. For a finite group Γ of even order where some $s \in S_\Gamma$ are not adjacent to some $t \in T_\Gamma$, there might be a pair of non-adjacent elements $s_1, s_2 \in S_\Gamma$ that are adjacent to different elements of T_Γ. Hence, the distance between s_1 and s_2, and also the diameter of $IG(\Gamma)$, might be greater than 2. Since $rc(IG(\Gamma))$ is greater than or equal to the diameter of $IG(\Gamma)$, the value of $rc(IG(\Gamma))$ might be greater than 2. Therefore, we need to determine the bounds for the rainbow connection number of a connected inverse graph of any finite group of even order.

Theorem 4.3. *Let Γ be a group of even order and $T_\Gamma \neq \emptyset$ be a set of generators of Γ. Then*

$$2 \leq rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$$

where m is the number of $s \in S_\Gamma$ that satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$. Furthermore, the lower bound is tight.

Proof. Let Γ be a group of even order and $T_\Gamma \neq \emptyset$ be a set of generators of Γ. Recall that if the order of Γ is even, then $|S_\Gamma|$ is also even, and hence $|S_\Gamma| > 1$. According to Theorem 3.2, since T_Γ is a set of generators of Γ, the inverse graph $IG(\Gamma)$ is connected. Because Γ is a nontrivial group, $IG(\Gamma)$ is not a complete graph. Therefore, we get $rc(IG(\Gamma)) \geq 2$. It is also known that in $IG(\Gamma)$, every element of T_Γ is adjacent to the identity element e and every element of S_Γ is not adjacent to e. Since $IG(\Gamma)$ is connected, there exist some $s \in S_\Gamma$ which is adjacent to some $t \in T_\Gamma$. Therefore, there exist some $s \in S_\Gamma$ such that $s * t \neq t^{-1} * s$ for some $t \in T_\Gamma$.

Let $S'_\Gamma = \{s \in S_\Gamma | s * t \neq t^{-1} * s \text{ for some } t \in T_\Gamma\}$ and $S''_\Gamma = S_\Gamma \setminus S'_\Gamma = \{s \in S_\Gamma | s * t = t^{-1} * s$ for all $t \in T_\Gamma\}$. According to its definition, every member of S'_Γ is adjacent to some elements of T_Γ and their inverse elements in $IG(\Gamma)$. Hence, every member of S'_Γ is connected to e in $IG(\Gamma)$ because every element of T_Γ is adjacent to e. Every $s \in S''_\Gamma$ satisfies $s * t = t^{-1} * s = (s * t)^{-1}$ for all $t \in T_\Gamma$. Hence, every $s \in S''_\Gamma$ is not adjacent to all $t \in T_\Gamma$ in $IG(\Gamma)$. Because $e * s = s * e = s$ for every $s \in S''_\Gamma$, all $s \in S''_\Gamma$ are not adjacent to e. Since $IG(\Gamma)$ is a connected graph, every element of S''_Γ is connected to all elements of S'_Γ in $IG(\Gamma)$.

Let $|S''_\Gamma| = m$. We color the edges of $IG(\Gamma)$ as follows:

1. for each $t_i \in T_\Gamma$ with $i \in \{1, 2, \ldots |T_\Gamma|\}$, the edge et_i are colored by color i,
2. for every $s \in S'_\Gamma$ and $t \in T_\Gamma$ which is adjacent to s, the edge st is colored by color $|T_\Gamma| + 1$ and the edge st^{-1} is colored by color $|T_\Gamma| + 2$,
3. for every path P that connects an $s_\alpha \in S''_\Gamma$ to an $s \in S'_\Gamma$, where $\alpha \in \{1, 2, \ldots, m\}$, the edge $\bar{s}s_\alpha$ in P is colored by color $|T_\Gamma| + 2 + \alpha$, where $\bar{s} = s$ or $\bar{s}$ is a member of S''_Γ whose distance from s on P is $d_P(s_\alpha, s)$–1,
4. the other edges in $IG(\Gamma)$ are colored by one of the colors above.

By using this edge coloring, the edges of $IG(\Gamma)$ are colored by $|T_\Gamma| + m + 2$ colors and every two distinct vertices of $IG(\Gamma)$ are connected by a rainbow path. Thus, $rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$. According to Theorem 4.2, if $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$. Therefore, the lower bound is tight. □

Theorem 4.3 leads us to obtain the upper and lower bounds for the rainbow connection number of the inverse graph of a direct product of some finite groups. The following corollary gives the result.

Corollary 4.2. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a direct product of finite groups with $S_\Gamma = S_{\Gamma_1} \times \ldots \times S_{\Gamma_n}$, $T_\Gamma = \Gamma \setminus S_\Gamma$, and $|\Gamma_i|$ is even for some $i \in \{1, 2, \ldots, n\}$. If for every $i \in \{1, 2, \ldots, n\}$, T_{Γ_i} is not empty and generates Γ_i, then $2 \leq rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$, where m is the number of $s \in S_\Gamma$ satisfying $s * t = t^{-1} * s$ for all $t \in T_\Gamma$. Furthermore, the lower bound is tight.*

Proof. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a direct product of finite groups, $S_\Gamma = S_{\Gamma_1} \times \cdots \times S_{\Gamma_n}$, $T_\Gamma = \Gamma \setminus S_\Gamma$, e_i is the identity element of Γ_i, $|\Gamma_i|$ is even for some $i \in \{1, 2, \ldots, n\}$, and $T_{\Gamma_i} \neq \emptyset$ be a set of generators of Γ_i for every $i \in \{1, 2, \ldots, n\}$. Obviously, $|\Gamma|$ is even. Since T_{Γ_i} is not empty for every $i \in \{1, 2, \ldots, n\}$, we get $T_{\Gamma_1} \times \cdots \times T_{\Gamma_n}$ is not empty. It is obvious that $T_{\Gamma_1} \times \cdots \times T_{\Gamma_n}$ is a subset of $T_\Gamma = \Gamma \setminus S_\Gamma$. Therefore, T_Γ is not empty. Consider a subset $\bar{T}_\Gamma = \{(\gamma_1, \ldots, \gamma_n) \in T_\Gamma | \gamma_i \in T_{\Gamma_i}$ for exactly one $i \in \{1, 2, \ldots, n\}$, and $\gamma_j = e_j$ for $j \in \{1, 2, \ldots, n\}$ where $j \neq i\}$. Each member of $\bar{T}_\Gamma$ is in the form $(t_1, e_2, \ldots, e_n)$, $(e_1, e_2, \ldots, t_n)$, or $(e_1, \ldots, e_{i-1}, t_i, e_{i+1}, \ldots, e_n)$, where $t_i \in T_{\Gamma_i}$

for $i \in \{1, 2, \ldots, n\}$. Since T_{Γ_i} generates Γ_i, every element of Γ_i is a finite product of some elements of T_{Γ_i}. Hence, every element of Γ is a finite product of some elements of $\bar{T}_\Gamma$. Therefore, $\bar{T}_\Gamma$ is a set of generators of Γ. It is clear that $\bar{T}_\Gamma$ is a subset of T_Γ. Thus, we get that T_Γ is also a set of generators of Γ, and $IG(\Gamma)$ is a connected graph.By applying Theorem 4.3, we get $2 \leq rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$, where m is the number of $s \in S_\Gamma$ which satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$, and the lower bound is tight. □

5. Conclusion

In this paper, we presented some findings on the inverse graph of a finite group and its rainbow connection number. For a group Γ of finite order, the inverse graph $IG(\Gamma)$ is connected if and only if Γ has a minimal set of generators whose all members are non-self-invertible. If Γ is a group of odd order, then $rc(IG(\Gamma)) = 2$. If Γ is a group of even order, $T_\Gamma \neq \emptyset$, and $s * t = t * s$ for every $s \in S_\Gamma$ and every $t \in T_\Gamma$, then $rc(IG(\Gamma)) = 2$. For any group Γ of even order whose set of generators is T_Γ, we obtained that $2 \leq rc(IG(\Gamma)) \leq |T_\Gamma| + m + 2$, where m is the number of $s \in S_\Gamma$ which satisfy $s * t = t^{-1} * s$ for all $t \in T_\Gamma$, and the lower bound is tight.

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