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On (F, H)-sim-magic labelings of graphs

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Abstract

A simple graph $G(V, E)$ admits an H-covering if every edge in G belongs to a subgraph of G isomorphic to H. In this case, G is called H-magic if there exists a bijective function $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$, such that for every subgraph H' of G isomorphic to H, $wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ is constant. Moreover, G is called H-supermagic if $f : V(G) \to \{1, 2, \ldots, |V|\}$. This paper generalizes the previous labeling by introducing the (F, H)-sim-(super) magic labeling. A graph admitting an F-covering and an H-covering is called (F, H)-sim-(super) magic if there exists a function f that is F-(super)magic and H-(super)magic at the same time. We consider such labelings for two product graphs: the join product and the Cartesian product. In particular, we establish a sufficient condition for the join product $G + H$ to be $(K_2 + H, 2K_2 + H)$-sim-supermagic and show that the Cartesian product $G \times K_2$ is (C_4, H)-sim-supermagic, for H isomorphic to a ladder or an even cycle. Moreover, we also present a connection between an α-labeling of a tree T and a (C_4, C_6)-sim-supermagic labeling of the Cartesian product $T \times K_2$.

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1. Introduction

The graphs considered in this paper are finite and simple. Let G be a graph, with the vertex set $V(G)$ and the edge set $E(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the order and the size of G, respectively. A *labeling* f of G is a map that assigns certain elements of G to positive or non-negative integers. In this paper, we consider a *total labeling* of G as a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$. Under a total labeling f, the weight of a vertex $v \in V(G)$ is $wt_f(v) = f(v) + \sum_{vw \in E(G)} f(vw)$ and the weight of an edge $vw \in E(G)$ is $wt_f(vw) = f(v) + f(vw) + f(w)$.

Simanjuntak et al. [27] introduced an (a, d)*-edge-antimagic total labeling* ((a, d)*-EAT)* as a total labeling f where the set of edge-weights $\{wt_f(vw)|vw \in E(G)\}$ constitutes a set of an arithmetic progression $\{a, a + d, \ldots, a + (|E(G)| - 1)d\}$ for two integers $a > 0$ and $d \geq 0$. When $d = 0$, the $(a, 0)$-edge(vertex)-antimagic labeling was previously known as the *edge-magic total labeling (EMT)* and was introduced by Kotzig and Rosa [15] in 1970. When G has EMT or (a, d)-EAT labelings and the corresponding f labeling has the property $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, we say that G is *super edge-magic total (SEMT)* or *super* (a, d)*-edge-antimagic total* ((a, d)*-SEAT)*, respectively.

Another variation of magic labeling called vertex-magic total labeling was introduced by MacDougal et al. [17]. A *vertex-magic total labeling (VMT)* of G is a total labeling where there exists a positive integer k such that the vertex-weight $wt_f(v) = k$ for every vertex v of G. If $\{wt_f(v)|v \in V(G)\} = \{a, a+d, \ldots, a+(|V(G)|-1)d\}$ for two integers $a > 0$ and $d \geq 0$, the labeling f of G is called (a, d)*-vertex-antimagic total labeling* ((a, d)*-VAT)*, that was first introduced by Bača et al. [3]. Comprehensive surveys about the existence of magic and antimagic graphs can be found in [4, 5, 11, 29].

In 2005, as an extension of the edge-magic total labeling, Gutiérez and Lladó [12] introduced an H-magic labeling of a graph. A graph G admits an H*-covering* if every edge in $E(G)$ belongs to a subgraph of G isomorphic to a given graph H. A total labeling f of G is an H*-magic labeling* if there exists a positive integer k such that $wt(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k$ for every subgraph H' of G isomorphic to H. In this case, G is called an H*-magic graph*. If $f(V) = \{1, 2, \ldots, |V(G)|\}$, then G is said to be an H*-supermagic graph*. Current results on H-magic labelings can be seen in the survey [11].

In 2005, Exoo et al. [9] asked whether there exists a labeling of a graph that is simultaneously vertex-magic and edge-magic and called such labeling *totally magic*. Subsequently, in 2005, Bača et al. [6] extended a similar question for (a, d)-EAT labeling and (a, d)-VAT labelings; and defined the *totally antimagic total (TAT) labeling*.

Motivated by the two notions above, it is interesting to ask a similar question by considering the subgraph covering in G. Suppose that G simultaneously admits an F-covering and an H-covering. We propose a new notion of a labeling called an (F, H)-sim-magic labeling as a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ where there exist two positive integers k_F and k_H (not necessarily the same) such that

$$wt_f(F') = \sum_{v \in V(F')} f(v) + \sum_{e \in E(F')} f(e) = k_F$$

and

$$wt_f(H') = \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k_H,$$

for each subgraph F' of G isomorphic to F and each subgraph H' of G isomorphic to H. We say that G is (F, H)-*sim-magic*. Furthermore, if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, G is said to be (F, H)-*sim-supermagic*.

The simplest example of a (F, H)-sim-magic graph can be deduced from previously known H-magic labelings. For odd m and n at least three, the disjoint union of m cycles mC_n is both SEMT [10] and C_n-supermagic [1, 18]. Although the C_n-supermagic labelings described in [1, 18] are not SEMT, the SEMT labeling of $3C_3$ described in [10] is also C_3-supermagic (see Figure 1). This implies that $3C_3$ is (K_2, C_3)-sim-supermagic.

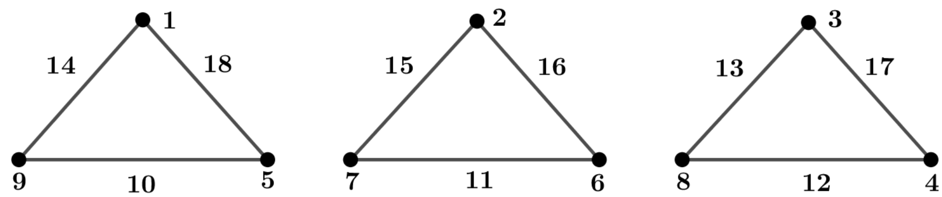

Figure 1. A (K_2, C_3)-sim-supermagic graph.

An interesting fact for (F, H)-sim-magic labeling is that although a graph is both F-magic and H-magic, such a graph does not need to be (F, H)-sim-magic. An example is the fan F_n with vertex-set $V(F_n) = \{v_i | 0 \leq i \leq n\}$ and edge-set $E(F_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_0 v_i | 1 \leq i \leq n\}$. It is known that, for every $n \geq 3$, F_n is EMT (see [28]) and C_3-supermagic (see [21]). However, for every $n \geq 3$, F_n is not (K_2, C_3)-sim-magic as stated in the following theorem.

Theorem 1.1. *Let $n \geq 3$ be a positive integer. A fan F_n is not (K_2, C_3)-sim-magic.*

Proof. Suppose that F_n is a (K_2, C_3)-sim-magic graph and let f be a (K_2, C_3)-sim-magic labeling of F_n with a magic constant pair (k_1, k_2). Consider the weights of two C_3 cycles v_0v_1, v_1v_2, v_2v_0 and v_0v_2, v_2v_3, v_3v_0. As these weights are equal, we have

$$\sum_{i=0}^{2} f(v_i) + f(v_0v_1) + f(v_1v_2) + f(v_2v_0) = \sum_{i=1}^{3} f(v_i) + f(v_0v_2) + f(v_2v_3) + f(v_3v_0),$$

and so

$$f(v_1) + f(v_1v_0) + f(v_1v_2) = f(v_3) + f(v_2v_3) + f(v_0v_3). \tag{1}$$

Adding $f(v_0)$ to both sides of Equation (1) and using the fact that all edges have the same edge weight, we obtain $f(v_1v_2) = f(v_2v_3)$, a contradiction. $\square$

In this paper, we study simultaneous labelings for two product graphs: the join product and Cartesian product graphs. In particular, we investigate a sufficient condition for the join product graph $G + H$ to be $(K_2 + H, 2K_2 + H)$-sim-supermagic (Section 3). We construct (C_4, H)-sim-supermagic labelings for the Cartesian product $G \times K_2$, where H is isomorphic to a ladder or an even cycle (Section 4). Finally, in the last section, we provide relationships between an α-labeling of a tree T and a (C_4, C_6)-sim-supermagic labeling of the Cartesian product $T \times K_2$.

Throughout the paper, we shall use the following definitions and notations. The degree of a vertex v is denoted by $\deg(v)$. For a connected graph H, a graph G is *H-free* if G does not contain H as a subgraph. Notations for some classes of graphs can be seen in Table 1.

Table 1. Classes of graphs

Notation	Notes
C_n	A cycle on n vertices, $n \geq 3$.
K_n	A complete graph on n vertices, $n \geq 1$.
$K_{1,n}$	A star with one internal vertex and n leaves, $n \geq 2$.
P_n	A path on n vertices, $n \geq 2$.
$S_{n_1,n_2,\ldots,n_k}$	A caterpillar is a graph derived from a path P_k, $k \geq 2$, where for $i \in \{1, 2, ..., k\}$, each $v_i \in V(P_k)$ is adjacent to $n_i \geq 0$ additional leaves.

2. Balanced and Anti Balanced Multisets

A *multiset* is a generalization of a set where repetition of elements is allowed. Let a and b be two integers. We use the notation $[a, b]$ to define the set of consecutive integers $\{a, a+1, \ldots, b\}$. So $[a, b] = \emptyset$, if $a > b$. For an integer k, the addition $k + [a, b]$ means $[a + k, b + k]$ and for a multiset of integers Y, we denote $\sum_{x \in Y} x$ by $\sum Y$. Let x be an element of a multiset Y. Then, the *multiplicity* of x, denoted by $m_Y(x)$, is the number of occurrences of x in Y. Let X and Y be two multisets. A *multiset sum* $X \biguplus Y$ is a union of X and Y, where $m_{X \biguplus Y}(x) = m_X(x) + m_Y(x)$ for each $x \in X \biguplus Y$. For example, if $X = \{a\}$ and $Y = \{a, a, b\}$, then, $X \biguplus Y = \{a, a, a, b\}$.

We shall utilize the notions of a k-balanced partition of a multiset introduced by Maryati et al. [19] and a (k, δ)-anti balanced partition of a multiset introduced by Inayah et al. [13] to construct labelings in Sections 3 and 4. Let k and δ be two positive integers, and X be a multiset containing positive integers. X is said to be (k, δ)*-anti balanced* if there exist k subsets of X, say $X_1, X_2, \ldots, X_k$, such that for every $i \in [1, k]$, $|X_i| = \frac{|X|}{k}$, $\biguplus_{i=1}^{k} X_i = X$, and for each $i \in [1, k-1]$, $\sum X_{i+1} - \sum X_i = \delta$. For every $i \in [1, k]$, X_i is called a (k, δ)*-anti balanced subset* of X. In the case that there exists a positive integer θ such that $\sum X_i = \theta$ for every $i \in [1, k]$, then X is called k*-balanced* with X_is as k*-balanced subsets* of X.

Lemma 2.1. *[18] Let x, y, and k be three integers, where $1 \leq x < y$ and $k > 1$. If $X = [x, y]$ and $|X|$ is a multiple of $2k$, then X is k-balanced with $\sum X_i = \frac{|X|}{2k}(x + y)$ for every $i \in [1, k]$.*

Lemma 2.2. *Let x and k be positive integers, $k \geq 2$. If*

$$X = \begin{cases} [x, x + 2k - 1], & \text{for odd } k; \\ [x, x + 2k] \setminus \{x + \frac{k}{2}\}, & \text{for even } k, \end{cases}$$

then X is $(k, 1)$-anti balanced with $\sum X_i = 2x + i + 3\left\lfloor \frac{k}{2} \right\rfloor$ for every $i \in [1, k]$.

Proof. For each $i \in [1, k]$, define $X_i = \{a^i, b^i\}$, where $a^i = x - 1 + \left\lceil \frac{i+1}{2} \right\rceil + 2(1 - (i \mod 2)) \left\lfloor \frac{k}{2} \right\rfloor$ and $b^i = x + \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil + 2(i \mod 2) \left\lfloor \frac{k}{2} \right\rfloor$. Thus, $\biguplus_{i=1}^{k} X_i = X$ and we have $|X_i| = 2$ and $\sum X_i = 2x + i + 3 \left\lfloor \frac{k}{2} \right\rfloor$ for each $i \in [1, k]$. Since $\sum X_{i+1} - \sum X_i = 1$ for every $i \in [1, k-1]$, X is a $(k, 1)$-anti balanced. □

Lemma 2.3. *Let x and k be positive integers, $k \geq 2$. If $X = [x, x + 2k - 1]$, then X is $(k, 2)$-anti balanced with $\sum X_i = 2(x + i - 1) + k$ for every $i \in [1, k]$.*

Proof. Define $X_i = \{x - 1 + i, x + i + k - 1\}$ for each $i \in [1, k]$. Hence, $\biguplus_{i=1}^{k} X_i = X$, $|X_i| = 2$, and $\sum X_i = 2(x + i - 1) + k$ for every $i \in [1, k]$. We have that X is $(k, 2)$-anti balanced since $\sum X_{i+1} - \sum X_i = 2$ for every $i \in [1, k - 1]$. □

3. Labelings for Join Product Graphs

Let $G \cup H$ denote the disjoint union of G and H. Then, the *join product* $G + H$ of two disjoint graphs G and H is the graph $G \cup H$ together with all the edges joining vertices of G and vertices of H. The study of H-magicness of join product graphs has been conducted for some particular families of graphs, as summarized in Table 2.

Table 2. Known join product graphs which are H-magic

Join product	H	Reference
$P_n + K_1, n \geq 3$	C_3	Ngurah et al. [21] and Ovais et al. [22]
	C_4	Ovais et al. [22]
$C_n + K_1$, n odd, $n \geq 5$	C_3	Lladó and Moragas [16]
n even, $n \geq 4$	C_3	Roswitha et al. [25]
$C_n + K_1, n \geq 3$	C_4	Semaničová-Feňovčíková et al. [26]
$K_{1,n} + K_1, n \geq 3$	C_3	Ngurah et al. [21]
$nK_2 + K_1, n \geq 2$	C_3	Lladó and Moragas [16]

The following theorem provides a sufficient condition for the join product graph $G + H$ to be $(K_2 + H, 2K_2 + H)$-sim-supermagic.

Theorem 3.1. *Let G and H be two connected graphs such that G admits a $2K_2$-covering and $G + H$ contains exactly $|E(G)|$ subgraphs isomorphic to $K_2 + H$. If G is SEMT, then $G + H$ is $(K_2 + H, 2K_2 + H)$-sim-supermagic.*

Proof. Let g be a super edge-magic total (SEMT) labeling of G with the magic constant m_g. Let $V(G) = \{v_i | v_i = g^{-1}(i)$ and $i \in [1, p]\}, V(H) = \{u_i | i \in [1, r]\}, E(G) = \{e_i | i \in [1, q]\}$, and $E(H) = \{k_i | i \in [1, s]\}$. Hence, $|V(G)| = p$, $|E(G)| = q$, $|V(H)| = r$, $|E(H)| = s$. Thus, $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{u_i v_j | i \in [1, r]$ and $j \in [1, p]\}$.

Consider $Y = [1, p + q + r + s + pr]$ as the set of all labels of vertices and edges in $G + H$. Then, we divide the proof into three cases.

Case 1. r is even.
Partition Y into five subsets, namely $A = [1, r]$, $B = r + [1, p]$, $C = (r + p) + [1, pr]$, $D = (r + p +$

$p\,r)+[1,q]$ and $E=(r+p+p\,r+q)+[1,s]$. Since r is even, $|C|$ is a multiple of $2p$. By Lemma 2.1, we have that C is p-balanced with $\sum C_i = \frac{pr}{2p}(r+p+1+r+p+pr) = \frac{1}{2}r(p(r+2)+2r+1)$ for each $i \in [1,p]$.

Next, label the vertices and edges in $G+H$ by total labeling f as defined in the following steps.

1. For each $i \in [1,r]$, $f(u_i) = i$.
2. For each $i \in [1,p]$, $f(v_i) = i + r$.
3. For each $j \in [1,p]$ and $i \in [1,r]$, $f(u_i v_j) = m_i$, where $m_i \in C_j$.
4. For each $e_i \in E(G)$ and $i \in [1,q]$, $f(e_i) = g(e_i) + r + pr$.
5. For each $k_i \in E(H)$ and $i \in [1,s]$, $f(k_i) = m_i$, where $m_i \in E$ and no two distinct edges in $E(H)$ are assigned the same number.

Thus, we get $\bigcup_{i=1}^{r}\{f(u_i)\} = A, \bigcup_{i=1}^{p}\{f(v_i)\} = B, \bigcup_{j=1}^{p}\{f(u_iv_j)|i\in[1,r]\} = C, \{f(v_iv_j)|v_iv_j \in E(G)\} = D$, and $\{f(u_iu_j)|u_iu_j \in E(H)\} = E$. Clearly, f is a bijective function from $V(G+H)\cup E(G+H)$ to Y.

Let F be a subgraph of $G+H$ isomorphic to K_2+H. It is clear that F contains exactly one edge of $E(G)$, say v_xv_y for some distinct $x, y \in [1,p]$. Then, $V(F) = V(H)\cup\{v_x, v_y\}$ and $E(F) = E(H)\cup\{v_xv_y\}\cup\{u_iv_j|j\in\{x,y\}, i\in[1,r]\}$. Thus,

$$\begin{aligned} wt_f(F) &= \sum_{i=1}^{r} f(u_i) + f(v_x) + f(v_y) + \sum_{e\in E(H)} f(e) + f(v_xv_y) + \sum_{i=1}^{r}[f(u_iv_x)+f(u_iv_y)] \\ &= [g(v_x)+g(v_y)+g(v_xv_y)] + 3r + pr + \tfrac{1}{2}r(r+1) + s(r+p+pr+q) + \sum_{i=1}^{s} i \\ &\quad + r(p(r+2)+2r+1). \end{aligned}$$

Since $[g(v_x)+g(v_y)+g(v_xv_y)] = m_g$, we see that $wt_f(F)$ is independent of F.

Now, let F' be a subgraph of $G+H$ isomorphic to $2K_2+H$. It is clear that F' contains two non-adjacent edges of $E(G)$. Then, $wt_f(F') = 2wt_f(F) - \left(\sum_{u\in V(H)} f(u) + \sum_{e\in E(H)} f(e)\right)$. So, $wt_f(F')$ is independent of F'.

Case 2. r is odd and p is odd.
Partition Y into five subsets, namely $A = [1,r]$, $B = r+[1,2p]$, $C = (r+2p)+[1,p(r-1)]$, $D = (r+p+p\,r)+[1,q]$, and $E = (r+p+p\,r+q)+[1,s]$. By Lemma 2.2, B is $(p,1)$-anti balanced with $\sum B_i = 2(r+1)+3\left\lfloor\frac{p}{2}\right\rfloor + i$ for every $i\in[1,p]$. Since g is an injective function, $g^{-1}(i) = v_i$ for every $i\in[1,p]$. This gives $\sum B_i = 2(r+1)+3\lfloor\frac{p}{2}\rfloor + i = 2(r+1)+3\lfloor\frac{p}{2}\rfloor + g(v_i)$ for every $i\in[1,p]$. The cardinality of C is a multiple of $2p$. By Lemma 2.1, C is p-balanced with $\sum C_i = \frac{p(r-1)}{2p}(r+2p+1+r+2p+p(r-1)) = \frac{1}{2}(r-1)(2r+3p+pr+1)$ for every $i\in[1,p]$.

Next, label the vertices and edges in $G+H$ by total labeling f as defined in the following steps.

1. For each $i \in [1,r]$, $f(u_i) = i$.
2. For each $i \in [1,p]$, $f(v_i) = \min\{x|x\in B_i\}$.
3. For each $i \in [1,p]$, $f(u_1v_i) = b_i$, where $b_i \in B_i \setminus \{f(v_i)\}$.

4. For each $j \in [1,p]$ and $i \in [2,r]$, $f(u_iv_j) = m_i$, where $m_i \in C_j$.
5. For each $e_i \in E(G)$ and $i \in [1,q]$, $f(e_i) = g(e_i) + r + pr$.
6. For each $k_i \in E(H)$ and $i \in [1,s]$, $k_i = m_i$, where $m_i \in E$ and no two distinct edges are assigned the same number.

Thus, we get $\bigcup_{i=1}^{r}\{f(u_i)\} = A$, $\bigcup_{i=1}^{p}\{f(v_i), f(u_1v_i)\} = B$, $\bigcup_{j=1}^{p}\{f(u_iv_j)|\ i \in [2,r]\} = C$, $\{f(v_iv_j)|v_iv_j \in E(G)\} = D$, and $\{f(u_iu_j)|u_iu_j \in E(H)\} = E$. Clearly, f is a bijective function from $V(G+H) \cup E(G+H)$ to Y.

Let F be a subgraph of $G+H$ isomorphic to $K_2 + H$. It is clear that F contains exactly one edge of $E(G)$, say v_xv_y for some distinct $x, y \in [1,p]$. Then, $V(F) = V(H) \cup \{v_x, v_y\}$ and $E(F) = E(H) \cup \{v_xv_y\} \cup \{u_iv_j|j \in \{x,y\}, i \in [1,r]\}$. Thus,

$$\begin{aligned} wt_f(F) &= \sum_{i=1}^{r} f(u_i) + f(v_x) + f(v_y) + \sum_{e\in E(H)} f(e) + f(v_xv_y) + \sum_{i=1}^{r}[f(u_iv_x) + f(u_iv_y)] \\ &= [g(v_x) + g(v_y) + g(v_xv_y)] + 4(r+1) + 6\left\lfloor \tfrac{p}{2}\right\rfloor + r + pr + \tfrac{1}{2}r(r+1) \\ &\quad + s(r+p+pr+q) + \tfrac{1}{2}s(s+1) + (r-1)(2r+3p+pr+1). \end{aligned}$$

Since $[g(v_x) + g(v_y) + g(v_xv_y)] = m_g$, we see that $wt_f(F)$ is independent on the choosing of F.

Now, let F' be a subgraph of $G+H$ isomorphic to $2K_2 + H$. It is clear that F' contains two non-adjacent edges of $E(G)$. Thus, $wt_f(F') = 2wt_f(F) - \left(\sum_{v\in V(H)} f(v) + \sum_{e\in E(H)} f(e)\right)$. So, $wt_f(F')$ is independent of F'.

Case 3. r is odd and p is even.
Partition Y into five subsets, $A = [1, r-1] \cup \{r + \frac{p}{2}\}$, $B = [r, r+2p] \setminus \{r + \frac{p}{2}\}$, $C = (r+2p) + [1, p\,(r-1)]$, $D = (r+p+p\,r) + [1,q]$, and $E = (r+p+pr+q) + [1,s]$. By Lemma 2.2, B is $(p,1)$-anti balanced with $\sum B_i = 2r + i + 3\lfloor \frac{p}{2}\rfloor$ for each $i \in [1,p]$. Since g is an injective function, $g^{-1}(i) = v_i$ for every $i \in [1,p]$. Therefore, $\sum B_i = 2r + 3\lfloor\frac{p}{2}\rfloor + i = 2r + 3\lfloor\frac{p}{2}\rfloor + g(v_i)$ for every $i \in [1,p]$.

Now, the cardinality of C is a multiple of $2p$. By Lemma 2.1, we have that C is p-balanced with $\sum C_i = \frac{p(r-1)}{2p}(r+2p+1+r+2p+pr) = \frac{1}{2}(r-1)(2r+3p+pr+1)$ for every $i \in [1,p]$.

Next, label the vertices and edges in $G+H$ by the total labeling f defined in the following steps.

1. For each $i \in [2,r]$, $f(u_i) = i-1$, and $f(u_1) = r + \frac{p}{2}$.
2. For each $i \in [1,p]$, $f(v_i) = \min\{x|x \in B_i\}$.
3. For each $i \in [1,p]$, $f(u_1v_i) = b_i$, where $b_i \in B_i \setminus \{f(v_i)\}$.
4. For each $j \in [1,p]$ and $i \in [2,r]$, $f(u_iv_j) = m_i$, where $m_i \in C_j$.
5. For each $e_i \in E(G)$ and $i \in [1,q]$, $f(e_i) = g(e_i) + r + pr$.
6. For each $k_i \in E(H)$ and $i \in [1,s]$, $f(k_i) = m_i$, where $m_i \in E$ and no two distinct edges are assigned the same number.

Then, $\bigcup_{i=1}^{r}\{f(u_i)\} = A, \bigcup_{i=1}^{p}\{f(v_i), f(u_1v_i)\} = B, \bigcup_{j=1}^{p}\{f(u_iv_j)|i \in [2,r]\} = C, \{f(v_iv_j)|v_iv_j \in E(G+H)\} = D$ and $\{f(u_iu_j)|u_iu_j \in E(G+H)\} = E$. Clearly, f is a bijective function from $V(G+H) \cup E(G+H)$ to Y.

Let F be a subgraph of $G + H$ isomorphic to $K_2 + H$. Then F contains exactly one edge of $E(G)$, say $v_x v_y$ for some distinct $x, y \in [1, p]$. Then, F has the form $V(F) = V(H) \cup \{v_x, v_y\}$ and $E(F) = E(H) \cup \{v_x v_y\} \cup \{u_i v_j | j \in \{x, y\}, i \in [1, r]\}$. Thus,

$$\begin{aligned} wt_f(F) &= \sum_{i=1}^{r} f(u_i) + f(v_x) + f(v_y) + \sum_{e \in E(H)} f(e) + f(v_x v_y) + \sum_{i=1}^{r} f(u_i v_x) + \sum_{i=1}^{r} f(u_i v_y) \\ &= [g(v_x) + g(v_y) + g(v_x v_y)] + \sum_{i=2}^{r} i + \sum_{i=1}^{s} i + p\left(r^2 + r(s+3) + s - \tfrac{5}{2}\right) + 6\left\lfloor \tfrac{p}{2} \right\rfloor \\ &\quad +2r^2 + rs + qs + 4r. \end{aligned}$$

Since $[g(v_x) + g(v_y) + g(v_x v_y)] = m_g$, we see that $wt_f(F)$ is independent on the choosing of F.

Now, let F' be a subgraph of $G + H$ isomorphic to $2K_2 + H$. F' contains two non-adjacent edges of $E(G)$. Thus, $wt_f(F') = 2wt_f(F) - \left(\sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)\right)$. So, $wt_f(F')$ is independent of F'. $\square$

An example of the labeling depicted in the proof of Theorem 3.1 can be seen in Figure 2 where a $(K_5, 2K_2 + C_3)$-sim-supermagic labeling of $S_{2,0,0,2} + C_3$ is presented.

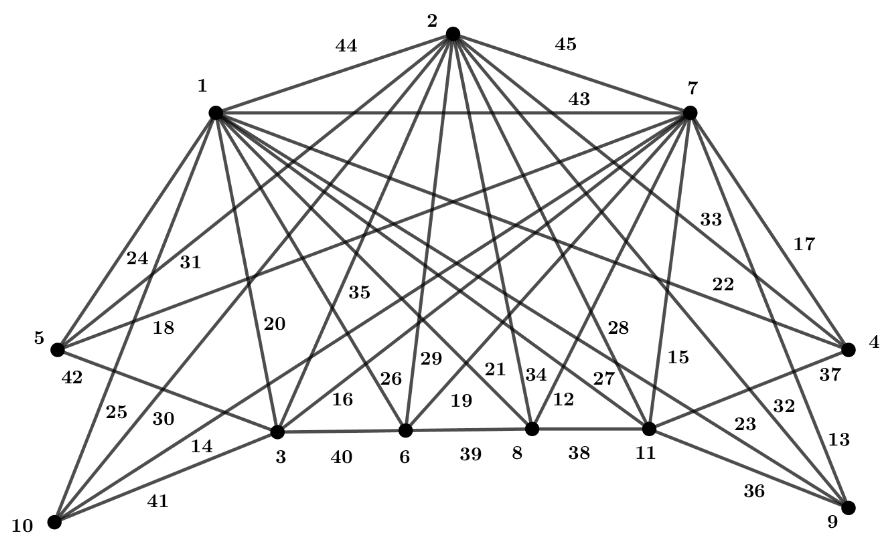

Figure 2. A $(K_5, 2K_2 + C_3)$-sim-supermagic labeling of $S_{2,0,0,2} + C_3$

The following corollary is a consequence of Theorem 3.1 with $H = K_1$.

Corollary 3.1. *Let G be a C_3-free connected graph containing a P_5. If G is SEMT graph, then $G + K_1$ is $(C_3, 2K_2 + K_1)$-sim-supermagic.*

This corollary enlarges the classes of graphs known to be C_3-supermagic; since up to date, only the following join product graphs were known to be C_3-supermagic: $P_n + K_1$, $C_n + K_1$, $K_{1,n} + K_1$, and $nK_2 + K_1$, where $n \geq 3$ [16, 21, 22, 25].

4. Labelings for Cartesian Product Graphs

The *Cartesian product* of two graphs G and H, denoted by $G \times H$, is a graph whose vertex set is $V(G) \times V(H) = \{(u, v)|u \in V(G), u \in V(H)\}$ and for which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $v_1v_2 \in E(H)$ and $u_1 = u_2$.

In this section, we shall study (F, H)-sim-supermagic labeling for the Cartesian product of an arbitrary graph G with K_2. The following notations are used or vertices and edges in $G \times K_2$ For each $x \in V(G)$, let x and x' be the corresponding vertices in the two copies of G in $G \times K_2$, and so $xx' \in E(G \times K_2)$. For each $xy \in E(G)$, denote by xy and $x'y'$ the corresponding edges in the two copies of G in $G \times K_2$.

We summarize the Cartesian product graphs $G \times K_2$ known to be H-magic in Table 3.

Table 3. $G \times K_2$ that are H-magic

Cartesian product	H	Conditions and Reference
$G \times K_2$	C_4	G is C_4-free and SEMT of odd size [16]
$P_m \times K_2$	C_4	$m \geq 3$ [21]
$mK_{1,n} \times K_2$	C_4	$m \geq 2$ and $n \geq 1$ [1]
$s(P_{n+1} \times K_2) \cup k(P_n \times K_2)$	C_4	$s \geq 1$, $k \geq 1$ and $n \geq 2$ [1]
$m(P_n \times K_2)$	C_4	$m \geq 2$ and $n \geq 2$ [23]
$P_n \times K_2$	C_{2m}	$n \geq 4$ and $m \in [3, \lfloor \frac{n}{2} \rfloor + 1]$ [20]
$P_n \times K_2$	$P_m \times K_2$	$n \geq 4$ and $m \in [3, n-1]$ [20]
$G \times K_2$	C_4	G is C_4-free, SEMT and a connected (p, q)-graph where p or q is odd [14]
$(2G) \times K_2$	C_4	G is C_4-free, connected, bipartite (with partite sets U and V) and G has a SEMT labeling f such that $f(U) = [1, \|U\|]$ [14]

In [20], Ngurah et al. constructed $(P_m \times K_2)$-supermagic labelings of the ladder $P_n \times K_2$ for every $m \in [3, n-1]$. A more general result by Baca et al. [7] established the following sufficient conditions for the Cartesian product $G_1 \times G_2$ to be $(H \times G_2)$-supermagic as stated in the following theorem. On the other hand, in [14] and [16] it was proved that if G is connected of odd order or size, C_4-free, and SEMT, then $G \times K_2$ admits a C_4-supermagic labeling.

Theorem 4.1. *[7] Let G_1 be a graph of odd order $p_1 \geq 3$ admitting an H-covering given by t subgraphs isomorphic to H. If G_2 is a graph of even order $q_2 \geq 2$ and odd size $p_2 \geq 3$ and the graph $G_1 \times G_2$ contains exactly t subgraphs isomorphic to $H \times G_2$, then $G_1 \times G_2$ is $(H \times G_2)$-supermagic.*

In the next theorem, we enlarge the classes of graphs known to be $(P_m \times K_2)$-supermagic [20] and extend sufficient conditions for the existence of a C_4-supermagic labeling of $G \times K_2$ [14, 16] without considering a SEMT labeling of G. Furthermore, our result settles the remaining cases of Theorem 4.1 for $p_2 = 1$ and $q_2 = 2$.

Theorem 4.2. *Let G be a C_4-free connected graph of odd order $p \geq 5$. If G admits a P_m-covering for some $m \in [3, p-1]$, then $G \times K_2$ is $(C_4, P_m \times K_2)$-sim-supermagic.*

Proof. Let p and q be the order and the size of G, respectively. Consider $A = [1, 3p + 2q]$ as the set of integers used to label vertices and edges in $G \times P_2$. Now, partition A into three sets $W = [1, 2p], X = [2p + 1, 3p]$, and $Y = [3p + 1, 3p + 2q]$. Since p is odd, by Lemma 2.2, W is $(p, 1)$-anti balanced with $\sum W_i = 2 + i + 3 \left\lfloor \frac{p}{2} \right\rfloor$ for every $i \in [1, p]$. Now, since $|Y| = 2q$, Lemma 2.1 ensures that Y is q-balanced with $\sum Y_j = \frac{2q}{2q}(3p + 1 + 3p + 2q) = 6p + 2q + 1$ for each $j \in [1, q]$.

Let g and h be bijections from $V(G)$ to $[1, p]$ and from $E(G)$ to $[1, q]$, respectively. Next, define a total labeling f of $G \times K_2$. For each $x \in V(G)$, label x and x' in $G \times K_2$ by the elements of $W_{g(x)}$ chosen so that $f(x) < f(x')$ and define $f(xx') = 3p - g(x) + 1$. For each $xy \in E(G)$, define f as a bijection from $\{xy, x'y'\}$ to $Y_{h(xy)}$ with $f(xy) < f(x'y')$. Hence, $\bigcup_{v \in V(G \times K_2)} \{f(v)\} = W$ and $\bigcup_{e \in E(G \times K_2)} \{f(e)\} = X \cup Y$. Consequently, f is a bijective function from $V(G \times K_2) \cup E(G \times K_2)$ to A.

Since G is C_4-free, there are q subgraphs of $G \times K_2$ isomorphic to C_4. Let F be a subgraph of $G \times K_2$ isomorphic to C_4. Then, $V(F) = \{x, x', y, y'\}$ and $E(F) = \{xx', yy', xy, x'y'\}$, where $x, y \in V(G)$ and $xy \in E(G)$. Therefore,

$$\begin{aligned} wt_f(F) &= f(x) + f(x') + f(y) + f(y') + f(xx') + f(yy') + f(xy) + f(x'y') \\ &= \sum W_{g(x)} + \sum W_{g(y)} + 3p - g(x) + 1 + 3p - g(y) + 1 + \sum Y_{h(xy)} \\ &= 12p + 6 \left\lfloor \tfrac{p}{2} \right\rfloor + 2q + 7, \end{aligned}$$

which is independent of F.

Moreover, as G admits a P_m-covering for some $m \in [3, p - 1]$, we have that $G \times K_2$ admits a $(P_m \times K_2)$-covering. Let $H = x_1 x_2 \ldots x_m$ be a subgraph of G isomorphic to P_m. For each H, denote by $H' = x'_1 x'_2 \ldots x'_m$ the corresponding subgraph in G'. Thus, for each H, we obtain H'' with $V(H'') = \{x_1, x_2, \ldots, x_m, x'_1, x'_2, \ldots, x'_m\}$ and $E(H'') = E(H) \cup E(H') \cup \{xx' | x \in V(H)\}$ as the corresponding subgraph in $G \times K_2$ isomorphic to $P_m \times K_2$. We can verify that there are exactly t subgraphs of $G \times K_2$ isomorphic to $P_m \times K_2$, where t is the number of subgraphs isomorphic to P_m in G. Thus,

$$\begin{aligned} wt_f(H'') &= \sum_{v \in V(H)} f(v) + \sum_{v \in V(H')} f(v) + \sum_{e \in E(H)} f(e) + \sum_{e \in E(H')} f(e) + \sum_{v \in V(H)} f(vv') \\ &= \sum_{v \in V(H)} [f(v) + f(v')] + \sum_{e \in E(H)} [f(e) + f(e')] + \sum_{v \in V(H)} [3p - g(v) + 1] \\ &= \sum_{v \in V(H)} \left[\sum W_{g(v)} \right] + \sum_{e \in E(H)} \left[\sum Y_{h(e)} \right] + \sum_{v \in V(H)} [3p - g(v) + 1] \\ &= 3m \left\lfloor \tfrac{p}{2} \right\rfloor + 4m + 9mp + 2mq - 6p - 2q - 1, \end{aligned}$$

which is independent of H''. Hence, $G \times K_2$ is $(C_4, P_m \times K_2)$-sim-supermagic. □

An example of the labeling in the proof of Theorem 4.2 is depicted in Figure 3.

In [20], Ngurah et al. showed that the ladder $P_n \times K_2$ is C_{2m}-supermagic for every $m \in [3, \left\lfloor \frac{n}{2} \right\rfloor + 1]$. Then it is natural to ask for which graphs G, the Cartesian product $G \times K_2$ is

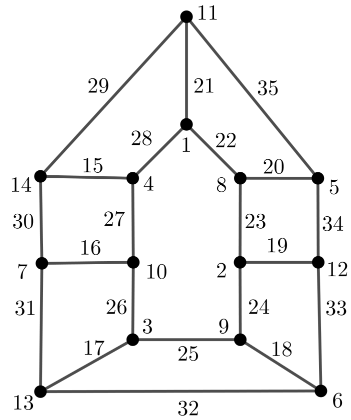

Figure 3. A $(C_4, P_m \times K_2)$-sim-supermagic labeling of $C_7 \times K_2$ for every $m \in [3, 6]$.

(C_{2x}, C_{2y})-sim-supermagic for some $x, y \in \left[3, \left\lfloor \frac{n}{2} \right\rfloor + 1\right]$. We will answer this question in Theorem 4.3, but to do so, we need to recall the following notion that was first introduced by Simanjuntak et al. [27]. An injective function f from $V(G)$ onto the set $\{1, 2, \ldots, |V(G)|\}$ is called (a, d)-*edge-antimagic vertex labeling ((a, d)-EAV)* if the set of edge-weights $\{w(xy) = f(x) + f(y) | xy \in E(G)\} = \{a, a + d, \ldots, a + (|E(G)| - 1)d\}$, where $a > 0$ and $d \geq 0$ are two integers. A graph G is said to be an (a, d)-*edge-antimagic vertex ((a, d)-EAV) graph* if G has an (a, d)-EAV labeling. In [4], it was shown that a connected graph G that is not a tree has no (a, d)-EAV labeling for $d \neq 1$.

Lemma 4.1. *[4] Let G be a connected graph that is not a tree. If G has an (a, d)-EAV labeling, then $d = 1$.*

The next theorem describes a construction of a (C_{2x}, C_{2y})-sim-supermagic labeling of $G \times K_2$ from an $(a, 2)$-EAV labeling for some $x, y \in \left[3, \left\lfloor \frac{n}{2} \right\rfloor + 1\right]$. Due to Lemma 4.1, we restrict our consideration to trees.

Theorem 4.3. *Let m, n and p be positive integers where $3 \leq m < p$. Let G be a tree on p vertices where $p \geq 5$, such that G admits a P_m-covering for some $m \in \left[3, \left\lfloor \frac{n}{2} \right\rfloor + 1\right]$. If G is an $(a, 2)$-EAV graph, then $G \times K_2$ is (C_{2x}, C_{2y})-sim-supermagic for all $x, y \in [2, m]$.*

Proof. Let p and q be the order and the size of G, respectively. Let $g : V(G) \to \{1, 2, \ldots, p\}$ be an $(a, 2)$-EAV labeling of G.

Since $|V(G \times K_2)| = 2p$ and $|E(G \times K_2)| = p + 2q$, the set of labels used to label vertices and edges of $G \times K_2$ is $A = [1, 3p+2q]$. Now, partition A into three sets $W = [1, 2p]$, $X = [2p+1, 3p]$ and $Y = [3p + 1, 3p + 2q]$. By Lemma 2.3, W is $(p, 2)$-anti balanced with $\sum W_i = 2i + p$ for every $i \in [1, p]$. According to Lemma 2.3, Y is $(q, 2)$-anti balanced with $\sum Y_j = 6p + q + 2j$ for each $j \in [1, q]$.

Next, define a total labeling f of $G \times K_2$. For each $x \in V(G)$, label the corresponding vertices x and x' in $G \times K_2$ by the elements of $W_{g(x)}$ chosen so that $f(x) < f(x')$. For each $x \in V(G)$, define $f(xx') = 3p + 1 - g(x)$. Now, for each $xy \in E(G)$, label the corresponding edges xy and

$x'y'$ in $G \times K_2$ by the elements of Y_r where $r = \frac{1}{2}(2q + a - g(x) - g(y))$ such that $f(xy) < f(x'y')$. It follows easily that f depends on g. Then, f is a bijective function from $V(G \times K_2) \cup E(G \times K_2)$ to A.

Since G admits a P_m-covering for some $m \in \left[3, \left\lfloor \frac{n}{2} \right\rfloor + 1\right]$, $G \times K_2$ admits C_{2z}-covering for every $z \in [2, m]$. Let $H = x_1x_2 \ldots x_z$ be a subgraph of G isomorphic to P_z for an arbitrary $z \in [2, m]$. For each H, denote by $H' = x'_1x'_2 \ldots x'_z$ the corresponding subgraph in G'. Thus, for each H, we obtain H'' as the corresponding subgraph in $G \times K_2$ isomorphic to C_{2z} where $V(H'') = V(H) \cup V(H')$ and $E(H'') = E(H) \cup E(H') \cup \{x_1x'_1, x_zx'_z\}$. We can verify that there are exactly t subgraphs of $G \times K_2$ isomorphic to C_{2z}, where t is the number of subgraphs in G isomorphic to P_z. Thus,

$$\begin{aligned} wt_f(H'') &= \sum_{v \in V(H)} f(v) + \sum_{v \in V(H')} f(v) + \sum_{uv \in E(H)} f(uv) + \sum_{uv \in E(H')} f(uv) + f(x_1x'_1) + f(x_zx'_z) \\ &= \sum_{v \in V(H)} [f(v) + f(v')] + \sum_{uv \in E(H)} [f(uv) + f(u'v')] + 3p + 1 - g(x_1) + 3p + 1 - g(x_z) \\ &= \sum_{v \in V(H)} \left[\sum W_{g(v)}\right] + \sum_{uv \in E(H)} \left[\sum Y_{\frac{1}{2}(2q+a-g(u)-g(v))}\right] + 6p + 2 - g(x_1) - g(x_z) \\ &= 7zp + 3zq - 3q + az - a + 2, \end{aligned}$$

which is independent of H''.

Therefore, $G \times K_2$ is (C_{2x}, C_{2y})-sim-supermagic for all $x, y \in [2, m]$. □

An example of the labeling in Theorem 4.3 can be seen in Figure 4.

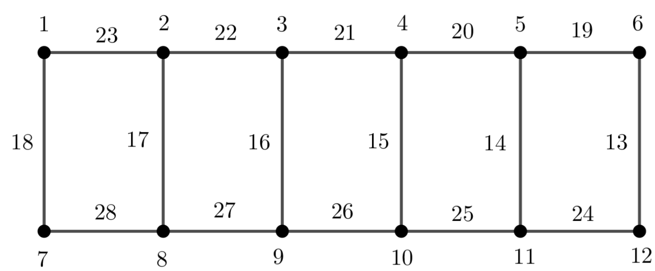

Figure 4. A (C_4, C_{2m})-sim-supermagic labeling of $P_6 \times K_2$ for $m = 3$ and 4

Note that the preceding theorem enlarges the classes of graphs known to be C_{2m}-supermagic, as stated in Table 3. For instance, since every path P_n was shown to be $(3, 2)$-EAV [27], an immediate consequence of Theorem 4.3 is that the ladder $P_n \times K_2$ is (C_4, C_{2m})-sim-supermagic for every $m \in [3, \lfloor \frac{n}{2} \rfloor + 1]$.

In [2], Bača and Barrientos described a connection between an α-labeling and an $(a, 2)$-EAV labeling of graphs. An injective mapping $f : V(G) \to [0, |E(G)|]$ is said to be *graceful labeling* if $|f(x) - f(y)|$ are distinct for each $xy \in E(G)$. A graceful labeling f is called an *α-labeling* if there exists an integer λ such that for each edge xy either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$ [24].

A graph G that admits an α-labeling is said to be an *α-graph*. From the definition of α-labeling, it follows that an α-graph is necessarily bipartite.

Let $\{A, B\}$ be the natural bipartition of the vertex set of an α-graph. Bača and Barrientos [2] presented the following theorem.

Theorem 4.4. *[2] A tree T is a $(3, 2)$-EAV graph if and only if T is an α-graph and $||A|-|B|| \leq 1$, where $\{A, B\}$ is the natural bipartition of the vertex set of T.*

Theorem 4.3 together with Theorem 4.4 implies the relationship between an α-labeling of a tree T and a (C_4, C_6)-sim-supermagic labeling of the Cartesian product $T \times K_2$. Let $n \geq 2$ be a positive integer and let T be an α-tree and $||A| - |B|| \leq 1$, where $\{A, B\}$ is the natural bipartition of the vertex set of T. It is clear that $T \times K_2$ admits a C_4-covering and a C_6-covering only if T is not isomorphic to a star.

Corollary 4.1. *Let T be an α-tree not isomorphic to a star on at least five vertices and let $||A| - |B|| \leq 1$, where $\{A, B\}$ is the natural bipartition of the vertex set of T. Then $T \times K_2$ is (C_4, C_6)-sim-supermagic.*

Figure 5 illustrates a (C_4, C_6)-sim-supermagic labeling of product graph $S_{2,1,0,1} \times K_2$.

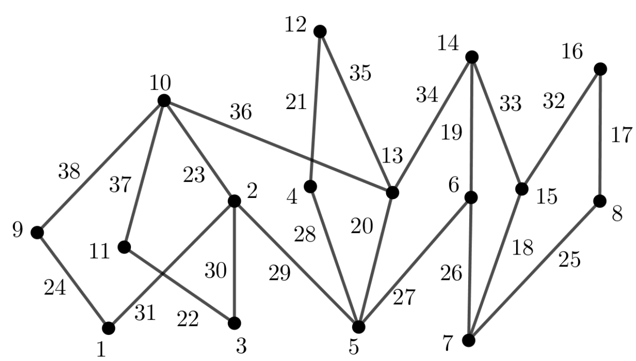

Figure 5. A (C_4, C_6)-sim-supermagic labeling of $S_{2,1,0,1} \times K_2$.

A *perfect matching* of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching. Brankovic et al. [8] posed the following conjecture for α-trees.

Conjecture 1. *[8] All trees with maximum degree three and a perfect matching have an α-labeling.*

Consider a tree T with a perfect matching. Since T is bipartite, by a perfect matching in T, we have a natural bipartition of the vertex-set of T, namely A and B, such that $||A| - |B|| \leq 1$. As a direct consequence of Corollary 4.1 and Conjecture 1, the following holds.

Theorem 4.5. *Let T be a tree on at least five vertices that are not isomorphic to a star, with a maximum degree three and containing a perfect matching. If Conjecture 1 is true, then $T \times K_2$ is (C_4, C_6)-sim-supermagic.*

Although all our results in this section are restricted to trees, the proof of Theorem 7 in [16] implied that $C_n \times K_2$ is (C_4, C_{2m})-sim-supermagic for each odd $n \geq 5$ and $m = 2$. Thus, it is interesting to seek conditions such that a Cartesian product of a non-tree graph G with K_2 admits a (C_4, C_{2m})-sim-supermagic labeling.

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References

[1] M. Asif, G. Ali, M. Numan, and A. Semaničová-Feňovčíková, Cycle-supermagic labeling for some families of graphs, *Util. Math.* **103** (2017), 51–59.

[2] M. Bača and C. Barrientos, Graceful and edge-antimagic labelings, *Ars Combin.* **96** (2010) 505–513.

[3] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak, and Slamin, Vertex-antimagic total labelings of graphs, *Discuss. Math. Graph Theory* **23** (2003), 67–83.

[4] M. Bača and M. Miller, Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solutions, *Brown Walker Press, Boca Raton* (2008).

[5] M. Bača, M. Miller, J. Ryan, and A. Semaničová-Feňovčíková, Magic and Antimagic Graphs: Attributes, Observations and Challenges in Graph Labelings, *Springer International Publushing, Switzerland AG* (2019).

[6] M. Bača, M. Miller, O. Phanalasy, J. Ryan, and A. Semaničová-Feňovčíková, A.A. Sillasen, Totally antimagic total graphs, *Australas. J. Combin.* **61** (2015), 42–56.

[7] M. Bača, A. Semaničová-Feňovčíková, M.A. Umar, and D. Welyyanti, On H-antimagicness of Cartesian product of graphs, *Turkish J. Math.* **42** (2018), 339–348.

[8] L. Brankovic, C. Murch, J. Pond, and A. Rosa, Alpha-size of trees with maximum degree three and perfect matching, *In Proceedings of the 16th Australasian Workshop on Combinatorial Algorithms, Ballarat, Australia* (2005), pp. 47–56.

[9] G. Exoo, A.C.H. Ling, and J.P. McSorley, N.C.K. Philips, W.D. Wallis, Totally magic graphs, *Discrete Math.* **254** (2002), 103–113.

[10] R.M. Figueroa-Centeno, R. Ichishima, and F.A. Muntaner-Batle, On super edge-magic graphs, *Ars Combin.* **64** (2002), 81–95.

[11] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **23** (2020), #DS6.

[12] A. Gutiérrez and A. Lladó, Magic coverings, *J. Combin. Math. Combin. Comput.* **55** (2005), 43–56.

[13] N. Inayah, R. Simanjuntak, A.N.M. Salman, and K.I.A. Syuhada, Super (a, d)-H-antimagic total labelings for shackles of a connected graph H, *Australas. J. Combin.* **57** (2013), 127–138.

[14] T. Kojima, On C_4-supermagic labelings of the Cartesian product of paths and graphs, *Discrete Math.* **313** (2013), 164–173.

[15] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[16] A. Lladó and J. Moragas, Cycle-magic graphs, *Discrete Math.* **307** (2007), 2925–2933.

[17] J.A. MacDougall, M. Miller, Slamin, and W.D. Wallis, Vertex-magic total labelings of graphs, *Util. Math.* **61** (2002), 3–21.

[18] T.K. Maryati, A.N.M. Salman, and E.T. Baskoro, Supermagic coverings of the disjoint union of graphs and amalgamations, *Discrete Math.* **313(4)** (2013), 397–405.

[19] T.K. Maryati, A.N.M. Salman, E.T. Baskoro, J. Ryan, and M. Miller, On H-supermagic labelings for certain shackles and amalgamations of a connected graph, *Util. Math.* **83** (2010), 333–342.

[20] A.A.G. Ngurah, A.N.M. Salman, and I.W. Sudarsana, On supermagic coverings of fans and ladders, *SUT J. Math.* **46** (2010), 67–78.

[21] A.A.G. Ngurah, A.N.M. Salman, and L.Susilowati, H-supermagic labelings of graphs, *Discrete Math.* **310** (2010), 1293–1300.

[22] A. Ovais, M.A. Umar, M. Bača, and A. Semaničová-Feňovčíková, Fans are cycle-antimagic, *Australas. J. Combin.* **68** (2017), 94-105.

[23] S.T.R. Rizvi, K. Ali, and M. Hussain, Cycle-supermagic labelings of the disjoint union of graphs, *Util. Math.* **104** (2017), 215–226.

[24] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs, Internat. Symposium, Rome, July 1996, Gordon and Breach, N. Y. and Dunod Paris* (1967), 349–355.

[25] M. Roswitha, E.T. Baskoro, T.K. Maryati, N.A. Kurdhi, and I. Susanti, Further results on cycle-supermagic labeling, *AKCE Int. J. Graphs Comb.* **10(2)** (2013), 211–220.

[26] A. Semaničová-Feňovčíková, M. Bača, M. Lascsáková, M. Miller, and J. Ryan, Wheels are cycle-antimagic, *Electron. Notes Discrete Math.* **48** (2015), 11–18.

[27] R. Simanjuntak, M. Miller, and F. Bertault, Two new (a, d)-antimagic graph labelings, *In Proceedings of the 11th Australasian Workshop on Combinatorial Algorithms, Hunter Valley, Australia* (2000), pp. 179–189.

[28] Slamin, M. Bača, Y. Lin, M. Miller, and R. Simanjuntak, Edge-magic total labelings of wheels, fans and friendship graphs, *Bull. Inst. Combin. Appl.* **35** (2002), 89–98.

[29] W.D. Wallis, Magic Graphs, *Birkäusher Basel, Boston* (2001).