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The complete list of Ramsey $(2K_2, K_4)$-minimal graphs

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Abstract

Let $F, G,$ and H be non-empty graphs. The notation $F \rightarrow (G, H)$ means that if all edges of F are arbitrarily colored by red or blue, then either the subgraph of F induced by all red edges contains a graph G or the subgraph of F induced by all blue edges contains a graph H. A graph F satisfying two conditions: $F \rightarrow (G, H)$ and for every $e \in E(F)$, $(F - e) \nrightarrow (G, H)$ is called a Ramsey $(G, H)-$minimal graph. In this paper, we determine all non-isomorphic Ramsey $(2K_2, K_4)-$minimal graphs.

Keywords: Ramsey minimal graph, red-blue coloring, complete graph
Mathematics Subject Classification : 05D10, 05C55
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1. Introduction

Let $F, G,$ and H be simple graphs. We write $F \rightarrow (G, H)$ to mean that any red-blue coloring on the edges of F contains a red copy of G or a blue copy of H. A red-blue coloring on the edges of F such that F contain neither a red G nor a blue H is called a $(G, H)-$*coloring*. If a graph F has a $(G, H)-$coloring, then we write $F \nrightarrow (G, H)$. A graph F is called a *Ramsey* $(G, H)-$*minimal* if $F \rightarrow (G, H)$ and for each $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. The set of all Ramsey $(G, H)-$minimal graphs will be denoted by $\mathfrak{R}(G, H)$.

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The problem of characterizing a member of the set $\mathfrak{R}(G, H)$ for given graphs G and H is very difficult, even though it is for small graphs G and H. There are many papers dealing with the determination of all graphs belonging to the set $\mathfrak{R}(G, H)$. Nešetřil and Rödl [15] proved that $\mathfrak{R}(G, H)$ is infinite if one of the following three cases is satisfied: (i) both G and H are forests containing a path of length three, (ii) both are $3-$connected graphs, or (iii) both have a chromatic number of at least three. Furthermore, Burr *et al.* [10] showed that the set $\mathfrak{R}(G, H)$ is Ramsey infinite when both G and H are forest, with at least one of G or H having a non-star component. They also constructed an infinite family of $\mathfrak{R}(P_n, P_n)$. Łuczak [12] showed that for every forest G other than a matching and every graph H containing a cycle, the set $\mathfrak{R}(G, H)$ is infinite. Moreover, Borowiecki *et al.* [5] gave two equivalent theorems which characterize the graphs in $\mathfrak{R}(K_{1,2}, K_{1,m})$ for $m \geq 3$. In particular, for $m = 3$, all graphs in $\mathfrak{R}(K_{1,2}, K_3)$ were determined by Borowiecki *et al.* [6]. Some infinite families of Ramsey $(K_{1,2}, C_4)-$minimal graphs have been also characterized by some researchers [1, 4, 17]. Yulianti *et al.* [18] constructed some infinite classes of graphs in $\mathfrak{R}(K_{1,2}, P_4)$. Next, Hałuszczak [11] characterized graphs belonging to $\mathfrak{R}(K_{1,2}, K_n)$. Moreover, Borowiecka-Olszewska and Hałuszczak [7] presented a procedure to generate an infinite family of $(K_{1,m}, \mathcal{G})-$minimal graphs, where $m \geq 2$ and $\mathcal{G}$ is a family of $2-$connected graphs.

Burr *et al.* [8] proved that if G is a matching then $\mathfrak{R}(G, H)$ is finite for any graph H. They also showed that for any graph H, $\mathfrak{R}(K_2, H) = \{H\}$, and gave $\mathfrak{R}(2K_2, 2K_2) = \{C_5, 3K_2\}$, and $\mathfrak{R}(2K_2, K_3) = \{K_5, 2K_3, G\}$ (see Figure 1). In the same paper, Burr *et al.* [8] gave a construction

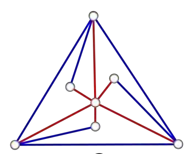

Figure 1. The graph $G \in \mathfrak{R}(2K_2, K_3)$.

of some graph G_r in $\mathfrak{R}(2K_2, K_n)$, for $n \geq 4$ and $1 \leq r \leq [(n+1)/2]$. A graph G_r is constructed from a graph K_{n+1} by adding new vertices and edges as follows. Let $V(K_{n+1}) = R \cup S$ be a partition of the vertices of K_{n+1}, where r denotes the cardinality of R. To each edge $e = xy$ with $\{x, y\} \subseteq R$ or $\{x, y\} \subseteq S$, associate a vertex v_e not in K_{n+1} and let v_e be adjacent to each vertex of K_{n+1} except for x and y. The graph G_r will have $|V(G_r)| = n + 1 + \binom{r}{2} + \binom{n+1-r}{2}$ and $|E(G_r)| = \binom{n+1}{2} + (n-1)\left(\binom{r}{2} + \binom{n+1-r}{2}\right)$. For example, the graphs G_1 and G_2 as depicted in Figure 2 are graphs in $\mathfrak{R}(2K_2, K_4)$.

Next, the set $\mathfrak{R}(G, H)$ with G is a $2K_2$ and H is a tK_2 was investigated by Burr *et al.* [9]. Mengersen and Oeckermann [13] characterized graphs in $\mathfrak{R}(2K_2, K_{1,n}), n > 3$ and determined all graphs in $\mathfrak{R}(2K_2, K_{1,n})$ for $n \leq 3$. All graphs in $\mathfrak{R}(2K_2, P_n)$ for $n = 4, 5$ were determined by Baskoro and Yulianti [3]. Moreover, all graphs belonging to $\mathfrak{R}(2K_2, 2P_3)$ were determined by Tatanto and Baskoro [16]. Mushi and Baskoro [14] derived the properties of graphs belonging to the class $\mathfrak{R}(3K_2, P_3)$ and determined all graphs belonging to $\mathfrak{R}(3K_2, P_3)$. These results proved the

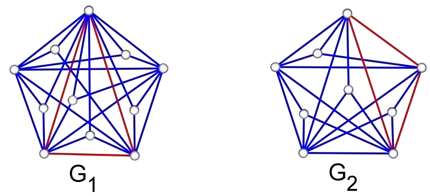

Figure 2. The graphs $G_1, G_2 \in \mathfrak{R}(2K_2, K_4)$.

previous claim given by Burr *et al.* in [9]. Recently, Baskoro and Wijaya [2] gave some necessary and sufficient conditions for graphs in $\mathfrak{R}(2K_2, H)$. They proved the following theorem.

Theorem 1.1. [2] *Let H be any connected graph. $F \in \mathfrak{R}(2K_2, H)$ if and only if the following conditions are satisfied:*

(*i*) *for every $v \in V(F)$, $F - v \supseteq H$,*

(*ii*) *for every K_3 in F, $F - E(K_3) \supseteq H$,*

(*iii*) *for every $e \in E(F)$, there exists $v \in V(F)$ or K_3 in F such that $(F - e) - v \not\supseteq H$ or $(F - e) - E(K_3) \not\supseteq H$.* □

If a graph F satisfies Theorem 1.1(i) and (ii), then $F \to (2K_2, H)$. While a graph F satisfying Theorem 1.1(iii) means that F satisfies the minimal property of F, that is for each $e \in F$, $F - e \nrightarrow (2K_2, H)$. Observe that if $F, G \in \mathfrak{R}(2K_2, H)$, then the minimality property implies that $F \nsubseteq G$ and $G \nsubseteq F$. In the same paper, Baskoro and Wijaya [2] gave all graphs of order at most 8 in $\mathfrak{R}(2K_2, K_4)$ as in the following theorem.

Theorem 1.2. [2]

1. *Graph $2K_4$ is the only disconnected graph in $\mathfrak{R}(2K_2, K_4)$.*
2. *Graph K_6 is the only graph of 6 vertices in $\mathfrak{R}(2K_2, K_4)$.*
3. *There is no connected graph with 7 vertices in $\mathfrak{R}(2K_2, K_4)$.*
4. *Graph F_1 in Figure 3 is the only graph of 8 vertices in $\mathfrak{R}(2K_2, K_4)$.* □

In [2], they also gave a graph of order 9, namely F_2, in $\mathfrak{R}(2K_2, K_4)$ (see Figure 3).

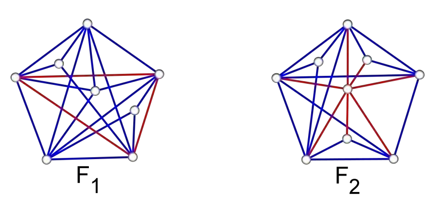

Figure 3. The graphs $F_1, F_2 \in \mathfrak{R}(2K_2, K_4)$.

In this paper, we determine all connected graphs of order at least 9 in $\mathfrak{R}(2K_2, K_4)$. These results will complete the earlier results regarding all graphs in $\mathfrak{R}(2K_2, K_4)$ discussed by Burr *et al.* [8] and Baskoro & Wijaya [2]. Additionally, we also give a general class graph which belong to $\mathfrak{R}(2K_2, K_n)$, for $n \geq 3$.

2. Main Results

In this section, we will construct all graphs in $\mathfrak{R}(2K_2, K_4)$.

Theorem 2.1. *Let H be a connected graph. Then, $2H \in \mathfrak{R}(2K_2, H)$.*

Proof. It is easy to see that $2H \rightarrow (2K_2, H)$. Let e be an edge of $2H$. Then, we need only consider when $e \in E(H)$. So, $2H - e = H \cup (H - e)$. We now consider the edge a in H. Let ϕ be a red-blue coloring on the edges of $2H - e$ such that $\phi(a) =$ red and $\phi(x) =$ blue otherwise. Under coloring ϕ, $2H - e$ contains a red K_2 and a blue $(H - a) \cup (H - e)$. We obtain a $(2K_2, H)-$coloring. Therefore, $2H \in \mathfrak{R}(2K_2, H)$. □

Lemma 2.1. *Let F and H be connected graphs. If $F \in \mathfrak{R}(2K_2, H)$, then*

(i) any two subgraphs of F isomorphic to H will intersect in at least one vertex,

(ii) F contains at least three subgraphs isomorphic to H.

Proof. Let F and H be connected graphs.

(i) By Theorem 1.1(i), we have two disjoint graphs H in F. Then, $F \supseteq 2H$. By Theorem 2.1, it contradicts to the minimality of F.

(ii) By Theorem 2.1, $2H \in \mathfrak{R}(2K_2, H)$. So, F does not contain $2H$. By case (i), F contains more than one subgraph H intersecting in at least one vertex. Let v be an intersecting vertex of some two subgraphs of F isomorphic to H. Since $F - v \supseteq H$, F must contain at least three subgraphs H in F. □

Lemma 2.2. *If $F \in \mathfrak{R}(2K_2, K_n)$, then $|V(F)| \geq n + 2$.*

Proof. By Theorem 1.1, F contains a K_n and every triangle K_3 in F, $F - E(K_3)$ must contain a K_n. This K_n can be formed by involving at most one vertex on K_3. So, to form a complete graph of order n, we need involve at least two vertices $u, v \in F$, where $u, v \notin V(K_n)$. Hence, F has order at least $n + 2$. □

By Lemma 2.1, if $F \in \mathfrak{R}(2K_2, K_4)$, then F will contain at least three subgraphs K_4 which any two K_4 are intersecting. Now, consider graphs A_1, A_2, and A_3 obtained by intersecting three K_4, as depicted in Figure 4. We then have the following lemma.

Lemma 2.3. *Let F be a connected graph of order at least 9 and $F \in \mathfrak{R}(2K_2, K_4)$. Then, F must contain A_1, A_2, or A_3.*

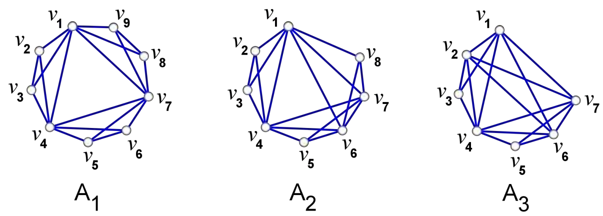

Figure 4. The graphs A_1, A_2, and A_3.

Proof. Let $F \in \mathfrak{R}(2K_2, K_4)$ and $V(F) = \{v_1, v_2, \ldots, v_n\}, n \geq 9$. By Theorem 1.1, F must contain a K_4, we may assume $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Next, we consider a K_3 with the vertex set $\{v_1, v_2, v_3\}$. There must be a K_4 in $F - E(K_3)$ by Theorem 1.1(ii). Since a complete graph of order 4 in $F - E(K_3)$ can only contain at most one vertex $v \in V(K_3)$, then (up to isomorphism) one of two vertex sets $\{v_4, v_5, v_6, v_7\}$ or $\{v_3, v_4, v_5, v_6\}$ form this K_4. We obtain that F contains a graph A or B as depicted in Figure 5.

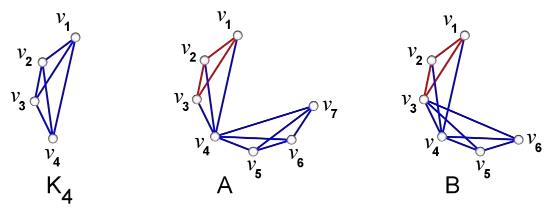

Figure 5. The possibilities of forming a K_4 in $F - E(K_3)$ for some $V(K_3) = \{v_1, v_2, v_3\}$ when F contains a K_4.

First, we consider F containing A. By Theorem 1.1(i), there must be a K_4 in $F - v_4$. This K_4 is formed by involving at most two vertices which are not in A, otherwise F contains a $2K_4$. Then, one of four vertex sets $\{v_1, v_7, v_8, v_9\}$, $\{v_1, v_6, v_7, v_8\}$, $\{v_1, v_2, v_6, v_7\}$ or $\{v_1, v_5, v_6, v_7\}$ will form a complete graph of order 4 in $F - v_4$. We obtain that F contains a graph A_1, A_2, A_3 (see Figure 4), or A_4 (see Figure 6). Now, we consider F containing A_4. By Theorem 1.1(ii), for $V(K_3) = \{v_1, v_4, v_5\}$, $F - E(K_3)$ must contain a K_4. Then, one of three vertex sets $\{v_4, v_6, v_7, v_8\}$, $\{v_2, v_5, v_7, v_8\}$, or $\{v_2, v_3, v_6, v_8\}$ will form this K_4. We obtain that F contains a graph A_{41}, A_{42}, or A_{43} as depicted in Figure 6. Since A_{41}, A_{42}, and A_{43} contain A_2 or A_3, it suffices to consider F

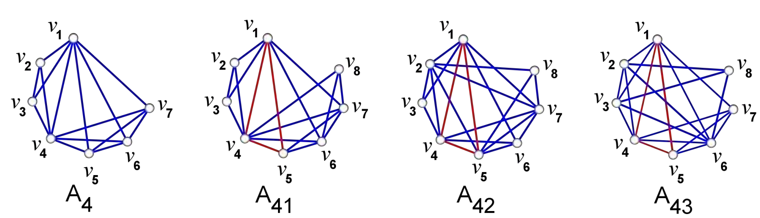

Figure 6. The possibilities of forming a K_4 in $F - E(K_3)$ for some $V(K_3) = \{v_1, v_4, v_5\}$ when $F \supseteq A_4$.

contains A_1, A_2, or A_3.

Second, we consider F containing B. To see this case, we have the following claim.

Claim: Let $F \in \mathfrak{R}(2K_2, K_n)$. If F contains $K_5 - e$, then F contains A.

Proof of Claim. Let $V(K_5 - e) = \{v_1, v_2, v_3, v_4, v_5\}$ where $d(v_2) = d(v_5) = 3$. By Theorem 1.1(ii), there must be a K_4 in $F - E(K_3)$ for a K_3 with the vertex set $\{v_1, v_3, v_4\}$. Up to isomorphism, the only vertex set $\{v_1, v_2, v_5, v_6\}$ will form this K_4. So, we now have F containing $K_6 \setminus P_3$. Consider a K_3 with the vertex set $\{v_1, v_2, v_5\}$. By Theorem 1.1(ii), $F - E(K_3)$ must contain a K_4. This K_4 is formed by involving one vertex v_7 in F, that is the vertex set $\{v_3, v_4, v_6, v_7\}$ or $\{u, v, w, v_7\}$ where $u \in \{v_1, v_2, v_5\}$ and $v, w \in \{v_3, v_4, v_6, v_7\}$. All resulted graphs contain A. $\diamondsuit$

If F contains B, then by Theorem 1.1(ii), for a K_3 with the vertex set $\{v_2, v_3, v_4\}$, $F - E(K_3)$ must contain a K_4. By the minimality, this K_4 is formed by the vertex set $\{v_1, v_3, v_5, v_6\}$ or $\{v_1, v_4, v_6, v_7\}$, otherwise F contains A. Both resulted graphs contain $K_5 - e$. So, by the Claim, F contains A. Thus, the claim follows immediately. $\square$

For the next theorem, we consider the graphs in Figures 2, 3, and 7. We will prove that $2K_4, K_6$, and all graphs in Figures 2, 3, and 7 are the only Ramsey $(2K_2, K_4)-$minimal graphs.

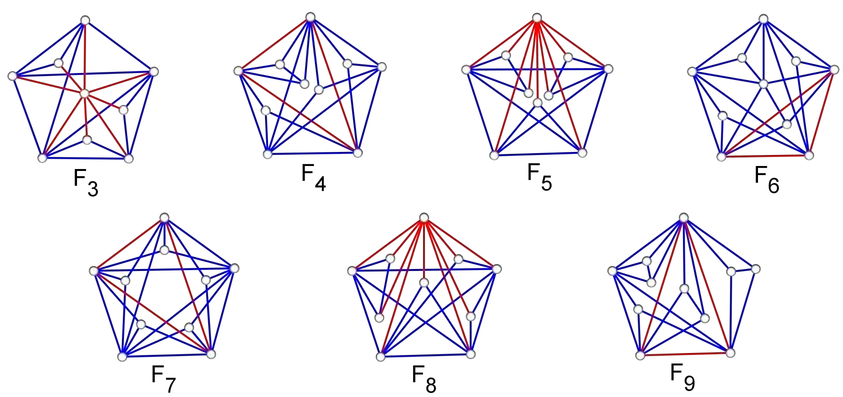

Figure 7. Some connected graphs in $\mathfrak{R}(2K_2, K_4)$.

Theorem 2.2.

$$\mathfrak{R}(2K_2, K_4) = \{2K_4\} \cup \{K_6, F_1, F_2\} \cup \{G_1, G_2\} \cup \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}.$$

Proof. First, we show that every $F \in \{2K_4\} \cup \{K_6, F_1, F_2\} \cup \{G_1, G_2\} \cup \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$ is a Ramsey $(2K_2, K_4)-$minimal graph. Baskoro and Wijaya [2] (see Theorem 1.2) showed that $2K_4$ is the only disconnected graph in $\mathfrak{R}(2K_2, K_4)$; K_6 and F_1 are the only connected graphs of order at most 8 in $\mathfrak{R}(2K_2, K_4)$. In the same paper, they also showed that $F_2 \in \mathfrak{R}(2K_2, K_4)$. Moreover, Burr *et al.* [8] showed that $G_1, G_2 \in \mathfrak{R}(2K_2, K_4)$. We can show easily that every $F \in \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$ satisfies Theorem 1.1(i) and (ii). The proof of the minimality of $F \in \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$ is depicted in Figure 8. We color all edges of F by red and blue. Observe that such a coloring induces a red K_2 and exactly a blue K_4 (drawn in bold blue). Thus, removing an arbitrary bold blue edge e in K_4 results in a $(2K_2, K_4)-$coloring of $F - e$.

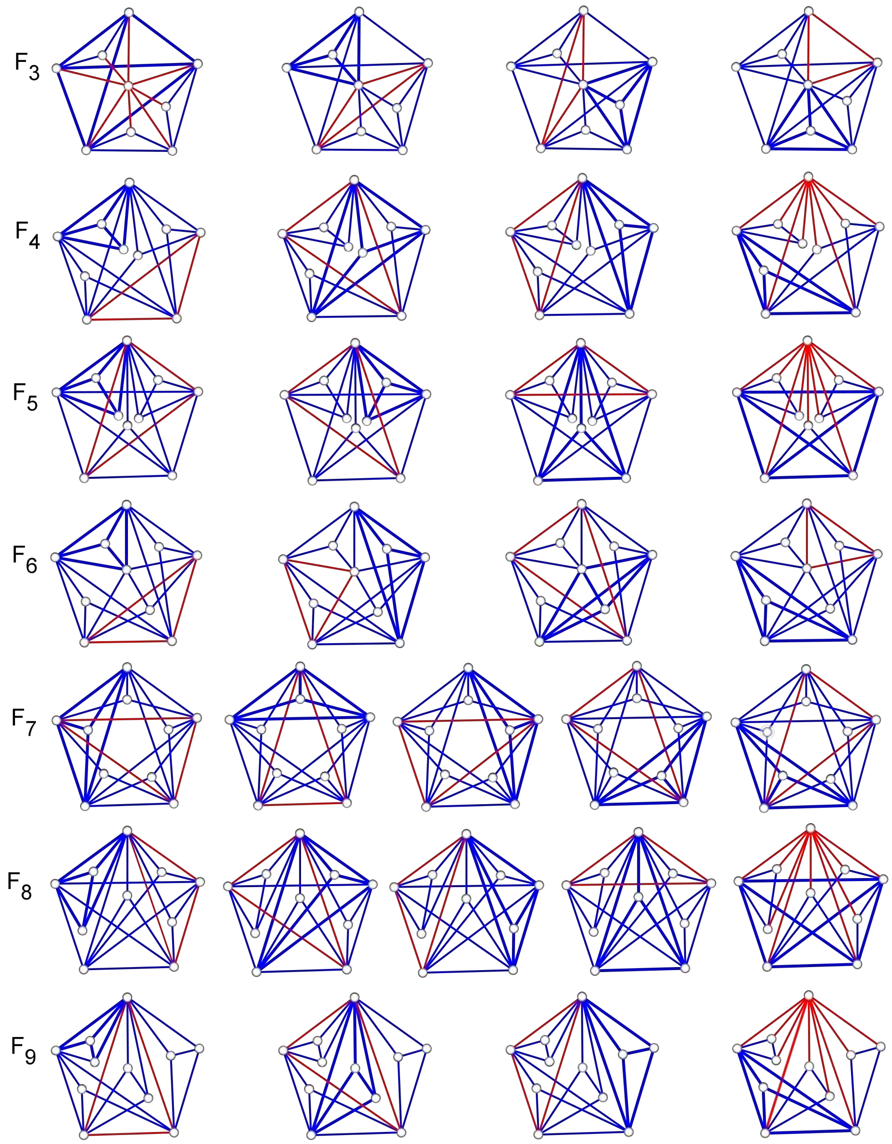

Figure 8. Some red-blue coloring of $F_3, F_4, F_5, F_6, F_7, F_8$, and F_9 contain a red K_2 and a unique blue K_4.

We now prove that all graphs in $\mathfrak{R}(2K_2, K_4)$ are $\{2K_4\} \cup \{K_6, F_1, F_2\} \cup \{G_1, G_2\} \cup \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$. By Theorem 1.2, we enough to show that $F_2, G_1, G_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9$ are the only graphs of order at least 9 in $\mathfrak{R}(2K_2, K_4)$. Let $F \in \mathfrak{R}(2K_2, K_4)$ be a connected graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$, $n \geq 9$. Then, F does not contain $2K_4$, K_6, or F_1. By Lemma 2.3, F contains a graph isomorphic to A_1, A_2, or A_3 in Figure 4. We can see that for $i = 1, 2, 3$, the graph A_i, satisfies Theorem 1.1(i) but each does not satisfy Theorem 1.1(ii) for some triangles in A_i. Therefore, to construct $F \in \mathfrak{R}(2K_2, K_4)$ we must form a complete graph K_4 in $F - E(K_3)$ by involving some edges or vertices (and edges) in F. Furthermore, there are three subcases follow.

(1) F contains A_1. The construction of graph F containing A_1 in $\mathfrak{R}(2K_2, K_4)$ is depicted in

Figure 9. The only K_3 in F not satisfying Theorem 1.1(ii) yet is the one with the vertex set $V_1 = \{v_1, v_4, v_7\}$.

(a) If F is of order 9, then a complete graph K_4 must contain exactly one vertex in V_1, otherwise F contains a graph $2K_4$. We may assume $v_1 \in K_4$. Up to isomorphism, there is only the vertex set $\{v_1, v_2, v_5, v_8\}$ forming a complete graph K_4. We obtain a graph A_{11} which is isomorphic to F_2 (in Figure 3).

(b) If F is of order 10 and $v_{10} \in F$, then v_{10} must be in K_4. Up to isomorphism, one of three vertex set $\{v_1, v_2, v_5, v_{10}\}$, $\{v_1, v_5, v_6, v_{10}\}$, or $\{v_2, v_5, v_8, v_{10}\}$ form a complete graph K_4. We obtain a graph A_{12}, A_{13}, or A_{14} as depicted in Figure 9 which is isomorphic to F_4, F_5, or F_6, respectively.

(c) If F is of order 11 and $v_{10}, v_{11} \in F$, then v_{10}, v_{11} must be in the graph K_4, another vertex must be one of v_1, v_4, or v_7 and the other vertex must be either v_5 or v_6. Up to isomorphism, there is only the vertex set $\{v_1, v_5, v_{10}, v_{11}\}$ forming a graph K_4. We obtain a graph A_{15} as depicted in Figure 9 which is isomorphic to F_9. Involving more than two vertices will make F not minimal.

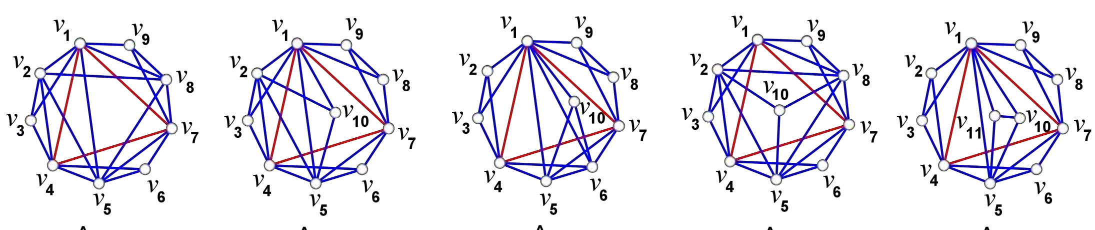

Figure 9. The possibilities of the form a K_4 in $F - E(K_3)$ for $V(K_3) = \{v_1, v_4, v_7\}$ when $F \supseteq A_1$.

(2) F contains A_2. All triangles in A_2 such that $F - E(K_3)$ not satisfying Theorem 1.1(ii) yet are K_3 with the vertex sets $V_1 = \{v_1, v_4, v_7\}$ or $V_2 = \{v_1, v_4, v_6\}$. We observe $V(K_3) = V_2$. We must form a complete graph K_4 in $F - E(K_3)$, by involving one, two, or more than two vertices in F. Moreover, forming this K_4 does not result in F containing A_1, since we have obtained all Ramsey $(2K_2, K_4)-$minimal graphs containing A_1.

(a) If F is of order 9 and $v_9 \in F$, then v_9 must be in K_4. Up to isomorphism, one of six vertex sets $\{v_1, v_2, v_5, v_9\}$, $\{v_1, v_2, v_7, v_9\}$, $\{v_1, v_5, v_7, v_9\}$, $\{v_1, v_7, v_8, v_9\}$, $\{v_2, v_5, v_6, v_9\}$, or $\{v_2, v_6, v_7, v_9\}$ form a complete graph K_4. We obtain the graphs A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, A_{26} as depicted in Figure 10. Now, we consider F containing A_{2i} for $i \in [1, 6]$.

(i) The graphs A_{21}, A_{25}, and A_{26} have satisfied Theorem 1.1(i) and (ii), since there exists a complete graph K_4 in $F - E(K_3)$ for $V(K_3) = V_1$. These graphs are isomorphic to F_3, F_2, and G_2, respectively (see Figure 7, 3, and 2, respectively).

(ii) The graphs A_{22}, A_{23}, and A_{24} have not satisfied Theorem 1.1(ii) yet, since there is no complete graph K_4 in $F - E(K_3)$ for $V(K_3) = V_1$. We must involve a vertex v_{10} in $F \supseteq A_{2i}$ $(i = 2, 3, 4)$ to form this K_4. Otherwise, F is not minimal. So, the complete graph K_4 in $F - E(K_3)$ is formed by the vertex set $\{v_2, v_4, v_6, v_{10}\}$ when $F \supseteq A_{22}$; $\{v_1, v_5, v_6, v_{10}\}$ when $F \supseteq A_{23}$, or $\{v_1, v_6, v_8, v_{10}\}$ when $F \supseteq A_{24}$.

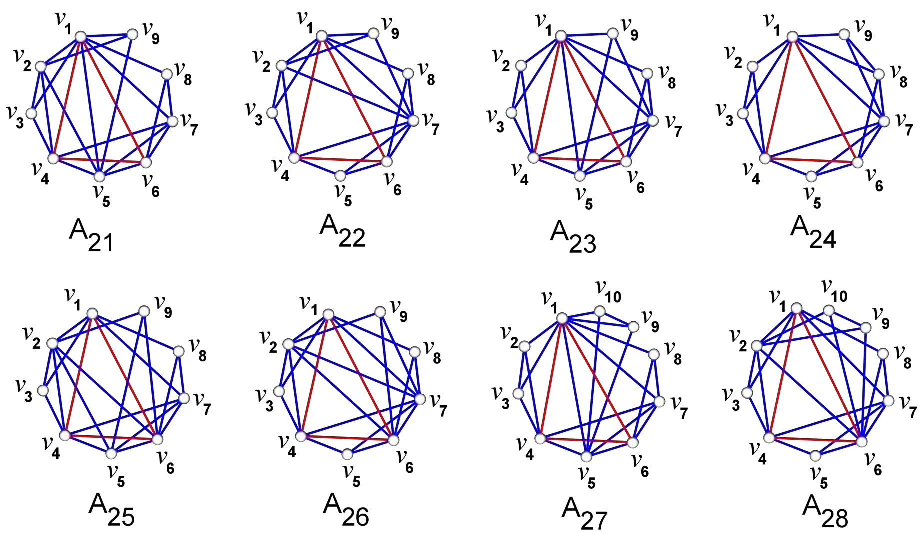

Figure 10. The possibilities form a K_4 in $F - E(K_3)$ for $V(K_3) = \{v_1, v_4, v_6\}$ when $F \supseteq A_2$.

Otherwise, F is not minimal. We obtain the graph F with the edge set $E(F) = E(A_{22}) \cup \{v_2v_6, v_iv_{10}| \ i = 2,4,6\}$, $E(F) = E(A_{23}) \cup \{v_iv_{10}| \ i = 1,5,6\}$, or $E(F) = E(A_{24}) \cup \{v_iv_{10}| \ i = 1,6,8\}$ which is isomorphic to F_7, F_8, or F_4, respectively.

(b) If F is of order 10 and $v_9, v_{10} \in F$, then v_9 and v_{10} must be in the graph K_4. Up to isomorphism, one of two vertex sets $\{v_1, v_5, v_9, v_{10}\}$ or $\{v_2, v_6, v_9, v_{10}\}$ will form a graph K_4. We obtain a graph A_{27} or A_{28}, respectively, as depicted in Figure 10, which is isomorphic to F_5 or F_4, respectively.

(c) F cannot have order more than 10, since involving more than two vertices in $F \supseteq A_2$ to form a complete graph K_4 in $F - E(K_3)$ will make F not minimal.

(3) F contains A_3. There are four triangles K_3 in A_3 such that F does not satisfy Theorem 1.1(ii) yet, namely the K_3 with the vertex set $V_1 = \{v_1, v_4, v_7\}$, $V_2 = \{v_1, v_4, v_6\}$, $V_3 = \{v_2, v_4, v_6\}$, or $V_4 = \{v_2, v_4, v_7\}$. We observe a triangle K_3 with the vertex set V_2. We must form a complete graph K_4 in $F - E(K_3)$. Since we have obtained all graphs containing A_1 or A_2 in $\mathfrak{R}(2K_2, K_4)$, we avoid forming a graph K_4 in $F - E(K_3)$ which contains A_1 or A_2. Therefore, a complete graph K_4 must contain three vertices in A_3. Suppose that v_8 is in K_4. There are two vertex sets forming this K_4, namely $\{v_2, v_4, v_5, v_8\}$ or $\{v_2, v_4, v_7, v_8\}$ (the graph A_{31} or A_{32} as depicted in Figure 11). Now, we consider F containing A_{31} or A_{32}. There is no complete graph K_4 in $F - E(K_3)$ for $V(K_3) = V_3$.

(a) Forming a complete graph K_4 in $F - E(K_3)$ when $F \supseteq A_{31}$ by involving one or more vertices (and edges) in F will cause F to contain $2K_4$, A_1, or A_2.

(b) One of two vertex sets $\{v_1, v_4, v_7, v_9\}$ or $\{v_3, v_4, v_7, v_9\}$ will form a graph K_4 in $F - E(K_3)$ when $F \supseteq A_{32}$. We obtain that F contains a graph A_a or A_b as depicted in Figure 11. Next, for $V(K_3) = V_4$, if we delete $E(K_3)$ of both graphs A_a and A_b, then

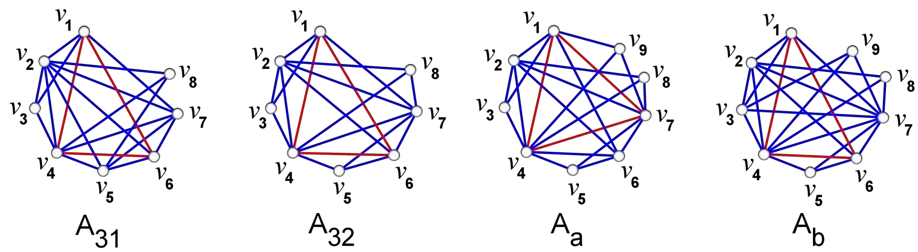

Figure 11. The possibilities form a K_4 in $F - E(K_3)$ for some $V(K_3) = \{v_1, v_4, v_6\}$ when $F \supseteq A_3$.

the resulted graphs do not contain a graph K_4. Thus, we must form the complete graph K_4 in $F - E(K_3)$.

(i) Consider F containing A_a. The graph K_4 in $F - E(K_3)$ must involve one vertex v_{10} in F, namely $\{v_1, v_4, v_6, v_{10}\}$. Otherwise, F is not minimal. We obtain that F contains a graph A_{a1} as depicted in Figure 12. By Theorem 1.1(ii), $F - E(K_3)$ must contain a graph K_4 for some $E(K_3) = V_1$. This K_4 is formed by the vertex set $\{v_2, v_4, v_6, v_{11}\}$. We obtain the graph A_{a2} as depicted in Figure 12 which is isomorphic to G_2 in Figure 2.

(ii) Consider F containing A_b. The graph K_4 in $F - E(K_3)$ cause F which contains the graph $2K_4$, A_1, or A_2.

So, all graphs in $\mathfrak{R}(2K_2, K_4)$ are $\{2K_4\} \cup \{K_6, F_1, F_2\} \cup \{G_1, G_2\} \cup \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$. □

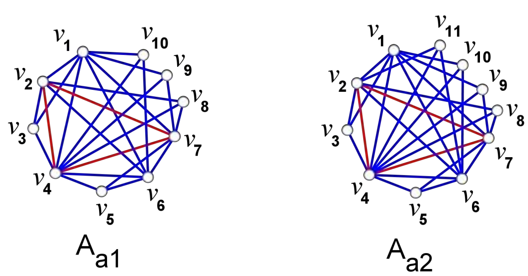

Figure 12. The possibility form a K_4 in $F - E(K_3)$ for $V(K_3) = \{v_2, v_4, v_7\}$ when $F \supseteq A_a$.

Now, we will give a class of graphs belonging to $\mathfrak{R}(2K_2, K_n)$ for $n \geq 3$.

Theorem 2.3. $K_{n+2} \in \mathfrak{R}(2K_2, K_n)$.

Proof. Since for every $v \in V(K_{n+2})$, we obtain $K_{n+2} - v \cong K_{n+1}$, then $K_{n+2} - v \supseteq K_n$. Furthermore, for each K_3 in K_{n+2}, we obtain $K_{n+2} - E(K_3) \cong K_{n-1} + 3K_1$, so $K_{n+2} - E(K_3) \supseteq K_n$. Next, for every $e \in E(K_{n+2})$, we obtain $K_{n+2} - e \cong K_n + 2K_1$. If we take an arbitrarily triangle in K_n, then we have $(K_{n+2} - e) - E(K_3) \cong K_{n-3} + 3K_1 + 2K_1$. So, $(K_{n+2} - e) - E(K_3)$ does not contain a K_n. □

By Theorem 2.2, we conclude that the set of Ramsey $(2K_2, K_4)-$minimal graphs does not contain two graphs with the same degree sequence. In general case, from [2], we can construct a graph $G \in \mathfrak{R}(2K_2, 2K_n)$, $n \geq 3$ by taking a disjoint union of any two connected graphs in $\mathfrak{R}(2K_2, K_n)$. We can also construct a graph $G \in \mathfrak{R}(2K_2, 2K_n)$ by identifying some vertices or edges of any two connected graphs in $\mathfrak{R}(2K_2, K_n)$.

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