Electronic Journal of Graph Theory and Applications **11** (1) (2023), 263–271

Electronic Journal of Graph Theory and Applications

Further results on local inclusive distance vertex irregularity strength of graphs

Fawwaz Fakhrurrozi Hadiputra[a], Eunike Setiawan[b], Tita Khalis Maryati[c], Denny Riama Silaban[d,e]

[a] *Master Program of Mathematics, Institut Teknologi Bandung, Bandung - Indonesia*
[b] *Master Program in Pure Mathematics, Imperial College London - United Kingdom*
[c] *Department of Mathematics Education, Universitas Islam Negeri Syarif Hidayatullah Jakarta, Ciputat - Indonesia*
[d] *Department of Mathematics, Universitas Indonesia, Depok - Indonesia*
[e] *Center for Research Collaboration on Graph Theory and Combinatorics - Indonesia*

fawwazfh@alumni.ui.ac.id, eunike.setiawan22@imperial.ac.uk, tita.khalis@uinjkt.ac.id, denny@sci.ui.ac.id

Abstract

Let $G = (V, E)$ be a simple undirected graph. A labeling $f : V(G) \to \{1, \dots, k\}$ is a local inclusive d-distance vertex irregular labeling of G if every adjacent vertices $x, y \in V(G)$ have distinct weights, with the weight $w(x), x \in V(G)$ is the sum of every labels of vertices whose distance from x is at most d. The local inclusive d-distance vertex irregularity strength of G, lidis(G), is the least number k for which there exists a local inclusive d-distance vertex irregular labeling of G. In this paper, we prove a conjecture on the local inclusive d-distance vertex irregularity strength for $d = 1$ for tree and we generalize the result for block graph using the clique number. Furthermore, we present several results for multipartite graphs and we also observe the relationship with chromatic number.

Keywords: Irregularity strength, tree, block graphs, multipartite graphs
Mathematics Subject Classification : 05C78, 05C05
DOI: 10.5614/ejgta.2023.11.1.21

1. Introduction

Let G be undirected and simple graph. For a vertex set $U \subseteq V(G)$, the set $N[U]$ is the inclusive neighborhood of U. If $U = \{v\}$, we simply write $N[v]$. Let $\omega(G)$ be the clique number of G and $\chi(G)$ be the chromatic number of G.

Received: 2 July 2022, Revised: 8 January 2023, Accepted: 4 March 2023.

Initially, Chartrand et al. [3] introduced a labeling $f : E(G) \to \{1, \ldots, k\}$, with k positive integer, to be *irregular labeling* if for every distinct $u, v \in V(G)$, $w(u) \neq w(v)$, with $w(u)$ is the sum of all labels in edges incident to u. The least number k satisfying the condition above is called *irregularity strength* of G. Some results regarding to irregularity strength of graphs may be seen in [5].

Then, Slamin [6] introduced a variant of labeling called distance irregular labeling. A labeling $f : V(G) \to \{1, \ldots, k\}$, for positive integer k, is a *distance irregular labeling* if for every distinct $u, v \in V(G)$, $w(u) \neq w(v)$, with $w(u) = \sum_{v \in N(u)} f(v)$. The minimum integer k satisfying such labeling f is called *distance vertex irregularity strength* of G. Then Bong et al. [2] generalizes the labeling into two version, inclusive d-distance irregular labeling and non-inclusive d-distance irregular labeling. If $d = 1$, then non-inclusive 1-distance irregular labeling is the same as the one introduced by Slamin [6]. Meanwhile, the weight of $u \in V(G)$ in inclusive 1-distance irregular labeling is $w(u) = f(u) + \sum_{v \in N(u)} f(v)$ (or simply put, $w(u) = \sum_{v \in N[u]} f(v)$). Some results regarding to non-inclusive and inclusive 1-distance irregular labeling can be seen in [1, 7].

Sugeng et al. [8] introduced a weaker version of inclusive d-distance irregular labeling. A labeling $f : V(G) \to \{1, \ldots, k\}$, with positive integer k, is *local inclusive d-distance irregular labeling* of a graph G if all two adjacent vertices $u, v \in V(G)$ have distinct weights, with the weight $w(u), u \in V(G)$ is the sum of every labels of vertices whose distance from u is at most d. The least number k for which there exists a local inclusive d-distance vertex irregular labeling of G is called the local inclusive d-distance vertex irregularity strength of G, denoted by lidis(G). Other authors refer local inclusive d-distance vertex irregular labeling as inclusive lucky labeling when $d = 1$ [4]. Throughout this paper, we discuss local inclusive d-distance vertex irregular strength of G for $d = 1$. For convenience, we always refer local inclusive distance vertex irregularity strength as LIDIS.

If there does not exists k for a local inclusive 1-distance irregular labeling for G then it is defined lidis$(G) = \infty$. A characterization of G with lidis$(G) = \infty$ have been determined by [8] written in Theorem 1.1. For convenience, we say graph G to be *locally irregular* if every two adjacent vertices have different degrees, otherwise the graph is called *non locally irregular*. Using this notation, we write this characterization result of [8] in Theorem 1.2. In Figure 1 we give an example of a local inclusive distance vertex irregular labeling on an locally irregular graph G. Since lidis(G) is a positive integer, we also have Corollary 1.1.

Theorem 1.1 ([8])**.** *For a graph G, it holds that lidis$(G) = \infty$ if and only if there exists an edge $uv \in E(G)$ such that $N[u] = N[v]$.*

Theorem 1.2 ([8])**.** *Let G be a graph. Then lidis$(G) = 1$ if and only if G is locally irregular.*

Corollary 1.1 ([8])**.** *Let G be a graph. Then lidis$(G) \geq 2$ if and only if G is non locally irregular.*

In section 2, we present a proof of conjecture given by [8] whether the value of lidis(T) would only yield 1 or 2 if T is a tree, a special case of block graph. Moreover, we improve the results of [8] regarding to LIDIS of complete multipartite graphs $K_{n_1, n_2, \ldots, n_m}$. In addition, we also determine the strength of certain bipartite graphs.

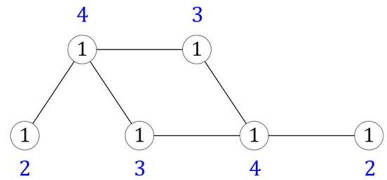

Figure 1. Local inclusive distance vertex irregular labeling on an locally irregular graph G with lidis$(G) = 1$.

2. Local Inclusive Distance Vertex Irregularity Strength of Block Graphs

A conjecture has been stated in [8] as follows.

Conjecture 1. *[8] For arbitrary tree T with $T \neq K_2$, lidis$(T) = 1$ or 2.*

We confirm this conjecture to be true by the following theorem.

Theorem 2.1. *For any tree T other than K_2, we have*

$$lidis(T) = \begin{cases} 1, & \text{if } T \text{ is locally irregular}, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. If G is locally irregular, then according to Theorem 1.2 lidis$(G) = 1$. Otherwise, observe the following algorithm.

1. Pick any vertex $s \in V(T)$.
2. Label vertex s and all vertices adjacent to s with 1.
3. For every vertex v having a label but adjacent to an unlabeled vertex, there exists u a labeled vertex adjacent to v. Then, check the following
   (a) If $w(u) - f(u) - f(v) \neq deg(v) - 1$, then label all unlabeled vertices adjacent to v with 1.
   (b) Otherwise, label any adjacent unlabeled vertex of v with 2 and label all other vertices adjacent to v (if it exists) with 1.

The process eventually terminates, giving all vertices with labels either 1 or 2 while keeping weights of adjacent vertices to be distinct. Therefore, lidis$(T) = 2$. □

In Figure 2 we give an example of a local inclusive distance vertex irregular labeling on a non locally irregular tree T.

A clique subgraph is defined to be *safe* if the clique has at most one vertex which is not a cut-vertex. A vertex v is called *fixed* if v and all adjacent vertices to v has a label (otherwise it is called *unfixed*). Notice that if a graph G has a clique which is not safe, then there exists two vertices in the clique satisfying the condition of Theorem 1.1 which implies lidis$(G) = \infty$. In Theorem 2.2, we give the lidis upper bound for block graphs G when lidis(G) is finite.

Theorem 2.2. *For any block graph G whose cliques are safe, we have*

$$lidis(G) \leq \omega(G).$$

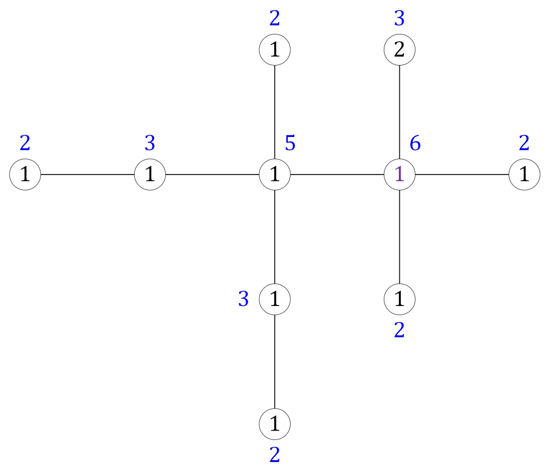

Figure 2. Local inclusive distance vertex irregular labeling on a tree T with lidis$(T) = 1$.

Proof. If G is locally irregular, then according to Theorem 1.2 lidis$(G) = 1$. Otherwise, observe the following algorithm.

1. Pick any vertex $s \in V(G)$.
2. Label vertex s and all vertices adjacent to s with 1.
3. Find a clique C which consists of fixed vertices u and unfixed vertices v in the clique.
4. Compile the degrees of all unfixed vertices $v \in C$ and create a monotone increasing sequence of the degrees D.
5. For every unfixed vertices $v \in C$, set $i = 1$. Check the following with the order of D
   (a) If $w(u) - \sum_{c \in C} f(c) \neq deg(v) - |C| + i$, for all fixed vertices $u \in C$, then label all unlabeled vertices adjacent to v with lowest value as possible such that the sum is equal to $deg(v) - |C| + i$. Now, the vertex v is a fixed vertex.
   (b) Otherwise, increase i to $i + 1$ and repeat the step (a).
6. If there exists a clique C which still consists of fixed vertices u and unfixed vertices v in the clique, repeat step 4.

The process eventually terminates, giving all vertices with labels at most $\omega(G)$ while keeping weights of adjacent vertices to be distinct. Therefore, lidis$(G) \leq \omega(G)$. □

In Figure 3 we give an example of a local inclusive distance vertex irregular labeling on a block graph G.

An interesting special case of block graph is $K_m \odot K_1$. It can be shown that if a graph G contains $K_m \odot K_1$ in some way, we can find the lower bound of lidis(G).

Theorem 2.3. *Let G be a non locally irregular block graph. If for some $U \subseteq V(G)$ with $m = |U|$, G contains induced subgraph $N[U]$ which is isomorphic to $K_m \odot K_1$ then*

$$lidis(G) \geq m.$$

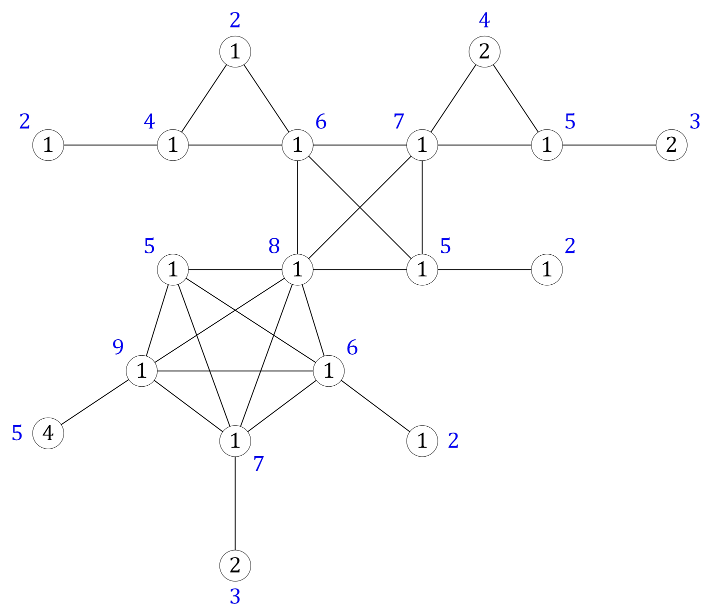

Figure 3. Local inclusive distance vertex irregular labeling on a block graph G with lidis$(G) \leq 5$.

Proof. Suppose f be local inclusive distance vertex irregular labeling of G. For $i \in \{1, \cdots, m\}$, let v_i be vertices of induced subgraph $N[U]$ which are not pendants and p_i be pendants which are only adjacent to v_i. In order to v_i, v_j be pairwise distinct for $j \in \{1, \cdots, m\}$, we have

$$\sum_{v \in N_G[v_i]} f(v) \neq \sum_{v \in N_G[v_j]} f(v)$$
$$f(p_i) + \sum_{k=1}^{m} f(v_k) \neq \sum_{k=1}^{m} f(v_k) + f(p_j)$$
$$f(p_i) \neq f(p_j)$$

Hence, $f(p_i) \neq f(p_j)$ for all distinct $i, j \in \{1, \cdots, m\}$. This implies that there exists j such that $f(p_j) \geq m$. Consequently, lidis$(G) \geq m$. □

Some graphs meet the equality conditions of Theorem 2.2 and Theorem 2.3. This result is depicted on the proceeding corollary.

Corollary 2.1. *Let m be positive integer and $\mathcal{G}(m)$ be collection of block graphs (may be trivial) with $\omega(G_i) \leq m$ for every $G_i \in \mathcal{G}(m)$. If the graph G is obtained from $K_m \odot K_1$ where each of its pendant is identified with a vertex of $G_i \in \mathcal{G}(m)$, then*

$$lidis(G) = m.$$

Proof. From the construction of G, it is obvious that $\omega(G) = m$ and $K_m \odot K_1$ exists as an induced subgraph $N[U]$ in G for some $U \subseteq V(G)$. Apply Theorem 2.2 and Theorem 2.3 on G. □

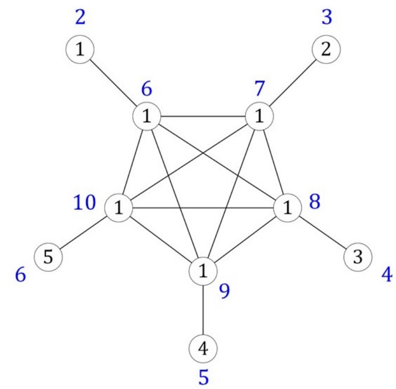

Figure 4. Local inclusive distance vertex irregular labeling on $K_5 \odot K_1$ with $\text{lidis}(K_5 \odot K_1) = 5$.

In Figure 4 we give an example of a local inclusive distance vertex irregular labeling on $K_5 \odot K_1$.

Another property which is interesting to discuss is how is the relationship between (finite) $\text{lidis}(G)$ and $\chi(G)$. For some classes of graph, such as locally irregular tree, cycle [8], and $K_n \odot K_1$, where both of the numbers are equal. However, we can find a block graph which tells us that the difference $\chi(G)$ and $\text{lidis}(G)$ can be arbitrarily large. Consider a block graph G obtained from a complete graph K_n with $V(K_n) = \{v_i \mid i \in [1, n]\}$ where every v_i is adjacent with i pendants. It can be seen that the graph G is locally irregular, hence $\text{lidis}(G) = 1$. However, the graph G contains a complete subgraph K_n implying $\chi(G) = n$.

Remark 2.1. The value $\chi(G) - \text{lidis}(G)$ can be arbitrarily large.

Meanwhile, we have not found any graphs G having $\text{lidis}(G)$ larger than $\chi(G)$. It is an open problem to find an existence of graphs satisfying this condition.

Problem 1. *Does there exists a graph G with lidis$(G) < \infty$ such that $\chi(G) <$ lidis(G)?*

3. Local Inclusive Distance Vertex Irregularity Strength of Complete Multipartite Graphs

In [8] Sugeng et al. gave a theorem about the lidis number for complete multipartite graphs. But in this theorem we proved a different result.

Theorem 3.1. *For complete multipartite graphs $K_{n_1,n_2,\ldots,n_m}$, we have*

$$\text{lidis}(K_{n_1,n_2,\ldots,n_m}) = \begin{cases} \infty, & \text{if } 1 = n_1 = n_2, \\ 1, & \text{if } n_1 < n_2 < \cdots < n_m, \\ \left\lceil \frac{m-1}{\lfloor \frac{n}{2} \rfloor} \right\rceil + 1, & \text{if } 2 \le n_1 = n_2 = \cdots = n_m. \end{cases}$$

Proof. Let us denote the vertices in the independent set V_i, $i = 1, 2, \ldots, m$ of a complete multipartite graph $K_{n_1,n_2,\ldots,n_m}$ by symbols $v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}$. If $1 = n_1 = n_2$ then for u, v vertices of the graph $N[u] = N[v]$. Thus, by Theorem 1.1 $\text{lidis}(K_{n_1,n_2,\ldots,n_m}) = \infty$. If $n_1 < n_2 < \cdots < n_m$

then the graph is locally irregular. By Theorem 1.2 we obtain lidis$(K_{n_1,n_2,\ldots,n_m}) = 1$. If $2 \le n_1 = n_2 = \cdots = n_m = n$ consider a vertex labeling f of $K_{n_1,n_2,\ldots,n_m}$ defined as follows

$$f(v_{i,j}) = \begin{cases} \left\lceil \frac{i-1}{\lfloor \frac{n}{2} \rfloor} \right\rceil, & \text{for } j \le n-c, \\ \left\lceil \frac{i-1}{\lfloor \frac{n}{2} \rfloor} \right\rceil + 1, & \text{for } j > n-c. \end{cases}$$

with

$$c = \begin{cases} 2(i-1 \mod \lfloor \frac{n}{2} \rfloor), & \text{for } i \not\equiv 1 \ (\text{mod} \lfloor \frac{n}{2} \rfloor), \\ n, & \text{for } i \equiv 1 \ (\text{mod} \lfloor \frac{n}{2} \rfloor). \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Note that in $K_{n_1,n_2,\ldots,n_m}$, every vertex $v_{i,j}$ is adjacent to every other vertex except the ones in V_i. Therefore, to simplify the weight calculation, we introduce another function $w^*(v_{i,j}) = \sum_{v \in V_i} f(v) - f(v_{i,j})$. With this function, the vertex weight $w(v_{i,j}) = \sum_{v \in V} f(v) - w^*(v_{i,j})$. For any vertex $v_{i,j}$ in the vertex set we get that

$$w^*(v_{i,j}) = \begin{cases} (n-1)\left\lceil \frac{i-1}{\lfloor \frac{n}{2} \rfloor} \right\rceil + 2(i-1 \mod \lfloor \frac{n}{2} \rfloor) - 1, & \text{for } j \le n-c \text{ and } i \not\equiv 1 \ (\text{mod} \lfloor \frac{n}{2} \rfloor), \\ (n-1)\left\lceil \frac{i-1}{\lfloor \frac{n}{2} \rfloor} \right\rceil + 2(i-1 \mod \lfloor \frac{n}{2} \rfloor), & \text{for } j > n-c \text{ and } i \not\equiv 1 \ (\text{mod} \lfloor \frac{n}{2} \rfloor), \\ (n-1)(\left\lceil \frac{i-1}{\lfloor \frac{n}{2} \rfloor} \right\rceil + 1), & \text{for } i \equiv 1 \ (\text{mod} \lfloor \frac{n}{2} \rfloor), \end{cases}$$

Therefore, it is clear that $w^*(v_{i,j})$ will be distinct for vertices from different partite sets. Thus, we obtain that the vertex weight are distinct for every adjacent pair of vertices. □

In Figure 5 we give an example of a local inclusive distance vertex irregular labeling on an $K_{4,4,4}$.

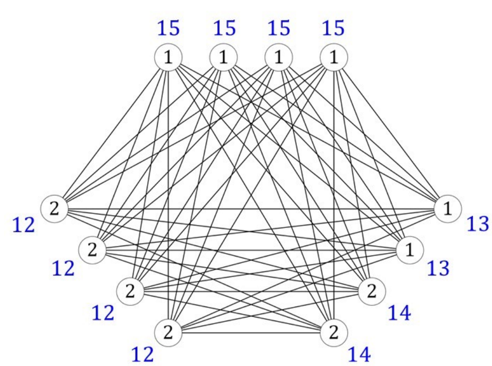

Figure 5. Local inclusive distance vertex irregular labeling on $K_{4,4,4}$ with lidis$(K_{4,4,4}) = 2$.

4. Local Inclusive Distance Vertex Irregularity Strength of Certain Bipartite Graphs

The set of vertices V of a bipartite graph can be divided into V_1 and V_2, such that all edges only incident to exactly one vertex in V_1 and one vertex in V_2. Sugeng et al. [8] have determined the lidis(G) when G is a complete bipartite. In the following theorem, we provide the strength when the bipartite graph G is regular.

Theorem 4.1. *For a regular bipartite graphs G other than K_2, we have lidis$(G) = 2$.*

Proof. The graph G is clearly non locally irregular, therefore, lidis$(G) \geq 2$. Since G is a bipartite graph, we can create a partition of vertices $V(G) = V_1 \cup V_2$ such that all edges $e = v_1v_2 \in E(G)$ satisfying $v_1 \in V_1$ and $v_2 \in V_2$. Define a labeling $f : V(G) \rightarrow \{1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } v \in V_1, \\ 2, & \text{if } v \in V_2. \end{cases}$$

We know that G is regular, hence there exists an positive integer $r \geq 2$ such that $\deg(v) = r$ for all vertices $v \in V(G)$. By evaluating, we have

$$w(v) = \begin{cases} 1 + 2r, & \text{if } v \in V_1, \\ 2 + 1r, & \text{if } v \in V_2. \end{cases}$$

We conclude that lidis$(G) = 2$. □

Some special cases of graphs which belongs to Theorem 4.1 can be seen in proceeding corollaries.

Corollary 4.1. *Let $n \geq 3$. For hypercubes Q_n, we have lidis$(Q_n) = 2$.*

Corollary 4.2. *Let $n \geq 4$ be an even integer. For prisms D_n, we have lidis$(D_n) = 2$.*

Corollary 4.3. *Let $n \geq 3$ be an odd integer. For möbius ladders M_{2n}, we have lidis$(M_{2n}) = 2$.*

The other result revolves bipartite graphs which have even degrees for one of the partition. This result is established in the following theorem.

Theorem 4.2. *Let G be a non locally irregular bipartite graph with $V(G) = V_1 \cup V_2$ as a partition of the vertices. If $\deg(v_2)$ is even for every $v_2 \in V_2$, we have lidis$(G) = 2$.*

Proof. Define a labeling $f : V(G) \rightarrow \{1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } v \in V_1, \\ 2, & \text{if } v \in V_2. \end{cases}$$

Notice that for every $v_1 \in V_1$, $w(v_1) = 1 + 2\deg(v_1)$, implying $w(v_1)$ would always be an odd number. Meanwhile, for every $v_2 \in V_2$, $w(v_2) = 2 + \deg(v_2)$. Since $\deg(v_2)$ is even, $w(v_2)$ would always yield an even number. Therefore, weights of adjacent vertices are pairwise distinct implying lidis$(G) = 2$. □

Let $S_m(G)$ be a graph obtained by splitting every edge in the graph into $m + 1$ edges which is incident to new vertices.

Corollary 4.4. *Let m be an odd positive integer. For every graph G, we can obtain the subdivided graph $S_m(G)$ satisfying*

$$\textit{lidis}(S_m(G)) = \begin{cases} 1, & \text{if } m = 1 \text{ and for every } v \in V(G), \deg(v) \neq 2, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. If for every $v \in V(G)$, $\deg(v) \neq 2$, then $S_1(G)$ is an locally irregular graph. We can apply Theorem 1.2 to conclude that lidis$(S_m(G)) = 1$. Otherwise, $S_m(G)$ is non locally irregular. Since $S_m(G)$ is bipartite, we can apply Theorem 4.2 to have lidis$(S_m(G)) = 2$. □

Theorem 2.1, Theorem 4.1, and Theorem 4.2 revolve around determining lidis(G) for bipartite graphs G. All of these cases of bipartite graphs G ensure that lidis$(G) \leq 2$. This motivates us to state a conjecture as follows.

Conjecture 2. *If G is a bipartite graph other than K_2, then lidis$(G) = 1$ or 2.*

Acknowledgment

The authors are pleased to thank Lucas Colucci and Andrea Semaničová-Feňovčíková for their supportive comments and suggestions.

References

[1] M. Bača, and A. Semaničová-Feňovčíková, Slamin, and K.A. Sugeng, On inclusive distance vertex irregular labelings, *Electron. J. Graph Theory Appl.* **6**(1) (2018), 61–83.

[2] N.H. Bong, Y. Lin, and S. Slamin, On inclusive and non-inclusive vertex irregular d-distance vertex labelings. submitted.

[3] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, and F. Saba, Irregular networks, *Congr. Numer.* **64** (1988), 187–192.

[4] N.J. Cutinho, S. Sudha, E. Soundarya, and P. Devaki, Inclusive lucky labeling for some special families of graphs, *International Journal of Innovative Research in Science, Engineering and Technology* **9**(5) (2020), 3360–3367.

[5] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **9** (2021) #DS6.

[6] S. Slamin, On distance irregular labelings of graphs, *Far East Journal of Mathematical Sciences*, **102** (2017) 919–932.

[7] F. Susanto, K. Wijaya, I. W. Sudarsana, and Slamin, Non-inclusive and inclusive distance irregularity strength for the join product of graphs, *Electron. J. Graph Theory Appl.* **10**(1) (2022), 1–13.

[8] K.A. Sugeng, D.R. Silaban, M. Bača, and A. Semaničová-Feňovčíková, Local inclusive distance vertex irregular graphs, *Mathematics* **9**(14) (2021) 1673.