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The locating chromatic number for m-shadow of a connected graph

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Abstract

Let $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a proper k-coloring of a simple connected graph G. Let $\Pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of $V(G)$, where C_i is the set of vertices of G receiving color i. The color code, $c_\Pi(v)$, of a vertex v with respect to Π is an ordered k-tuple $(d(v, C_2), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) : x \in C_i\}$ for $i = 1, 2, \ldots, k$. If distinct vertices have distinct color codes then c is called a locating coloring of G. The minimum k for which c is a locating coloring is the locating chromatic number of G, denoted by $\chi_L(G)$. Let G be a non trivial connected graph and let $m \geq 2$ be an integer. The m-*shadow* of G, denoted by $D_m(G)$, is a graph obtained by taking m copies of G, say $G_1, G_2, \ldots, G_m$, and each vertex v in G_i, $i = 1, 2, \ldots, m-1$, is joined to the neighbors of its corresponding vertex v' in G_{i+1}. In the present paper, we deal with the locating chromatic number for m-shadow of connected graphs. Sharp bounds on the locating chromatic number of $D_m(G)$ for any non trivial connected graph G and any integer $m \geq 2$ are obtained. Then the values of locating chromatic number for m-shadow of complete multipartite graphs and paths are determined, some of which are considered to be optimal.

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1. Introduction

All graphs considered in this paper are only connected, finite and undirected containing no loops nor multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use P_n to denote a path of order n and $K_{n_1,n_2,\ldots,n_r}$ to denote a complete r-partite graph where $n_i \leq n_j$ for $i < j$. A complete graph on n vertices and a star on $n+1$ vertices are denoted by K_n and $K_{1,n}$, respectively. For $u, v \in V(G)$ and a non negative integer k, a $u - v$ *walk* of length k is a sequence of vertices $(u = v_0, v_1, \ldots, v_k = v)$, where $v_i v_{i+1}$ is an edge of G for every $i = 0, 1, \ldots, k-1$. When no vertex is repeated in the sequence, it is called a $u - v$ *path*. The *distance* between u and v, denoted by $d_G(u, v)$ (or $d(u, v)$ in short), is defined as the length of the shortest $u - v$ path.

Let $c : V(G) \to \{1, 2, \ldots, k\}$ be a proper k-coloring of G and let $\Pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of $V(G)$, where C_i is the set of vertices in G receiving color i. The *color code* $c_\Pi(v)$ of v with respect to Π is an ordered k-tuple $(d(v, C_1), d(v, C_2), \ldots, (v, C_k))$, where $d(v, C_i) = \min\{d(v, x) : x \in C_i\}$ for $i = 1, 2, \ldots, k$. If distinct vertices have distinct color codes then c is called a *locating k-coloring* of G. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the minimum k for which c is a locating coloring.

The concept of locating chromatic number of graphs was introduced in 2002 by Chartrand *et al.* [15] as a combination of two concepts in graph theory namely graph coloring and partition dimension of graphs. Computing the locating chromatic number for general graphs is an NP-complete. It means that there is no efficient algorithm of calculating the locating chromatic number for a general graph. However, a number of researches restricted on specific classes of graphs have been carried out. For instance, the locating chromatic number for some families of graphs has been found such as paths, cycles, complete multipartite, and bistars in [15], amalgamation of stars in [3], firecracker graphs in [4], Barbell graphs in [5], Kneser graphs in [12], powers of paths and cycles in [17], book graphs in [19], Origami graphs in [20], and Möobius ladder graphs in [21]. Characterizations of graphs having certain locating chromatic number were studied such as trees with locating chromatic number 3 in [8], trees of order n with locating chromatic number $n - t$ for $2 \leq t < \frac{n}{2}$ in [22], unicyclic graphs of order n with locating chromatic number $n - 3$ or $n - 2$ in [2, 7], and graphs of order n with locating chromatic number $n - 1$ in [14]. In [10, 11, 13], it was investigated the locating chromatic number for the join product, Cartesian product and corona product of graphs. Recently, some upper bounds for the locating chromatic number of trees were also developed by some authors (see [6, 9, 16, 18]).

In [1], Agustin *et al.* defined an m-shadow of a graph as follows. Let G be a non trivial connected graph and let an integer $m \geq 2$ be given. The *m-shadow* of G, denoted by $D_m(G)$, is a graph obtained by taking m copies of G, say $G_1, G_2, \ldots, G_m$, and each vertex v in G_i, $i = 1, 2, \ldots, m-1$, is joined to the neighbors of its corresponding vertex v' in G_{i+1}.

In this paper, we study the locating chromatic number for m-shadow of a connected graph. We give sharp lower and upper bound of the parameter $\chi_L(G)$ for m-shadow of any non trivial connected graph. The values of locating chromatic number for m-shadow of complete multipartite graphs and paths are then determined, some of which are considered to be optimal.

2. Lower and upper bound

We begin with the following result that will be used later.

Proposition 2.1. *Let $u_i \in V(G_i)$ and $v_j \in V(G_j)$, $i \neq j$, be two vertices of $H \cong D_m(G)$ corresponding to vertices u and v in G, respectively. Then*

$$d_H(u_i, v_j) = \begin{cases} |i-j|, & \text{if there exists a } u-v \text{ walk of length } |i-j| \text{ in } G, \\ \max\{d_G(u,v), |i-j|+1\}, & \text{otherwise}. \end{cases}$$

Proof. For each $u, v \in V(G)$, let $(u = x^0, x^1, \ldots, x^k = v)$ be a $u - v$ walk of length k in G. Let $H \cong D_m(G)$. Consider two vertices $u_i, v_j \in V(H)$, $i \neq j$, with $u_i \in V(G_i)$ and $v_j \in V(G_j)$, corresponding to u and v in G, respectively. Say, without loss of generality, $i < j$. Let $k = |i-j|$. Then $(u_i = x_i^0, x_{i+1}^1, \ldots, x_j^k = v_j)$ is a $u_i - v_j$ path of length k in H. So $d_H(u_i, v_j) = |i-j|$. Now suppose $k \neq |i-j|$. We have two cases.

Case 1. $d_G(u,v) > |i-j|$. Then $d_G(u,v) = \max\{d_G(u,v), |i-j|+1\}$. So $(u_i = x_i^0, x_{i+1}^1, \ldots, x_j^{|i-j|}, x_j^{|i-j|+1}, \ldots, x_j^{d_G(u,v)} = v_j)$ is a $u_i - v_j$ path of length $d_G(u,v)$ in H. Thus $d_H(u_i, v_j) = d_G(u,v)$.

Case 2. $d_G(u,v) < |i-j|$. Then $|i-j|+1 = \max\{d_G(u,v), |i-j|+1\}$. Suppose that w is a vertex adjacent to v in G. Then $(u_i = x_i^0, x_{i+1}^1, \ldots, x_{i+d_G(u,v)}^{d_G(u,v)} = v_{i+d_G(u,v)})$ is a $u_i - v_{i+d_G(u,v)}$ path of length $d_G(u,v)$ in H and $(v_{i+d_G(u,v)}, w_{i+d_G(u,v)+1}, v_{i+d_G(u,v)+2}, w_{i+d_G(u,v)+3}, \ldots, v_{j-1}, w_j, v_j)$ is a $v_{i+d_G(u,v)} - v_j$ path of length $j - i - d_G(u,v) + 1$ in H. Hence

$$d_H(u_i, v_j) \leq d_G(u,v) + j - i - d_G(u,v) + 1 = |i-j| + 1. \tag{1}$$

As $k \neq |i-j|$,

$$d_H(u_i, v_j) \geq |i-j| + 1. \tag{2}$$

By (1) and (2), $d_H(u_i, v_j) = |i-j| + 1$. □

The next theorem presents the bounds on the locating chromatic number for m-shadow of a connected graph G.

Theorem 2.1. *Let G be a non trivial connected graph and $m \geq 2$. Then*

$$\chi_L(G) + 1 \leq \chi_L(D_m(G)) \leq 2\chi_L(G).$$

Proof. First, we will prove the lower bound. We here will use a contradiction strategy. Thus, assume that $\chi_L(D_m(G)) = \chi_L(G)$. Consider the first and the second copy of G in the graph $D_m(G)$, i.e., G_1 and G_2, respectively. If there is no mutual color used in G_1 and G_2, then the number of colors used in G_1 must be less than $\chi_L(G)$ and number of colors used in G_2 must be less than $\chi_L(G)$. By definition of locating coloring, at least two vertices in G_1 or G_2 belonging to the same color class have the same color codes, a contradiction. Suppose that at least one mutual color is used in G_1 and G_2. Then there must be two vertices in $G_1 \cup G_2$ belonging to the same color class having the same color codes, a contradiction. Thus $\chi_L(D_m(G)) \geq \chi_L(G) + 1$.

Next, we will prove the upper bound. Let us denote by the symbol v_i, $i = 1, 2, \dots, m$, a vertex in G_i, $G_i \subset D_m(G)$, corresponding to the vertex v in G. Define a coloring c on the vertices of $D_m(G)$ in the following way. Let $c(v_m) = c'(v) + \chi_L(G)$, and $c(v_i) = c'(v)$ for $i = 1, 2, \dots, m-1$, where c' is a locating $\chi_L(G)$-coloring of G.

We show that c is a locating coloring of $D_m(G)$. To do so, it suffices to show that every two distinct vertices u_i and v_j of $D_m(G)$ in each color class have distinct color codes. If $1 \le i = j \le m$ then $c_\Pi(u_i) \ne c_\Pi(v_j)$ since c' is a locating coloring of G. Let us consider $1 \le i < j \le m-1$. First, let $j - i \ge 2$. We have $d_{D_m(G)}(u_i, v_m) \ge m - i \ge m - j + 2 > m - j + 1 \ge d_{D_m(G)}(v_j, v_m)$. Then u_i and v_j are distinguished by the color class $C_{\chi_L(G)+c(v_j)}$. Now let $j - i = 1$. By Proposition 2.1, $m - j \le d_{D_m(G)}(v_j, v_m) \le m - j + 1$. If $d_{D_m(G)}(v_j, v_m) = m - j$ then $d_{D_m(G)}(u_i, v_m) \ge m - i = m - j + 1 > m - j = d_{D_m(G)}(v_j, v_m)$. So u_i and v_j are distinguished by the color class $C_{\chi_L(G)+c(v_j)}$. If $d_{D_m(G)}(v_j, v_m) = m - j + 1$ then $d_{D_m(G)}(v_j, w_m) = m - j$, where w_m is a vertex in G_m adjacent to v_m. We get $d_{D_m(G)}(u_i, w_m) \ge m - j + 1 > m - j = d_{D_m(G)}(v_j, w_m)$. So u_i and v_j are distinguished by the color class $C_{\chi_L(G)+c(w_j)}$. Thus all the vertices have distinct color codes and the proof is concluded. □

As we will see the results in (3), (4) and Lemma 4.1, the bounds in Theorem 2.1 are sharp.

3. The m-shadow of complete multipartite graphs

Let $G \cong D_m(K_{n_1,n_2,\dots,n_r})$ be the m-shadow of a complete r-partite graph consisting of mr vertex subsets namely $V_1^1, V_1^2, \dots, V_1^r, V_2^1, V_2^2, \dots, V_2^r, \dots, V_m^1, V_m^2, \dots, V_m^r$ such that $|V_i^j| = n_j$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, r$, which form a partition in G. The graph G has vertex set $V(G) = \bigcup_{i=1}^m \bigcup_{j=1}^r V_i^j$, where $V_i^j = \{v_{i,k}^j : j = 1, 2, \dots, r \text{ and } k = 1, 2, \dots, n_j\}$ is the jth partite in the ith copy of $K_{n_1,n_2,\dots,n_r}$.

It was proved in [15] that $\chi_L(K_{n_1,n_2,\dots,n_r}) = \sum_{z=1}^r n_z$. Since the 2-shadow of a complete r-partite $K_{n_1,n_2,\dots,n_r}$ is a complete r-partite graph $K_{2n_1,2n_2,\dots,2n_r}$, we have

$$\chi_L(D_2(K_{n_1,n_2,\dots,n_r})) = 2\sum_{z=1}^{r} n_z. \tag{3}$$

Let us consider the graph $D_m(K_{n_1,n_2,\dots,n_r})$, for $m \ge 3$. In this case, we claim that it is not possible to attain the lower bound in Theorem 2.1. We need to add at least one more color in order to produce a locating coloring. The next lemma proves our claim. Please note that every proper $\sum_{z=1}^r n_z$-coloring of $D_m(K_{n_1,n_2,\dots,n_r})$ has the property that for every $1 \le i \ne i' \le m$ and $1 \le j \le r$, the sets of colors used to color the vertices on V_i^j are the same as those used to color the vertices on $V_{i'}^j$.

Lemma 3.1. *Let $m \ge 3$, $r \ge 2$ and $n_1 \le n_2 \le \dots \le n_r$. Then $\chi_L(D_m(K_{n_1,n_2,\dots,n_r})) \ge \sum_{z=1}^r n_z + 2$.*

Proof. It suffices to show that there is no locating $(\sum_{z=1}^r n_z + 1)$-coloring of $D_m(K_{n_1,n_2,\dots,n_r})$. Assume that such a coloring c exists. The vertex $(\sum_{z=1}^r n_z + 1)$-coloring c of $D_m(K_{n_1,n_2,\dots,n_r})$ can be obtained from the vertex $\sum_{z=1}^r n_z$-coloring of $D_m(K_{n_1,n_2,\dots,n_r})$ by replacing 'old' color of at least one vertex with the 'new' color $\sum_{z=1}^r n_z + 1$. Say, the color c_0 of

$v_{i,k}^j \in V_i^j$ for some c_0, i, j, k, is replaced with the color $\sum_{z=1}^r n_z + 1$. As c is proper, the sets of colors used to color the vertices in $V_{i-1}^{j'}$, $V_i^{j'}$ and $V_{i+1}^{j'}$ for every j', $1 \leq j' \neq j \leq r$, are not replaced. This implies that at least two vertices in $V_{i-1}^{j'} \cup V_i^{j'} \cup V_{i+1}^{j'}$ belonging to the same color class have the same color codes, a contradiction. Thus $\chi_L(D_m(K_{n_1,n_2,\ldots,n_r})) \geq \sum_{z=1}^r n_z + 2$. □

Lemma 3.2. *Let $r \geq 2$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$. Then $\chi_L(D_3(K_{n_1,n_2,\ldots,n_r})) \leq \sum_{z=1}^r n_z + 3$.*

Proof. Define a vertex $(\sum_{z=1}^r n_z + 3)$-coloring c of $D_3(K_{n_1,n_2,\ldots,n_r})$ in the following way. Let $c(v_{1,n_1}^1) = c(v_{3,n_2}^2) = \sum_{z=1}^r n_z + 1$, $c(v_{1,n_2}^2) = \sum_{z=1}^r n_z + 2$ and $c(v_{3,n_1}^1) = \sum_{z=1}^r n_z + 3$. Next, let $c(v_{i,k}^j) = \sum_{z=1}^{j-1} n_z + k$ for $i = 1, 2, 3$, $j = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, n_j$, where $(i, k) \neq (1, n_1), (3, n_1), (1, n_2)$ or $(3, n_2)$. It is easy to see that c is a locating coloring. □

Lemma 3.3. *Let $r \geq 2$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$. Then $\chi_L(D_4(K_{n_1,n_2,\ldots,n_r})) \leq \sum_{z=1}^r n_z + 3$.*

Proof. Define a vertex $(\sum_{z=1}^r n_z + 3)$-coloring c of $D_4(K_{n_1,n_2,\ldots,n_r})$ as follows. Let $c(v_{1,n_1}^1) = \sum_{z=1}^r n_z + 1$, $c(v_{2,n_1}^1) = c(v_{4,n_2}^2) = \sum_{z=1}^r n_z + 2$ and $c(v_{3,n_2}^2) = \sum_{z=1}^r n_z + 3$. Next, let $c(v_{i,k}^j) = \sum_{z=1}^{j-1} n_z + k$ for $i = 1, 2, \ldots, 4$, $j = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, n_j$, where $(i, k) \neq (1, n_1), (2, n_1), (3, n_2)$ or $(4, n_2)$. It is not hard to show that c is a locating coloring. □

Lemma 3.4. *Let $m \geq 3$, $r \geq 2$ and $1 = n_1 \leq n_2 \leq \cdots \leq n_r$. Then $\chi_L(D_m(K_{n_1,n_2,\ldots,n_r})) = \sum_{z=1}^r n_z + 2$.*

Proof. Due to Lemma 3.1, it is sufficient to show the existence of a locating $(\sum_{z=1}^r n_z+2)$-coloring of $D_m(K_{n_1,n_2,\ldots,n_r})$. Let us define a coloring $c : V(D_m(K_{n_1,n_2,\ldots,n_r})) \to \{1, 2, \ldots, \sum_{z=1}^r n_z + 2\}$ in the following way.

$$
\begin{aligned}
c(v_{i,1}^1) &= 1 \quad \text{for } i = 1, 2, \ldots, m-2, \\
c(v_{i,1}^1) &= \sum_{z=1}^{r} n_z - m + 2 + i \quad \text{for } i = m-1, m, \\
c(v_{i,k}^j) &= \sum_{z=1}^{j-1} n_z + k \quad \text{for } i = 1, 2, \ldots, m,\ j = 2, 3, \ldots, r \text{ and } k = 1, 2, \ldots, n_j.
\end{aligned}
$$

Let u and v be any two vertices of $D_m(K_{n_1,n_2,\ldots,n_r})$ such that u and v belong to the same color class. Consider the following cases.

- If $u = v_{i,k}^j$ and $v = v_{i',k}^j$ for $1 \leq i \neq i' \leq m-1$, $2 \leq j \leq r$ and $1 \leq k \leq n_j$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{\sum_{z=1}^r n_z+2}) \neq d(v, C_{\sum_{z=1}^r n_z+2})$.
- If $u = v_{i,k}^j$ and $v = v_{m,k}^j$ for $1 \leq i \leq m-1$, $2 \leq j \leq r$ and $1 \leq k \leq n_j$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_1) \neq d(v, C_1)$.
- If $u = v_{i,1}^1$ and $v = v_{i',1}^1$ for $1 \leq i \neq i' \leq m-2$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{\sum_{z=1}^r n_z+2}) \neq d(v, C_{\sum_{z=1}^r n_z+2})$.

The color codes of vertices are all distinct and so c is a locating coloring of $D_m(K_{n_1,n_2,\dots,n_r})$. Hence $\chi_L(D_m(K_{n_1,n_2,\dots,n_r})) = \sum_{z=1}^{r} n_z + 2$. □

Lemma 3.5. *Let $m \geq 5$ and $2 \leq n_1 \leq n_2$. Then $\chi_L(D_m(K_{n_1,n_2})) \leq n_1 + n_2 + 3$.*

Proof. Define a coloring $c : V(D_m(K_{n_1,n_2})) \rightarrow \{1, 2, \dots, n_1 + n_2 + 3\}$ in the following way.

$$
\begin{aligned}
c(v^1_{i,k}) &= k \quad \text{for } i = 1, 2, \dots, m \text{ and } k = 1, 2, \dots, n_1, \text{ where } (i,k) \neq (1, n_1) \text{ or } (m, n_1),\\
c(v^2_{i,k}) &= n_1 + k \quad \text{for } i = 1, 2, \dots, m \text{ and } k = 1, 2, \dots, n_2, \text{ where } (i,k) \neq (3, n_2) \text{ or } (m, n_2),\\
c(v^1_{1,n_1}) &= n_1 + n_2 + 1,\\
c(v^2_{3,n_2}) &= n_1 + n_2 + 2,\\
c(v^1_{m,n_1}) &= n_1 + n_2 + 3 \quad \text{if } m \text{ is odd},\\
c(v^1_{m,n_1}) &= n_1 \quad \text{if } m \text{ is even},\\
c(v^2_{m,n_2}) &= n_2 \quad \text{if } m \text{ is odd},\\
c(v^2_{m,n_2}) &= n_1 + n_2 + 3 \quad \text{if } m \text{ is even}.
\end{aligned}
$$

Let u and v be any two vertices of $D_m(K_{n_1,n_2})$ such that u and v belong to the same color class. Consider the following cases.

- If $u = v^j_{i,k}$ and $v = v^j_{i',k}$ for $1 \leq i \neq i' \leq m$, $1 \leq j \leq 2$ and $1 \leq k \leq n_j$, $|i - i'| \geq 2$, then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+3}) \neq d(v, C_{n_1+n_2+3})$.
- If $u = v^1_{1,k}$ and $v = v^1_{2,k}$ for $1 \leq k \leq n_1 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+2}) \neq d(v, C_{n_1+n_2+2})$.
- If $u = v^1_{i,k}$ and $v = v^1_{i+1,k}$ for odd i, $2 \leq i \leq m-2$, and $1 \leq k \leq n_1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$.
- If $u = v^1_{i,k}$ and $v = v^1_{i+1,k}$ for even i, $2 \leq i \leq m-2$, and $1 \leq k \leq n_1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+3}) \neq d(v, C_{n_1+n_2+3})$.
- If $u = v^1_{m-1,k}$ and $v = v^1_{m,k}$ for $1 \leq k \leq n_1 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+2}) \neq d(v, C_{n_1+n_2+2})$ if m is odd and $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$ if m is even.
- If $u = v^1_{m-1,n_1}$ and $v = v^1_{m,n_1}$ for even m then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$.
- If $u = v^2_{1,k}$ and $v = v^2_{2,k}$ for $1 \leq k \leq n_2$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+3}) \neq d(v, C_{n_1+n_2+3})$.
- If $u = v^2_{2,k}$ and $v = v^2_{3,k}$ for $1 \leq k \leq n_2 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$.
- If $u = v^2_{3,k}$ and $v = v^2_{4,k}$ for $1 \leq k \leq n_2 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+3}) \neq d(v, C_{n_1+n_2+3})$.

- If $u = v^2_{i,k}$ and $v = v^2_{i+1,k}$ for odd i, $4 \leq i \leq m-2$, and $1 \leq k \leq n_2$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+3}) \neq d(v, C_{n_1+n_2+3})$.
- If $u = v^2_{i,k}$ and $v = v^2_{i+1,k}$ for even i, $4 \leq i \leq m-2$, and $1 \leq k \leq n_2$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$.
- If $u = v^2_{m-1,k}$ and $v = v^2_{m,k}$ for $1 \leq k \leq n_2 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$ if m is odd and $d(u, C_{n_1+n_2+2}) \neq d(v, C_{n_1+n_2+2})$ if m is even.
- If $u = v^2_{m-1,n_2}$ and $v = v^2_{m,n_2}$ for odd m then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1+n_2+1}) \neq d(v, C_{n_1+n_2+1})$.

Thus distinct vertices have distinct color codes. Hence c is a locating coloring of $D_m(K_{n_1,n_2})$. $\square$

Lemma 3.6. *Let $m \geq 5$, $r \geq 3$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$. Then $\chi_L(D_m(K_{n_1,n_2,\ldots,n_r})) = \sum_{z=1}^r n_z + 2$.*

Proof. According to Lemma 3.1, it suffices to construct an optimal locating $(\sum_{z=1}^r n_z+2)$-coloring of $D_m(K_{n_1,n_2,\ldots,n_r})$. Let us define a coloring $c : V(D_m(K_{n_1,n_2,\ldots,n_r})) \to \{1, 2, \ldots, \sum_{z=1}^r n_z + 2\}$ in the following way. Let $c(v^1_{m,n_1}) = \sum_{z=1}^r n_z + 2$ and for $i = 2, 3, \ldots, m-1$, let $c(v^1_{i,n_1}) = \sum_{z=1}^r n_z + 1$. Next, let $c(v^j_{i,k}) = \sum_{z=1}^{j-1} n_z + k$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, n_j$, where $(i,k) \neq (2, n_1), (3, n_1), \ldots, (m, n_1)$.

We show that the color codes for each vertex are distinct. Let us consider any two distinct vertices $u, v \in V(D_m(K_{n_1,n_2,\ldots,n_r}))$ such that u and v belong to the same color class. Consider the following cases.

- If $u = v^1_{i,k}$ and $v = v^1_{i',k}$ for $1 \leq i \neq i' \leq m-2$ and $1 \leq k \leq n_1 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{\sum_{z=1}^r n_z+2}) \neq d(v, C_{\sum_{z=1}^r n_z+2})$.
- If $u = v^1_{i,k}$ and $v = v^1_{i',k}$ for $1 \leq i \leq m-2$, $m-1 \leq i' \leq m$, $1 \leq k \leq n_1 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1}) \neq d(v, C_{n_1})$.
- If $u = v^1_{m-1,k}$ and $v = v^1_{m,k}$ for $1 \leq k \leq n_1 - 1$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1}) \neq d(v, C_{n_1})$.
- If $u = v^1_{i,n_1}$ and $v = v^1_{i',n_1}$ for $2 \leq i \neq i' \leq m-2$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{\sum_{z=1}^r n_z+2}) \neq d(v, C_{\sum_{z=1}^r n_z+2})$.
- If $u = v^1_{i,n_1}$ and $v = v^1_{m-1,n_1}$ for $1 \leq i \leq m-2$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1}) \neq d(v, C_{n_1})$.
- If $u = v^j_{i,k}$ and $v = v^j_{i',k}$ for $1 \leq i \neq i' \leq m-1$, $2 \leq j \leq r$ and $1 \leq k \leq n_j$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{\sum_{z=1}^r n_z+2}) \neq d(v, C_{\sum_{z=1}^r n_z+2})$.
- If $u = v^j_{i,k}$ and $v = v^j_{m,k}$ for $1 \leq i \leq m-1$, $2 \leq j \leq r$ and $1 \leq k \leq n_j$ then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, C_{n_1}) \neq d(v, C_{n_1})$.

The color codes of vertices are all distinct. So $\chi_L(D_m(K_{n_1,n_2,\ldots,n_r})) = \sum_{z=1}^{r} n_z + 2$. $\square$

Summarizing the results from (3) and Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 we have the following theorem which gives the locating chromatic number for m-shadow of complete multipartite graphs.

Theorem 3.1. *Let $m \geq 2$, $r \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r$. Then*

$$\chi_L(D_m(K_{n_1,n_2,\ldots,n_r})) = \begin{cases} 2\sum_{z=1}^{r} n_z, & \text{if } m = 2,\ r \geq 2 \text{ and } 1 \leq n_1 \leq n_2 \leq \cdots \leq n_r, \\ \sum_{z=1}^{r} n_z + 2, & \text{if } m \geq 3,\ r \geq 2 \text{ and } 1 = n_1 \leq n_2 \leq \cdots \leq n_r \text{ or} \\ & m \geq 5,\ r \geq 3 \text{ and } 2 \leq n_1 \leq n_2 \leq \cdots \leq n_r. \end{cases}$$

Moreover, $\sum_{z=1}^{r} n_z + 2 \leq \chi_L(D_m(K_{n_1,n_2,\ldots,n_r})) \leq \sum_{z=1}^{r} n_z + 3$ if $3 \leq m \leq 4$, $r \geq 2$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ or $m \geq 5$, $r = 2$ and $2 \leq n_1 \leq n_2$.

The next two corollaries presenting the locating chromatic number for m-shadow of stars and complete graphs follow from Theorem 3.1.

Corollary 3.1. *Let $m \geq 3$ and $n \geq 1$. Then $\chi_L(D_m(K_{1,n})) = n + 3$.*

Corollary 3.2. *Let $m \geq 3$ and $n \geq 2$. Then $\chi_L(D_m(K_n)) = n + 2$.*

4. The m-shadow of paths

Let $G \cong D_m(P_n)$ be the m-shadow of a path with vertex set $V(G) = \bigcup_{i=1}^{m} V(P_n^i)$, where $V(P_n^i) = \{v_i^j : j = 1, 2, \ldots, n\}$ is the vertex subset in the ith copy of P_n, $i = 1, 2, \ldots, m$.

In [13], Behtoei and Omoomi proved that $\chi_L(P_m \times P_n) = 4$ for $n \geq m \geq 2$. In fact, we have $D_m(P_2) \cong P_m \times P_2$ for $m \geq 2$. So

$$\chi_L(D_m(P_2)) = 4. \tag{4}$$

For $m \geq 3$ and $n = 3$ we obtain $D_m(P_3) \cong D_m(K_{1,2})$, and for $m = 2$ and $n = 3$ we get $D_2(P_3) \cong D_2(K_{1,2})$. So from Corollary 3.1 and (3),

$$\chi_L(D_m(P_3)) = 5 \tag{5}$$

and

$$\chi_L(D_2(P_3)) = 6, \tag{6}$$

respectively.

In [15], it was proved that $\chi_L(P_n) = 3$ for $n \geq 3$.

Lemma 4.1. *Let $m \geq 3$. Then $\chi_L(D_m(P_4)) = 4$.*

Proof. By Theorem 2.1, $\chi_L(D_m(P_4)) \geq 4$. Define a vertex 4-coloring c of $D_m(P_4)$ in the following way.

$$
\begin{aligned}
c(v_i^j) &= 1 && \text{for } i = 1, 2, \ldots, m-2 \text{ and } j = 1, 4,\\
c(v_i^j) &= 2 && \text{for } i = 1, 2, \ldots, m \text{ and } j = 3 \text{ or } i = m-1 \text{ and } j = 1,\\
c(v_i^j) &= 3 && \text{for } i = m \text{ and } j = 1, 4,\\
c(v_i^j) &= 4 && \text{for } i = 1, 2, \ldots, m \text{ and } j = 2 \text{ or } i = m-1 \text{ and } j = 4.
\end{aligned}
$$

Then we obtain the color codes of vertices below.

$$
\begin{aligned}
c_\Pi(v_i^1) &= (0, 2, m-i, 1) && \text{for } i = 1, 2, \ldots, m-2,\\
c_\Pi(v_{m-1}^1) &= (2, 0, 2, 1),\\
c_\Pi(v_m^1) &= (2, 2, 0, 1),\\
c_\Pi(v_i^2) &= (1, 1, m-i, 0) && \text{for } i = 1, 2, \ldots, m-1,\\
c_\Pi(v_m^2) &= (2, 1, 1, 0),\\
c_\Pi(v_i^3) &= (1, 0, m-i, 1) && \text{for } i = 1, 2, \ldots, m-1,\\
c_\Pi(v_m^3) &= (2, 0, 1, 1),\\
c_\Pi(v_i^4) &= (0, 1, m-i, 2) && \text{for } i = 1, 2, \ldots, m-2,\\
c_\Pi(v_{m-1}^4) &= (2, 1, 2, 0),\\
c_\Pi(v_m^4) &= (2, 1, 0, 2).
\end{aligned}
$$

Clearly the color codes are distinct for all vertices. So $\chi_L(D_m(P_4)) = 4$ and the proof is completed. □

Lemma 4.2. *Let $n \geq 4$. Then $\chi_L(D_2(P_n)) = 5$.*

Proof. We have $\chi_L(D_2(P_n)) \geq 4$ according to Theorem 2.1. Assume now, $\chi_L(D_2(P_n)) = 4$. Let c' be a locating 4-coloring of $D_2(P_n)$. We may assume, without loss of generality, that $\{c'(v_1^1), c'(v_2^1)\} = \{1, 2\}$ and $\{c'(v_1^2), c'(v_2^2)\} = \{3, 4\}$. Then $\{c'(v_1^j), c'(v_2^j) : j \neq 1 \text{ is odd and } c'(v_1^j) \neq c'(v_2^j)\} = \{1, 2\}$ and $\{c'(v_1^j), c'(v_2^j) : j \neq 2 \text{ is even and } c'(v_1^j) \neq c'(v_2^j)\} = \{3, 4\}$. It is easy to see that the color codes of vertices are not unique. So $\chi_L(D_2(P_n)) \geq 5$.

Let c be a vertex 5-coloring of $D_2(P_n)$ defined such that

$$
\begin{array}{ll}
c(v_1^j) = 1 & \text{if } n \text{ is odd, for } j = 1, 3, \dots, n \text{ or} \\
 & n \text{ is even, for } j = 2, 4, \dots, n, \\
c(v_i^j) = 2 & \text{if } n \text{ is odd, for } i = 1 \text{ and } j = 4, 6, \dots, n-1 \text{ or} \\
 & n \text{ is odd, for } i = 2 \text{ and } j = 2 \text{ or} \\
 & n \text{ is even, for } i = 1 \text{ and } j = 3, 5, \dots, n-1 \text{ or} \\
 & n \text{ is even, for } i = 2 \text{ and } j = 1, \\
c(v_i^j) = 3 & \text{if } n \text{ is odd, for } i = 1 \text{ and } j = 2 \text{ or} \\
 & n \text{ is odd, for } i = 2 \text{ and } j = 5, 7, \dots, n \text{ or} \\
 & n \text{ is even, for } i = 1 \text{ and } j = 1 \text{ or} \\
 & n \text{ is even, for } i = 2 \text{ and } j = 4, 6, \dots, n, \\
c(v_2^j) = 4 & \text{if } n \text{ is odd, for } j = 1, 4, 6, \dots, n-1 \text{ or} \\
 & n \text{ is even, for } j = 3, 5, \dots, n-1, \\
c(v_2^j) = 5 & \text{if } n \text{ is odd, for } j = 3 \text{ or} \\
 & n \text{ is even, for } j = 2.
\end{array}
$$

We show that the color codes for all vertices of $D_m(P_n)$ are distinct. Let us divide the cases depending on the parity of n. First let $n \geq 5$ be odd. We have the color codes as follows. $c_\Pi(v_1^1) = (0, 1, 1, 2, 2)$, $c_\Pi(v_2^1) = (2, 1, 1, 0, 2)$, $c_\Pi(v_1^2) = (1, 2, 0, 1, 1)$, $c_\Pi(v_2^2) = (1, 0, 2, 1, 1)$, $c_\Pi(v_1^3) = (0, 1, 1, 1, 2)$ and $c_\Pi(v_2^3) = (2, 1, 1, 1, 0)$. For odd $j \geq 5$ we obtain $c_\Pi(v_1^j) = (0, 1, 2, 1, j-3)$ and $c_\Pi(v_2^j) = (2, 1, 0, 1, j-3)$. For even $j \geq 4$ we obtain $c_\Pi(v_1^j) = (1, 0, 1, 2, j-3)$ and $c_\Pi(v_2^j) = (1, 2, 1, 0, j-3)$.

Now let $n \geq 4$ be even. Then we get the color codes as follows. $c_\Pi(v_1^1) = (1, 2, 0, 2, 1)$, $c_\Pi(v_2^1) = (1, 0, 2, 2, 1)$, $c_\Pi(v_1^2) = (0, 1, 1, 1, 2)$ and $c_\Pi(v_2^2) = (2, 1, 1, 1, 0)$. For odd $j \geq 3$ we have $c_\Pi(v_1^j) = (1, 0, 1, 2, j-2)$ and $c_\Pi(v_2^j) = (1, 2, 1, 0, j-2)$. For even $j \geq 4$ we have $c_\Pi(v_1^j) = (0, 1, 2, 1, j-2)$ and $c_\Pi(v_2^j) = (2, 1, 0, 1, j-2)$.

Since the color codes are distinct for all cases, c is a locating coloring of $D_m(P_n)$. Hence $\chi_L(D_m(P_n)) = 5$. □

Lemma 4.3. *Let $m \geq 3$ and $n \geq 5$. Then $\chi_L(D_m(P_n)) \leq 5$.*

Proof. Define a coloring $c : V(D_m(P_n)) \to \{1, 2, \dots, 5\}$ as follows.

$$
\begin{array}{rl}
c(v_i^1) = 1 & \text{for } i = 1, 2, \dots, m, \\
c(v_i^j) = 2 & \text{for } i = 1, 2, \dots, m \text{ and odd } j \neq 1, \\
c(v_i^j) = 3 & \text{for } i = 1, 2, \dots, m-2 \text{ and even } j, \\
c(v_{m-1}^j) = 4 & \text{for even } j, \\
c(v_m^j) = 5 & \text{for even } j.
\end{array}
$$

Under the coloring c, we obtain the color codes of vertices as follows.

$$
\begin{aligned}
c_\Pi(v_m^1) &= (0,2,3,1,1), & & \\
c_\Pi(v_{m-i}^1) &= (0,2,1,i,i) & & \text{for odd } i, \\
c_\Pi(v_{m-i}^1) &= (0,2,1,i-1,i+1) & & \text{for even } i, \\
c_\Pi(v_m^j) &= (j-1,0,3,1,1) & & \text{for odd } j \neq 1, \\
c_\Pi(v_{m-i}^j) &= (j-1,0,1,i,i) & & \text{for odd } i \text{ and odd } j \neq 1, \\
c_\Pi(v_{m-i}^j) &= (j-1,0,1,i-1,i+1) & & \text{for even } i \text{ and odd } j \neq 1, \\
c_\Pi(v_{m-i}^j) &= (j-1,1,0,i-1,i+1) & & \text{for odd } i \neq 1 \text{ and even } j, \\
c_\Pi(v_{m-i}^j) &= (j-1,1,0,i,i) & & \text{for even } i \text{ and even } j, \\
c_\Pi(v_{m-1}^j) &= (j-1,1,2,0,2) & & \text{for even } j, \\
c_\Pi(v_m^j) &= (j-1,1,2,2,0) & & \text{for even } j.
\end{aligned}
$$

It is easy to see that distinct vertices have distinct color codes. Thus $\chi_L(D_m(P_n)) \leq 5$. □

Immediately from Theorem 2.1, (4), (5), (6), and Lemmas 4.1, 4.2 and 4.3, we obtain the following theorem giving the locating chromatic number for m-shadow of paths.

Theorem 4.1. *Let $m \geq 2$ and $n \geq 2$. Then*

$$
\chi_L(D_m(P_n)) = \begin{cases} 4, & \text{if } m \geq 2 \text{ and } n = 2 \text{ or } m \geq 3 \text{ and } n = 4, \\ 5, & \text{if } m = 2 \text{ and } n \geq 4 \text{ or } m \geq 3 \text{ and } n = 3, \\ 6, & \text{if } m = 2 \text{ and } n = 3. \end{cases}
$$

Moreover, $4 \leq \chi_L(D_m(P_n)) \leq 5$ *if* $m \geq 3$ *and* $n \geq 5$.

5. Remark and Conclusion

Here, we state the new result corresponding to the tight lower bound and upper bound of locating chromatic number on m-shadow of a graph G, that is $\chi_L(G) + 1 \leq \chi_L(D_m(G)) \leq 2\chi_L(G)$. Moreover, we also obtain the exact value of $\chi_L(D_m(G))$, when G is a path or a complete multipartite graph on some particular orders. In future, we need to classify the families of graphs in which the lower or upper bound holds.

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