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On d-antimagic labelings of plane graphs

Martin Bača[a], Ljiljana Brankovic[b], Marcela Lascsáková[a], Oudone Phanalasy[c], Andrea Semaničová–Feňovčíková[a]

[a]*Department of Applied Mathematics and Informatics, Technical University, Košice, Slovak Republic*
[b]*School of Electrical Engineering and Computer Science, The University of Newcatle, Australia*
[c]*Department of Mathematics, National University of Laos, Vientiane, Laos*

martin.baca@tuke.sk, ljiljana.brankovic@newcastle.edu.au, marcela.lascsakova@tuke.sk,
oudone.phanalasy@newcastle.edu.au, andrea.fenovcikova@tuke.sk

Abstract

The paper deals with the problem of labeling the vertices and edges of a plane graph in such a way that the labels of the vertices and edges surrounding that face add up to a weight of that face. A labeling of a plane graph is called d*-antimagic* if for every positive integer s, the s-sided face weights form an arithmetic progression with a difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices.
In the paper we examine the existence of such labelings for several families of plane graphs.

Keywords: plane graph, d-antimagic labeling, super d-antimagic labeling
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1. Introduction

Let $G = (V, E, F)$ be a finite connected plane graph without loops and multiple edges, where V, E and F are its vertex set, edge set and face set, respectively. Let $|V(G)| = p$, $|E(G)| = q$ and $|F(G)| = r$ be the number of the vertices, the edges and the faces, respectively.

A labeling of type $(1, 1, 1)$ assigns labels from the set $\{1, 2, \ldots, p+q+r\}$ to the vertices, edges and faces of a plane graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label.

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A labeling of type $(1,1,0)$ is a bijection from the set $\{1,2,\dots,p+q\}$ to the vertices and edges of a graph G.

The *weight of a face* under a labeling is the sum of labels (if present) carried by that face and the edges and vertices on its boundary.

A labeling of a plane graph G is called *d-antimagic* if for every positive integer s the set of s-sided face weights is $W_s = \{a_s, a_s+d, a_s+2d, \dots, a_s+(r_s-1)d\}$ for some integers a_s and $d \geq 0$, where r_s is the number of s-sided faces. We allow different sets W_s for different s.

If $d = 0$ then Ko-Wei Lih in [16] called such labeling *magic*. Ko-Wei Lih [16] described magic (0-antimagic) labelings of type $(1,1,0)$ for the wheels, the friendship graphs and the prisms. The magic labelings of type $(1,1,1)$ for the grid graphs and the honeycomb are given in [2] and [3], respectively.

The concept of the d-antimagic labeling of the plane graphs was defined in [10], where it was also proved that the prism D_n has d-antimagic labelings of type $(1,1,1)$ for $d \in \{2,3,4,6\}$ and $n \equiv 3 \pmod 4$. The d-antimagic labelings of type $(1,1,1)$ for the hexagonal planar maps, the generalized Petersen graph $P(n,2)$ and the grids can be found in [5], [7] and [8], respectively. Lin *et al.* in [17] showed that prism D_n, $n \geq 3$, admits d-antimagic labelings of type $(1,1,1)$ for $d \in \{2,4,5,6\}$. The d-antimagic labelings of type $(1,1,1)$ for D_n and for several $d \geq 7$ are described in [19].

A d-antimagic labeling is called *super* if the smallest possible labels appear on the vertices. The super d-antimagic labelings of type $(1,1,1)$ for antiprisms and for $d \in \{0,1,2,3,4,5,6\}$ are described in [4], and for disjoint union of prisms and for $d \in \{0,1,2,3,4,5\}$ are given in [1]. The existence of super d-antimagic labelings of type $(1,1,1)$ for disconnected plane graphs and for plane graphs containing a special Hamilton path is examined in [6] and [12], respectively.

In this paper we examine the existence of super d-antimagic labelings of type $(1,1,0)$ for several families of plane graphs. To label the vertices and edges of plane graphs we will use an edge-antimagic vertex labeling and an edge-antimagic total labeling.

Simanjuntak, Bertault and Miller in [18] define an *(a,d)-edge-antimagic vertex labeling* of a (p,q)-graph $G = (V,E)$ as an injective mapping $\beta : V(G) \to \{1,2,\dots,p\}$ such that the set of edge-weights $\{\beta(u)+\beta(v) : uv \in E(G)\}$ is $\{a, a+d, a+2d, \dots, a+(q-1)d\}$ for two non-negative integers a and d. A bijection $\alpha : V(G) \cup E(G) \to \{1,2,\dots,p+q\}$ is called an *(a,d)-edge-antimagic total labeling* of G if the edge-weights $\{\alpha(u)+\alpha(uv)+\alpha(v) : uv \in E(G)\}$ form an arithmetic sequence starting at a and having a common difference d, where $a > 0$ and $d \geq 0$ are two fixed integers. An (a,d)-edge-antimagic total labeling is a natural extension of a notion of *magic valuation* defined by Kotzig and Rosa in [15].

An (a,d)-edge-antimagic total labeling is called *super* if the smallest possible labels appear on the vertices. A super (a,d)-edge-antimagic total labeling is a natural extension of a notion of *super edge-magic labeling* defined by Enomoto *et al.* in [13].

More comprehensive information on magic valuations and (a,d)-edge-antimagic total labelings can be found in [11], [14] and [20], respectively.

2. Edge-antimagic labelings of paths

Let P_n be the path on n vertices. It is known (see [9]), that P_n is super (a, d)-edge-antimagic total if and only if $d \leq 3$. We denote the vertices of P_n by $v_1, v_2, \ldots, v_n$ and describe these labelings $\alpha_d^n : V(P_n) \cup E(P_n) \to \{1, 2, \ldots, 2n-1\}$ in the following way.

a) The super $(2n + \lceil \frac{n}{2} \rceil + 1, 0)$-edge-antimagic total labeling α_0^n of P_n:

$$\alpha_0^n(v_i) = \begin{cases} \frac{i+1}{2} & \text{for} \quad i \equiv 1 \pmod 2 \text{ and } 1 \leq i \leq n, \\ \lceil \frac{n}{2} \rceil + \frac{i}{2} & \text{for} \quad i \equiv 0 \pmod 2 \text{ and } 2 \leq i \leq n, \end{cases}$$
$$\alpha_0^n(v_i v_{i+1}) = 2n - i \qquad \text{for} \quad i = 1, 2, \ldots, n-1.$$

The common weight for all edges of P_n is

$$w_{\alpha_0^n}(v_i v_{i+1}) = 2n + \left\lceil \frac{n}{2} \right\rceil + 1 = C_{\alpha,0}^n, \;\; i = 1, 2, \ldots, n-1.$$

b) The super $(2n + 2, 1)$-edge-antimagic total labeling α_1^n of P_n:

$$\alpha_1^n(v_i) = i \qquad \text{for} \quad i = 1, 2, \ldots, n,$$
$$\alpha_1^n(v_i v_{i+1}) = 2n - i \qquad \text{for} \quad i = 1, 2, \ldots, n-1.$$

The set of edge-weights of P_n consists of the consecutive integers

$$\{w_{\alpha_1^n}(v_i v_{i+1}) = 2n + 1 + i = C_{\alpha,1}^n + i : i = 1, 2, \ldots, n-1\}.$$

c) The super $(n + \lceil \frac{n}{2} \rceil + 3, 2)$-edge-antimagic total labeling α_2^n of P_n:

$$\alpha_2^n(v_i) = \begin{cases} \frac{i+1}{2} & \text{for} \quad i \equiv 1 \pmod 2, \\ \lceil \frac{n}{2} \rceil + \frac{i}{2} & \text{for} \quad i \equiv 0 \pmod 2, \end{cases}$$
$$\alpha_2^n(v_i v_{i+1}) = n + i \qquad \text{for} \quad i = 1, 2, \ldots, n-1.$$

The edge-weights of P_n constitute the arithmetic progression of difference 2:

$$\{w_{\alpha_2^n}(v_i v_{i+1}) = n + \left\lceil \frac{n}{2} \right\rceil + 1 + 2i = C_{\alpha,2}^n + 2i : i = 1, 2, \ldots, n-1\}.$$

d) The super $(n + 4, 3)$-edge-antimagic total labeling α_3^n of P_n:

$$\alpha_3^n(v_i) = i \qquad \text{for} \quad i = 1, 2, \ldots, n,$$
$$\alpha_3^n(v_i v_{i+1}) = n + i \qquad \text{for} \quad i = 1, 2, \ldots, n-1.$$

The edge-weights of P_n constitute the arithmetic progression of difference 3:

$$\{w_{\alpha_3^n}(v_i v_{i+1}) = n + 1 + 3i = C_{\alpha,3}^n + 3i : i = 1, 2, \ldots, n-1\}.$$

Now, we define the super (a,d)-edge-antimagic total labelings of P_n also for negative differences d in the following way

$$\alpha^n_{-k}(v_i) = \alpha^n_k(v_{n+1-i}) \qquad \text{for} \quad i = 1,2,\dots,n,$$
$$\alpha^n_{-k}(v_iv_{i+1}) = \alpha^n_k(v_{n+1-i}v_{n-i}) \qquad \text{for} \quad i = 1,2,\dots,n-1,$$

where $k = 0,1,2,3$.

In this paper we will use also (a,d)-edge-antimagic vertex labelings of P_n for two differences $d = 1$ and $d = 2$. These labelings $\beta^n_d : V(P_n) \to \{1,2,\dots,n\}$ we define in the following way:

e) The $(\lceil \frac{n}{2} \rceil + 2, 1)$-edge-antimagic vertex labeling β^n_1 of P_n:

$$\beta^n_1(v_i) = \begin{cases} \frac{i+1}{2} & \text{for} \quad i \equiv 1 \pmod 2, \\ \lceil \frac{n}{2} \rceil + \frac{i}{2} & \text{for} \quad i \equiv 0 \pmod 2. \end{cases}$$

The set of edge-weights of P_n consists of the consecutive integers

$$\{w_{\beta^n_1}(v_iv_{i+1}) = \left\lceil \frac{n}{2} \right\rceil + 1 + i = C^n_{\beta,1} + i : i = 1,2,\dots,n-1\}.$$

f) The $(3,2)$-edge-antimagic vertex labeling β^n_2 of P_n:

$$\beta^n_2(v_i) = i \qquad \text{for} \quad i = 1,2,\dots,n.$$

The edge-weights of P_n constitute the arithmetic progression of difference 2:

$$\{w_{\beta^n_2}(v_iv_{i+1}) = 1 + 2i = C^n_{\beta,2} + 2i : i = 1,2,\dots,n-1\}.$$

The (a,d)-edge-antimagic vertex labelings of P_n for d negative we define as follows

$$\beta^n_{-l}(v_i) = \beta^n_l(v_{n+1-i}) \qquad \text{for} \quad i = 1,2,\dots,n,$$

where $l = 1,2$.

3. Partitions with determined differences

For construction of vertex and edge labelings of plane graphs we will use the partitions of a set of integers with determined differences.

Let n,k,d and i be positive integers. We will consider the partition $\mathcal{P}^n_{k,d}$ of the set $\{1,2,\dots, kn\}$ into n, $n \geq 2$, k-tuples such that the difference between the sum of the numbers in the $(i+1)$th k-tuple and the sum of the numbers in the ith k-tuple is always equal to the constant d, where $i = 1,2,\dots,n-1$. Thus they form an arithmetic sequence with the difference d. By the symbol $\mathcal{P}_{k,d}(i)$ we denote the ith k-tuple in the partition with the difference d, where $i = 1,2,\dots,n$.

Let $\sum \mathcal{P}^n_{k,d}(i)$ be the sum of the numbers in $\mathcal{P}^n_{k,d}(i)$. Evidently $\sum \mathcal{P}^n_{k,d}(i+1) - \sum \mathcal{P}^n_{k,d}(i) = d$. It is obvious that if there exists a partition of the set $\{1,2,\dots,kn\}$ with the difference d, there also exists a partition with the difference $-d$. By the notation $\mathcal{P}^n_{k,d}(i) \oplus c$ we mean that we add the constant c to every number in $\mathcal{P}^n_{k,d}(i)$.

If $k = 1$ then only the following partition of the set $\{1, 2, \ldots, n\}$ is possible

$$\mathcal{P}^n_{1,1}(i) = \{i\} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

If $k = 2$ then we have several partitions of the set $\{1, 2, \ldots, 2n\}$. Let us define the partitions into 2-tuples in the following way:

$$\begin{aligned} \mathcal{P}^n_{2,0}(i) &= \{i, 2n+1-i\}, & & \\ \sum \mathcal{P}^n_{2,0}(i) &= 2n+1, & &\text{for} \quad i = 1, 2, \ldots, n. \\ \mathcal{P}^n_{2,2}(i) &= \{i, n+i\}, & & \\ \sum \mathcal{P}^n_{2,2}(i) &= n+2i, & &\text{for} \quad i = 1, 2, \ldots, n. \\ \mathcal{P}^n_{2,4}(i) &= \{2i-1, 2i\}, & & \\ \sum \mathcal{P}^n_{2,4}(i) &= 4i-1, & &\text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Moreover, for $3 \le n \equiv 1 \pmod 2$

$$\begin{aligned} \mathcal{P}^n_{2,1}(i) &= \begin{cases} \{\frac{n+1}{2} + \frac{i-1}{2}, n+1+\frac{i-1}{2}\} & \text{for} \quad i \equiv 1 \pmod 2, \\ \{\frac{i}{2}, n + \frac{n+1}{2} + \frac{i}{2}\} & \text{for} \quad i \equiv 0 \pmod 2, \end{cases} \\ \sum \mathcal{P}^n_{2,1}(i) &= n + \frac{n+1}{2} + i, \qquad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Note that we are also able to obtain the partitions into 2-tuples $\mathcal{P}^n_{2,0}(i)$ and $\mathcal{P}^n_{2,2}(i)$ as $\mathcal{P}^n_{1,s}(i) \cup \left(\mathcal{P}^n_{1,t}(i) \oplus n\right)$, where $s, t = \pm 1$. We can use this idea to construct the other partitions. More precisely,

$$\mathcal{P}^n_{k,d}(i) = \mathcal{P}^n_{l,s}(i) \cup \left(\mathcal{P}^n_{m,t}(i) \oplus ln\right),$$

where $k = l + m$.

For example, we are able to obtain $\mathcal{P}^n_{3,d}(i)$ from the partitions $\mathcal{P}^n_{1,s}(i)$, $s = \pm 1$ and $\mathcal{P}^n_{2,t}(i)$, $t = 0, \pm 2, \pm 4$ and also $t = \pm 1$ for n odd. It means, $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5$ and if $n \equiv 1 \pmod 2$ also for $d = 0, \pm 2$. Moreover, we are able to construct $\mathcal{P}^n_{3,9}$ in the following way

$$\begin{aligned} \mathcal{P}^n_{3,9}(i) &= \{3(i-1)+1, 3(i-1)+2, 3(i-1)+3\}, \\ \sum \mathcal{P}^n_{3,9}(i) &= 9i - 3, \quad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned}$$

Thus $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5, \pm 9$ and if $n \equiv 1 \pmod 2$ also for $d = 0, \pm 2$.

For the partition into 4-tuples we can use the following fact

$$\mathcal{P}^n_{4,d}(i) = \mathcal{P}^n_{l,s}(i) \cup \left(\mathcal{P}^n_{m,t}(i) \oplus ln\right),$$

where $l = 3, m = 1$ or $l = 2, m = 2$. Also

$$\mathcal{P}^n_{4,16}(i) = \{4(i-1)+1, 4(i-1)+2, 4(i-1)+3, 4(i-1)+4\},$$
$$\sum \mathcal{P}^n_{4,16}(i) = 16i - 6, \quad \text{for} \quad i = 1, 2, \dots, n.$$

Thus $\mathcal{P}^n_{4,d}$ exists for $d = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$ and if $n \equiv 1 \pmod 2$ also for $d = \pm 1, \pm 3, \pm 5$.

Let us note that each of the defined partition $\mathcal{P}^n_{k,d}$ has the property that

$$\sum \mathcal{P}^n_{k,d}(i) = C^n_{k,d} + di,$$

where $C^n_{k,d}$ is a constant depending on the parameters k and d.

4. d-antimagic labelings for certain families of plane graphs

In this section, we shall use the edge-antimagic labelings of paths P_n and the partitions of the set $\{1, 2, \dots, kn\}$ with determined differences described in the previous two sections to examine the existence of a super d-antimagic labeling for several families of plane graphs.

The *friendship graph* F_n is a set of n triangles having a common central vertex, say v, and otherwise disjoint. The friendship graph F_n has $2n$ vertices of degree 2, say v_i, u_i for $i = 1, 2, \dots, n$, and $3n$ edges, say v_iv, u_iv, v_iu_i for $i = 1, 2, \dots, n$. Let us define the 3-sided face f_i, $i = 1, 2, \dots, n$, as the face bounded by the edges vv_i, v_iu_i and u_iv and let f be the external unbounded face.

Theorem 4.1. *The friendship graph F_n, $n \geq 2$, has a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{1, 3, 5, 7, 9, 11, 13\}$.*
Moreover, if $n \equiv 1 \pmod 2$ then the graph F_n also admits a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{0, 2, 4, 6, 8, 10\}$.

Proof. We define the bijection $g_1 : V(F_n) \cup E(F_n) \to \{1, 2, \dots, 5n+1\}$ as follows:

$$\begin{aligned} &\{g_1(v_i), g_1(u_i)\} = \mathcal{P}^n_{2,k}(i), && i = 1, 2, \dots, n, \\ &g_1(v) = 2n+1, \\ &\{g_1(v_iv), g_1(v_iu_i), g_1(u_iv)\} = \mathcal{P}^n_{3,l}(i) \oplus (2n+1) && i = 1, 2, \dots, n, \end{aligned}$$

where $k = 0, \pm 2, \pm 4$ or for $n \equiv 1 \pmod 2$ also $k = \pm 1$, and $l = \pm 1, \pm 3, \pm 5, \pm 9$ or for $n \equiv 1 \pmod 2$ also $l = 0, \pm 2$.

It is not difficult to check that the vertices are labeled by the smallest possible numbers $1, 2, \dots, 2n+1$. Moreover, for the weight of the face f_i, $i = 1, 2, \dots, n$, we obtain

$$\begin{aligned} w_{g_1}(f_i) &= \big(g_1(v_i) + g_1(u_i)\big) + g_1(v) + \big(g_1(v_iv) + g_1(v_iu_i) + g_1(u_iv)\big) \\ &= \sum \mathcal{P}^n_{2,k}(i) + (2n+1) + \sum (\mathcal{P}^n_{3,l}(i) \oplus (2n+1)) \\ &= \big(C^n_{2,k} + ki\big) + (2n+1) + \big(C^n_{3,l} + li + 3(2n+1)\big) \\ &= C^n_{2,k} + C^n_{3,l} + 4(2n+1) + (k+l)i. \end{aligned}$$

As $k = 0, \pm 2, \pm 4$ or for $n \equiv 1 \pmod 2$ also $k = \pm 1$, and $l = \pm 1, \pm 3, \pm 5, \pm 9$ or for $n \equiv 1 \pmod 2$ also $l = 0, \pm 2$, we obtain that $k + l \in \{1, 3, 5, 7, 9, 11, 13\}$ and for n odd we also get $k + l \in \{0, 2, 4, 6, 8, 10\}$. Thus under the labeling g_1 the weights of the 3-sided faces form an arithmetic sequence with the desired difference, which completes the proof. □

If we replace every edge $v_i u_i$, $i = 1, 2, \dots, n$, of the friendship graph F_n by a path of length two with vertices v_i, w_i, u_i, then we obtain a graph, say B_n, with the vertex set $V(B_n) = \{v_i, w_i, u_i, v : i = 1, 2, \dots, n\}$ and the edge set $E(B_n) = \{v_i v, u_i v, v_i w_i, w_i u_i : i = 1, 2, \dots, n\}$. Let us define the 4-sided face f_i, $i = 1, 2, \dots, n$, as the face bounded by the edges vv_i, $v_i w_i$, $w_i u_i$ and $u_i v$ and let f be the external unbounded face.

Theorem 4.2. *The graph B_n, $n \geq 2$, has a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{1, 3, 5, \dots, 21, 25\}$.*
Moreover, if $n \equiv 1 \pmod 2$ then the graph B_n also admits a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{0, 2, 4, \dots, 18\}$.

Proof. We define the bijection $g_2 : V(B_n) \cup E(B_n) \to \{1, 2, \dots, 7n + 1\}$ in the following way:

$$\begin{aligned} &\{g_2(v_i), g_2(w_i), g_2(u_i)\} = \mathcal{P}^n_{3,k}(i), && i = 1, 2, \dots, n, \\ &g_2(v) = 3n + 1, && \\ &\{g_2(v_i v), g_2(v_i w_i), g_2(w_i u_i), g_2(u_i v)\} = \mathcal{P}^n_{4,l}(i) \oplus (3n + 1), && i = 1, 2, \dots, n. \end{aligned}$$

It is not difficult to see that the vertices are labeled by the numbers $1, 2, \dots, 3n + 1$. Moreover, for the weight of the face f_i, $i = 1, 2, \dots, n$, we have

$$\begin{aligned} w_{g_2}(f_i) =& \big(g_2(v_i) + g_2(w_i) + g_2(u_i)\big) + g_2(v) \\ &+ \big(g_2(v_i v) + g_2(v_i w_i) + g_2(w_i u_i) + g_2(u_i v)\big) \\ =& \sum \mathcal{P}^n_{3,k}(i) + (3n + 1) + \sum (\mathcal{P}^n_{4,l}(i) \oplus (3n + 1)) \\ =& \big(C^n_{3,k} + ki\big) + (3n + 1) + \big(C^n_{4,l} + li + 4(3n + 1)\big) \\ =& C^n_{3,k} + C^n_{4,l} + 5(3n + 1) + (k + l)i, \end{aligned}$$

where $k = \pm 1, \pm 3, \pm 5, \pm 9$ or for $n \equiv 1 \pmod 2$ also $k = 0, \pm 2$, and $l = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$ or for $n \equiv 1 \pmod 2$ also $l = \pm 1, \pm 3, \pm 5$. It means that g_2 is a super d-antimagic labeling of type $(1, 1, 0)$ of B_n, for $d = 1, 3, 5, \dots, 21, 25$ and if $n \equiv 1 \pmod 2$ then $d = 0, 2, 4, \dots, 18$. □

A *triangular snake* E_n is a triangular cactus whose block-cutpoint graph is a path, i.e. E_n is obtained from a path $v_1, v_2, \dots, v_{n+1}$ by joining v_i and v_{i+1} to a new vertex u_i, for $i = 1, 2, \dots, n$. Let f_i be the 3-sided face, $i = 1, 2, \dots, n$, bounded by the edges $v_i u_i$, $u_i v_{i+1}$ and $v_i v_{i+1}$. We denote the external unbounded face by the symbol f.

Theorem 4.3. *The graph E_n, $n \geq 2$, has a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{0, 1, 2, \dots, 12\}$.*

Proof. Define the bijection $g_3 : V(E_n) \cup E(E_n) \to \{1, 2, \dots, 5n+1\}$ as follows:

$$\begin{aligned} &g_3(v_i) = \alpha_k^{n+1}(v_i), && i = 1, 2, \dots, n+1,\\ &g_3(u_i) = \alpha_k^{n+1}(v_i v_{i+1}), && i = 1, 2, \dots, n,\\ &\{g_3(v_i u_i), g_3(u_i v_{i+1}), g_3(v_{i+1} v_i)\} = \mathcal{P}_{3,l}^n(i) \oplus (2n+1), && i = 1, 2, \dots, n. \end{aligned}$$

The labeling g_3 assigns the numbers $1, 2, \dots, 2n+1$ to the vertices of the graph E_n. For the weight of the face f_i, $i = 1, 2, \dots, n$, we have

$$\begin{aligned} w_{g_3}(f_i) &= \big(g_3(v_i) + g_3(u_i) + g_3(v_{i+1})\big) + \big(g_3(v_i u_i) + g_3(u_i v_{i+1}) + g_3(v_{i+1} v_i)\big)\\ &= w_{\alpha_k^{n+1}}(v_i v_{i+1}) + \sum \mathcal{P}_{3,l}^n(i) \oplus (2n+1)\\ &= \big(C_{\alpha,k}^{n+1} + ki\big) + \big(C_{3,l}^n + li + 3(2n+1)\big)\\ &= C_{\alpha,k}^{n+1} + C_{3,l}^n + 3(2n+1) + (k+l)i, \end{aligned}$$

where $k = 0, \pm1, \pm2, \pm3$ and $l = \pm1, \pm3, \pm5, \pm9$, moreover for $n \equiv 1 \pmod 2$ also $l = 0, \pm2$. Analogously as in the proof of the previous theorem we obtain that for $d \in \{0, 1, 2, \dots, 12\}$ the bijection g_3 is a super d-antimagic labeling of type $(1, 1, 0)$ of the graph E_n. □

If we replace every edge $v_i v_{i+1}$, $i = 1, 2, \dots, n$, of the triangular snake E_n by a path of length two with vertices v_i, w_i, v_{i+1}, then we obtain a graph, say G_n, with the vertex set $V(G_n) = \{v_1, v_2, \dots, v_{n+1}, u_1, u_2, \dots, u_n, w_1, w_2, \dots w_n\}$ and the edge set $E(G_n) = \{v_i u_i, u_i v_{i+1}, v_i w_i, w_i v_{i+1} : i = 1, 2, \dots, n\}$. Let us define the 4-sided face f_i, $i = 1, 2, \dots, n$, as the face bounded by the edges $v_i u_i$, $u_i v_{i+1}$, $v_i w_i$ and $w_i v_{i+1}$ and let f be the external unbounded face.

Theorem 4.4. *The graph G_n, $n \geq 2$, has a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{0, 1, 2, \dots, 22\}$.*

Proof. Define the bijection $g_4 : V(G_n) \cup E(G_n) \to \{1, 2, \dots, 7n+1\}$ in the following way:

$$\begin{aligned} &g_4(v_i) = \beta_k^{n+1}(v_i), && i = 1, 2, \dots, n+1,\\ &\{g_4(u_i), g_4(w_i)\} = \mathcal{P}_{2,l}^n(i) \oplus (n+1), && i = 1, 2, \dots, n,\\ &\{g_4(v_i u_i), g_4(u_i v_{i+1}), g_4(v_i w_i), g_4(w_i v_{i+1})\} &&\\ &\qquad = \mathcal{P}_{4,m}^n(i) \oplus (3n+1), && i = 1, 2, \dots, n. \end{aligned}$$

It is easy to verify that the labeling g_4 assigns integers $1, 2, \dots, 3n+1$ to the vertices. By direct computation we obtain that the weight of the face f_i, $i = 1, 2, \dots, n$, admits a value

$$\begin{aligned} w_{g_4}(f_i) =& \big(g_4(v_i) + g_4(v_{i+1})\big) + \big(g_4(u_i) + g_4(w_i)\big)\\ &+ \big(g_4(v_i u_i) + g_4(u_i v_{i+1}) + g_4(v_i w_i) + g_4(w_i v_{i+1})\big)\\ =& w_{\beta_k^{n+1}}(v_i v_{i+1}) + \sum \mathcal{P}_{2,l}^n(i) \oplus (n+1) + \sum \mathcal{P}_{4,m}^n(i) \oplus (3n+1)\\ =& \big(C_{\beta,k}^{n+1} + ki\big) + \big(C_{2,l}^n + li + 2(n+1)\big) + \big(C_{4,m}^n + mi + 4(3n+1)\big)\\ =& C_{\beta,k}^{n+1} + C_{2,l}^n + C_{4,m}^n + 14n + 6 + (k+l+m)i, \end{aligned}$$

where $k = \pm 1, \pm 2$, $l = 0, \pm 2, \pm 4$ and $m = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$. After some manipulations we obtain that there exists a super d-antimagic labeling of the graph G_n for every difference $d \in \{0, 1, 2, \ldots, 22\}$. □

Let ladder L_n be a Cartesian product $L_n \simeq P_n \times P_2$ of a path on n vertices with a path on two vertices. Let $V(L_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ be the vertex set and $E(L_n) = \{v_i v_{i+1}, u_i u_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_i u_i : i = 1, 2, \ldots, n\}$ be the edge set of ladder. Let us define the 4-sided face f_i, $i = 1, 2, \ldots, n$, as the face bounded by the edges $v_i v_{i+1}$, $v_{i+1} u_{i+1}$, $u_i u_{i+1}$ and $v_i u_i$ and let f be the external unbounded face.

Theorem 4.5. *The ladder $L_n \simeq P_n \times P_2$, $n \geq 3$, admits a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{0, 1, 2, \ldots, 10\}$.*

Proof. Construct the bijective function $g_5 : V(L_n) \cup E(L_n) \to \{1, 2, \ldots, 5n-2\}$ as follows:

$$\begin{aligned}
&g_5(v_i) = \beta_k^n(v_i), && i = 1, 2, \ldots, n,\\
&g_5(u_i) = \beta_l^n(v_i) + n, && i = 1, 2, \ldots, n,\\
&g_5(v_i u_i) = \beta_m^n(v_i) + 2n, && i = 1, 2, \ldots, n,\\
&\{g_5(v_i v_{i+1}), g_5(u_i u_{i+1})\} = \mathcal{P}_{2,t}^{n-1}(i) \oplus (3n), && i = 1, 2, \ldots, n-1.
\end{aligned}$$

It is a routine procedure to verify that the vertices are labeled by the smallest possible numbers $1, 2, \ldots, 2n$. Moreover, for the weight of the face f_i, $i = 1, 2, \ldots, n-1$, we obtain

$$\begin{aligned}
w_{g_5}(f_i) =& \big(g_5(v_i) + g_5(v_{i+1})\big) + \big(g_5(u_i) + g_5(u_{i+1})\big) + \big(g_5(v_i u_i) + g_5(v_{i+1} u_{i+1})\big)\\
&+ \big(g_5(v_i v_{i+1}) + g_5(u_i u_{i+1})\big)\\
=& w_{\beta_k^n}(v_i v_{i+1}) + (w_{\beta_l^n}(v_i v_{i+1}) + 2n) + (w_{\beta_m^n}(v_i v_{i+1}) + 4n)\\
&+ \sum \mathcal{P}_{2,t}^{n-1}(i) \oplus (3n)\\
=& \big(C_{\beta,k}^n + ki\big) + \big(C_{\beta,l}^n + li + 2n\big) + \big(C_{\beta,m}^n + mi + 4n\big)\\
&+ \big(C_{2,t}^{n-1} + ti + 6n\big)\\
=& C_{\beta,k}^n + C_{\beta,l}^n + C_{\beta,m}^n + C_{2,t}^{n-1} + 12n + (k + l + m + t)i.
\end{aligned}$$

As $k = \pm 1, \pm 2$, $l = \pm 1, \pm 2$, $m = \pm 1, \pm 2$ and $t = 0, \pm 2, \pm 4$ we obtain that $k + l + m + t \in \{0, 1, 2, 3, \ldots, 10\}$. This completes the proof. □

Another variation of a ladder graph is specified as follows. A *ladder* $\mathbb{L}_n$, $n \geq 2$, is a graph obtained by completing the ladder $L_n \simeq P_n \times P_2$ by $n-1$ edges such that $V(\mathbb{L}_n) = \{v_1, v_2, \ldots, v_{2n}\}$ is the vertex set and $E(\mathbb{L}_n) = \{v_1 v_2, v_2 v_3, \ldots, v_{2n-1} v_{2n}, v_1 v_3, v_2 v_4, \ldots, v_{2n-2} v_{2n}\}$ is the edge set.

Theorem 4.6. *The graph $\mathbb{L}_n$, $n \geq 2$, admits a super d-antimagic labeling of type $(1, 1, 0)$ for $d \in \{0, 1, 2, \ldots, 6\}$.*

Proof. Construct the bijective function $g_6 : V(\mathbb{L}_n) \cup E(\mathbb{L}_n) \to \{1, 2, \ldots, 6n-1\}$ in the following way:

$$
\begin{aligned}
&g_6(v_i) = i, && i = 1, 2, \dots, 2n,\\
&g_6(v_iv_{i+1}) = \beta_k^{2n-1}(v_i) + 2n, && i = 1, 2, \dots, 2n-1,\\
&g_6(v_iv_{i+2}) = \mathcal{P}_{1,l}^{2n-2}(i) \oplus (4n-1), && i = 1, 2, \dots, 2n-2.
\end{aligned}
$$

The reader can easily verify that the labeling g_6 assigns integers $1, 2, \dots, 2n$ to the vertices and a weight of the face f_i, $i = 1, 2, \dots, 2n-2$, is

$$
\begin{aligned}
w_{g_6}(f_i) =&\big(g_6(v_i) + g_6(v_{i+1}) + g_6(v_{i+2})\big) + \big(g_6(v_iv_{i+1}) + g_6(v_{i+1}v_{i+2})\big)\\
&+ g_6(v_iv_{i+2})\\
=&\big(i + (i+1) + (i+2)\big) + \big(w_{\beta_k}^{2n-1}(v_iv_{i+1}) + 4n\big)\\
&+ \sum \mathcal{P}_{1,l}^{2n-2}(i) \oplus (4n-1)\\
=&\big(3+3i\big) + \big(C_{\beta,k}^{2n-1} + ki + 4n\big) + \big(C_{1,l}^{2n-2} + li + (4n-1)\big)\\
=&C_{\beta,k}^{2n-1} + C_{1,l}^{2n-2} + 8n + 2 + (3+k+l)i.
\end{aligned}
$$

Since $k = \pm1, \pm2$, and $l = \pm1$ we are able to show that $3+k+l \in \{0, 1, 2, \dots, 6\}$. This implies that the labeling g_6 is a super d-antimagic labeling of type $(1,1,0)$ for $d \in \{0, 1, 2, \dots, 6\}$ of the graph $\mathbb{L}_n$. □

If we replace every edge v_iv_{i+1} (respectively, every edge u_iu_{i+1}), $i = 1, 2, \dots, n-1$, of the ladder $L_n \simeq P_n \times P_2$ by a path of length two with vertices v_i, w_i, v_{i+1} (respectively, u_i, w_{n-1+i}, u_{i+1}) then we obtain a graph, say H_n, with the vertex set $V(H_n) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_{2n-2}\}$ and the edge set $E(H_n) = \{v_iw_i, w_iv_{i+1}, u_iw_{n-1+i}, w_{n-1+i}u_{i+1} : i = 1, 2, \dots, n-1\} \cup \{v_iu_i : i = 1, 2, \dots, n\}$.

Let us define the 6-sided face f_i, $i = 1, 2, \dots, n-1$, as the face bounded by the edges v_iw_i, w_iv_{i+1}, $v_{i+1}u_{i+1}$, $u_{i+1}w_{n-1+i}$, $w_{n-1+i}u_i$, u_iv_i and let f be the external unbounded face.

Theorem 4.7. *The graph H_n, $n \geq 2$, admits a super d-antimagic labeling of type $(1,1,0)$ for $d \in \{0, 1, 2, \dots, 26\}$.*

Proof. We define the bijection $g_7 : V(H_n) \cup E(H_n) \to \{1, 2, \dots, 9n-2\}$ in the following way:

$$
\begin{aligned}
&g_7(v_i) = \beta_k^n(v_i), && i = 1, 2, \dots, n,\\
&g_7(u_i) = \beta_l^n(v_i), && i = 1, 2, \dots, n,\\
&\{g_7(w_i), g_7(w_{n-1+i})\} = \mathcal{P}_{2,m}^{n-1}(i) \oplus (2n), && i = 1, 2, \dots, n-1,\\
&g_7(v_iu_i) = \beta_t^n(v_i) + 4n - 2, && i = 1, 2, \dots, n,\\
&\{g_7(v_iw_i), g_7(w_iv_{i+1}), g_7(u_iw_{n-1+i}), g_7(w_{n-1+i}u_{i+1})\} &&\\
&\qquad = \mathcal{P}_{4,s}^{n-1}(i) \oplus (5n-2), && i = 1, 2, \dots, n-1.
\end{aligned}
$$

It is easy to see that the vertices of H_n are labeled by the smallest possible integers $1, 2, \dots, 4n-2$. For the weight of the face f_i, $i = 1, 2, \dots, n-1$, we get

$$\begin{aligned}
w_{g_7}(f_i) =& \big(g_7(v_i) + g_7(v_{i+1})\big) + \big(g_7(u_i) + g_7(u_{i+1})\big) + \big(g_7(w_i) + g_7(w_{n-1+i})\big) \\
&+ \big(g_7(v_iu_i) + g_7(v_{i+1}u_{i+1})\big) + \big(g_7(v_iw_i) + g_7(w_iv_{i+1}) + g_7(u_iw_{n-1+i}) \\
&+ g_7(w_{n-1+i}u_{i+1})\big) \\
=& w_{\beta_k^n}(v_iv_{i+1}) + \big(w_{\beta_l^n}(v_iv_{i+1}) + 2n\big) + \sum \mathcal{P}_{2,m}^{n-1}(i) \oplus (2n) \\
&+ \big(w_{\beta_t}^n(v_iv_{i+1}) + 8n - 4\big) + \sum \mathcal{P}_{4,s}^{n-1}(i) \oplus (5n-2) \\
=& \big(C_{\beta,k}^n + ki\big) + \big(C_{\beta,l}^n + li + 2n\big) + \big(C_{2,m}^{n-1} + mi + 2(2n)\big) \\
&+ \big(C_{\beta,t}^n + ti + 8n - 4\big) + \big(C_{4,s}^{n-1} + si + 4(5n-2)\big) \\
=& C_{\beta,k}^n + C_{\beta,l}^n + C_{\beta,t}^n + C_{2,m}^{n-1} + C_{4,s}^{n-1} + 34n - 12 + (k+l+m+t+s)i.
\end{aligned}$$

Since $k = \pm1, \pm2$, $l = \pm1, \pm2$, $m = 0, \pm2, \pm4$, $t = \pm1, \pm2$ and $s = 0, \pm2, \pm4, \pm6, \pm8, \pm10$, ±16 we get that $k + l + m + t + s \in \{0, 1, 2, \ldots, 26\}$. Thus g_7 is the required super d-antimagic labeling of type $(1, 1, 0)$ of the graph H_n. This completes the proof. □

5. Concluding Remarks

In the foregoing section we studied the super d-antimagic labelings for the seven families of plane graphs. We have shown that there exist super d-antimagic labelings of type $(1, 1, 0)$ for these graphs for wide variety of difference d. We conclude with the following open problems.

Problem 1. *Find the upper bound for the feasible values of the difference d which makes a super d-antimagic labelings of type $(1, 1, 0)$ possible for the studied families of plane graphs.*

Problem 2. *Find other feasible values of the difference d and the corresponding super d-antimagic labelings of type $(1, 1, 0)$ for the studied families of plane graphs.*

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