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Symmetric colorings of $G \times \mathbb{Z}_2$

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Abstract

Let G be a finite group and let $r \in \mathbb{N}$. An *r-coloring* of G is any mapping $\chi : G \to \{1, \dots, r\}$. A coloring χ is *symmetric* if there is $g \in G$ such that $\chi(gx^{-1}g) = \chi(x)$ for every $x \in G$. We show that if $f(r)$ is the polynomial representing the number of symmetric r-colorings of G, then the number of symmetric r-colorings of $G \times \mathbb{Z}_2$ is $f(r^2)$.

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1. Introduction

Let G be a finite group and let $r \in \mathbb{N}$. An *r-coloring* of G is any mapping $\chi : G \to \{1, \dots, r\}$. The group G naturally acts on its r-colorings. For every coloring χ and for every $g \in G$, the coloring χg is defined by

$$\chi g(x) = \chi(xg^{-1}).$$

Colorings χ and ψ are *equivalent* if there is $g \in G$ such that $\chi g = \psi$ (that is, if χ and ψ belong to the same orbit). Let $c_r(G)$ denote the number of equivalence classes of r-colorings of G (= the number of orbits). Applying Burnside's Lemma [1, I, §3] gives us that

$$c_r(G) = \frac{1}{|G|} \sum_{g \in G} r^{\frac{|G|}{|\langle g \rangle|}},$$

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where $\langle g \rangle$ is the subgroup generated by g. For $G = \mathbb{Z}_n$, the cyclic group of order n, this formula simplifies to

$$c_r(\mathbb{Z}_n) = \frac{1}{n} \sum_{d|n} \varphi(d) r^{\frac{n}{d}},$$

where φ is the Euler function [2].

For every $g \in G$, the *symmetry* on G with respect to g is the mapping

$$G \ni x \mapsto gx^{-1}g \in G.$$

This is an old notion, which can be found in the book [5]. We say that a coloring χ of G is *symmetric* if it is invariant under some symmetry, that is, if there is $g \in G$ such that $\chi(gx^{-1}g) = \chi(x)$ for all $x \in G$. A coloring equivalent to a symmetric one is also symmetric. Let $S_r(G)$ denote the number of symmetric r-colorings of G and $s_r(G)$ the number of equivalence classes of symmetric r-colorings of G (= the number of symmetric orbits). For every finite Abelian group G and for every $r \in \mathbb{N}$,

$$S_r(G) = \sum_{X \le G} \sum_{Y \le X} \frac{\mu(Y, X)|G/Y|}{|B(G/Y)|} r^{\frac{|G/X| + |B(G/X)|}{2}},$$

$$s_r(G) = \sum_{X \le G} \sum_{Y \le X} \frac{\mu(Y, X)}{|B(G/Y)|} r^{\frac{|G/X| + |B(G/X)|}{2}},$$

where X runs over subgroups of G, Y over subgroups of X, $\mu(Y, X)$ is the Möbius function on the lattice of subgroups of G, and $B(H) = \{x \in H : 2x = 0\}$ [3]. Similar but more complicated formulas hold also in the non-Abelian case [6]. For $G = \mathbb{Z}_n$ the formulas above simplify to

$$S_r(\mathbb{Z}_n) = \begin{cases} \sum_{d|n} d \prod_{p|\frac{n}{d}} (1-p) r^{\frac{d+1}{2}}, & \text{if } n \text{ is odd,} \\ \sum_{d|\frac{n}{2}} d \prod_{p|\frac{n}{2d}} (1-p) r^{d+1}, & \text{if } n \text{ is even,} \end{cases}$$

$$s_r(\mathbb{Z}_n) = \begin{cases} r^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \\ \frac{1}{2}(r^{\frac{n}{2}+1} + r^{\frac{m+1}{2}}), & \text{if } n \text{ is even,} \end{cases}$$

where p is a variable of prime value and m is the greatest odd divisor of n [3]. For $G = D_n$, the dihedral group of order $2n$, the numbers $S_r(D_n)$ and $s_r(D_n)$, were computed in [7]. (See also [4].)

In [8] it was shown that, for every finite Abelian group G, if $f(r)$ is the polynomial representing $S_r(G)$, that is $S_r(G) = f(r)$, then $S_r(G \times \mathbb{Z}_2) = f(r^2)$, and so, for every $n \in \mathbb{N}$, $S_r(G \times \prod_n \mathbb{Z}_2) = f\left(r^{2^n}\right)$.

In this paper we extend this result to all finite groups.

Theorem 1.1. *Let G be a finite group and let $f(r)$ be the polynomial representing $S_r(G)$. Then $S_r(G \times \mathbb{Z}_2) = f(r^2)$.*

The proof of Theorem 1.1 is a combination of that from [8] and the optimal partitions method for counting $S_r(G)$ from [6].

2. Preliminaries

In this section we recall the main result and related notions from [6].

For every coloring $\chi : G \to \{1, 2, \dots, r\}$, let $[\chi]$ and $St(\chi)$ denote the orbit and the stabilizer of χ, that is,

$$[\chi] = \{\chi g : g \in G\} \text{ and } St(\chi) = \{g \in G : \chi g = \chi\}.$$

As in the general case of an action,

$$|[\chi]| = \frac{|G|}{|St(\chi)|} \text{ and } St(\chi g) = g^{-1} St(\chi) g.$$

In counting $S_r(G)$ and $s_r(G)$ an important role is played also by the sets

$$Z(\chi) = \{g \in G : \chi \text{ is symmetric with respect to } g\},$$

$$[\chi]_e = \{\psi \in [\chi] : \psi \text{ is symmetric with respect to } e\},$$

where e is the identity of G. The set $Z(\chi)$ is a union of left cosets of G by $St(\chi)$ and

$$|[\chi]_e| = \frac{|Z(\chi)|}{|St(\chi)|}.$$

Similarly to colorings, these notions naturally extend to partitions of G. In particular, for every partition π of G, $St(\pi)$ is the set of all $g \in G$ such that every cell of π is invariant under right translation by g^{-1}, and $Z(\pi)$ is the set of all $g \in G$ such that every cell of π is invariant under symmetry with respect to g. We say that a partition π of G is *optimal* if $e \in Z(\pi)$ and for every partition π' of G with $St(\pi') = St(\pi)$ and $Z(\pi') = Z(\pi)$, one has $\pi \leq \pi'$. The latter means that every cell of π is contained in some cell of π'.

Let P be the set of all pairs $x = (St(x), Z(x))$ such that $St(x) = St(\chi)$ and $Z(x) = Z(\chi)$ for some coloring χ of G symmetric with respect to e. Define the order $\leq$ on P by

$$x \leq y \Leftrightarrow St(x) \subseteq St(y) \text{ and } Z(x) \subseteq Z(y).$$

For every $x \in P$, let π_x denote the finest partition of G with $St(\pi) = St(x)$ and $Z(\pi) = Z(x)$. Then $\{\pi_x : x \in P\}$ is the set of all optimal partitions of G and $\pi_x \leq \pi_y \Leftrightarrow x \leq y$, so $\{\pi_x : x \in P\}$ can be identified with P.

For every partition π, we write $|\pi|$ to denote the number of cells of π.

Theorem 2.1. *Let P be the partially ordered set of optimal partitions of G. Then*

$$S_r(G) = |G| \sum_{x \in P} \sum_{y \leq x} \frac{\mu(y, x)}{|Z(y)|} r^{|x|},$$

$$s_r(G) = \sum_{x \in P} \sum_{y \leq x} \frac{\mu(y, x) St(y)}{|Z(y)|} r^{|x|}.$$

3. Proof of Theorem 1.1

Lemma 3.1. *Let $\chi : G \times \mathbb{Z}_2 \to \{1, 2, \ldots, r\}$. For each $j \in \mathbb{Z}_2$, define $\chi_j : G \to \{1, 2, \ldots, r\}$ by $\chi_j(x) = \chi(x, j)$. Then χ is symmetric if and only if there is $g \in G$ such that each χ_j is symmetric with respect to g.*

Proof. Suppose that χ is symmetric. Then there is $(g, i) \in G \times \mathbb{Z}_2$ such that $\chi((g, i)(x, j)^{-1}(g, i)) = \chi(x, j)$ for every $(x, j) \in G \times \mathbb{Z}_2$. Since

$$(g, i)(x, j)^{-1}(g, i) = (g, i)(x^{-1}, j)(g, i) = (gx^{-1}g, iji) = (gx^{-1}g, j),$$

we have that $\chi(gx^{-1}g, j) = \chi(x, j)$, so $\chi_j(gx^{-1}g) = \chi_j(x)$.

Conversely, suppose that there is $g \in G$ such that each χ_j is symmetric with respect to g. Then χ is symmetric with respect to (g, i) for any $i \in \mathbb{Z}_2$. Indeed,

$$\chi((g, i)(x, j)^{-1}(g, i)) = \chi(gx^{-1}g, j) = \chi_j(gx^{-1}g) = \chi_j(x) = \chi(x, j).$$

□

We shall say that a pair (χ_0, χ_1) of colorings of G is *symmetric* if there is $g \in G$ such that each χ_j is symmetric with respect to g, that is, $\chi_j(gx^{-1}g) = \chi_j(x)$ for each j and for every $x \in G$.

Clearly, the correspondence $\chi \mapsto (\chi_0, \chi_1)$ defined in Lemma 3.1 is a bijection between the set of r-colorings of $G \times \mathbb{Z}_2$ and the set of pairs of r-colorings of G, and by Lemma 3.1, it maps the set of symmetric r-colorings of $G \times \mathbb{Z}_2$ onto the set of symmetric pairs of r-colorings of G. Thus, we obtain that

Corollary 3.1. *$S_r(G \times \mathbb{Z}_2)$ is equal to the number of symmetric pairs of r-colorings of G.*

Define the action of G on the pairs (χ_0, χ_1) of r-colorings of G by

$$(\chi_0, \chi_1)g = (\chi_0 g, \chi_1 g).$$

For every pair (χ_0, χ_1), let $[(\chi_0, \chi_1)]$ and $St(\chi_0, \chi_1)$ denote the orbit and the stabilizer of (χ_0, χ_1), that is,

$$[(\chi_0, \chi_1)] = \{(\chi_0, \chi_1)g : g \in G\} \text{ and } St(\chi_0, \chi_1) = \{g \in G : (\chi_0, \chi_1)g = (\chi_0, \chi_1)\}.$$

As in the general case of an action,

$$|[(\chi_0, \chi_1)]| = \frac{|G|}{|St(\chi_0, \chi_1)|}.$$

For every pair (χ_0, χ_1), let $Z(\chi_0, \chi_1)$ denote the set of all $g \in G$ such that (χ_0, χ_1) is symmetric with respect to g.

Lemma 3.2. *$Z((\chi_0, \chi_1)g) = Z(\chi_0, \chi_1)g$ for every $g \in G$.*

Proof. To see that $Z(\chi_0, \chi_1)g \subseteq Z((\chi_0, \chi_1)g)$, let $a \in Z(\chi_0, \chi_1)$. Then for each j and for every $x \in G$,

$$\chi_j g(agx^{-1}ag) = \chi_j(agx^{-1}a) = \chi_j(xg^{-1}) = \chi_j g(x).$$

Consequently, $ag \in Z((\chi_0, \chi_1)g)$.

Now, conversely,

$$Z((\chi_0, \chi_1)g) = Z((\chi_0, \chi_1)g)g^{-1}g \subseteq Z((\chi_0, \chi_1)gg^{-1})g = Z(\chi_0, \chi_1)g.$$

□

It follows from Lemma 3.2, in particular, that if a pair (χ_0, χ_1) is symmetric, then the whole orbit $[(\chi_0, \chi_1)]$ is symmetric.

The next lemma tells us that, for every symmetric pair (χ_0, χ_1), $Z(\chi_0, \chi_1)$ is a union of left cosets of G by $St(\chi_0, \chi_1)$.

Lemma 3.3. $Z(\chi_0, \chi_1) \cdot St(\chi_0, \chi_1) = Z(\chi_0, \chi_1)$.

Proof. Clearly, $Z(\chi_0, \chi_1) \subseteq Z(\chi_0, \chi_1) \cdot St(\chi_0, \chi_1)$. To see the converse inclusion, let $g \in Z(\chi_0, \chi_1)$ and $h \in St(\chi_0, \chi_1)$. Then for each j and for every $x \in G$,

$$\chi_j(ghx^{-1}gh) = \chi_j h^{-1}(ghx^{-1}g) = \chi_j(ghx^{-1}g) = \chi_j(xh^{-1}) = \chi_j h(x) = \chi_j(x).$$

Consequently, $gh \in Z(\chi_0, \chi_1)$. □

For every symmetric pair (χ_0, χ_1), let $[(\chi_0, \chi_1)]_e$ denote the subset of $[(\chi_0, \chi_1)]$ consisting of all pairs symmetric with respect to e.

Lemma 3.4. $[(\chi_0, \chi_1)]_e = \{(\chi_0, \chi_1)a^{-1} : a \in Z(\chi_0, \chi_1)\}$.

Proof. To see that $\{(\chi_0, \chi_1)a^{-1} : a \in Z(\chi_0, \chi_1)\} \subseteq [(\chi_0, \chi_1)]_e$, let $a \in Z(\chi_0, \chi_1)$. Then for each j and for every $x \in G$,

$$\chi_j a^{-1}(x^{-1}) = \chi_j(x^{-1}a) = \chi_j(aa^{-1}xa) = \chi_j(xa) = \chi_j a^{-1}(x).$$

To see the converse inclusion, let $g \in G$ and suppose that $(\chi_0, \chi_1)g$ is symmetric with respect to e. Then for each j and for every $x \in G$,

$$\chi_j(g^{-1}x^{-1}g^{-1}) = \chi_j g(g^{-1}x^{-1}) = \chi_j g(xg) = \chi_j(xgg^{-1}) = \chi_j(x).$$

Consequently, $g^{-1} \in Z(\chi_0, \chi_1)$. □

From Lemma 3.4 and Lemma 3.3 we obtain that

Corollary 3.2. *For every symmetric pair* (χ_0, χ_1)*,*

$$|[(\chi_0, \chi_1)]_e| = \frac{|Z(\chi_0, \chi_1)|}{|St(\chi_0, \chi_1)|}.$$

Now we can count $S_r(G \times \mathbb{Z}_2)$.

Theorem 3.1. *Let P be the partially ordered set of optimal partitions of G. Then*

$$S_r(G \times \mathbb{Z}_2) = |G| \sum_{x \in P} \sum_{y \le x} \frac{\mu(y,x)}{|Z(y)|} r^{2|x|}.$$

Proof. By Corollary 3.1, we count the number of symmetric pairs of r-colorings of G. Let C denote the set of all pairs (χ_0, χ_1) symmetric with respect to e, and for every $x \in P$, let

$$C(x) = \{(\chi_0,\chi_1) \in C : St(\chi_0,\chi_1) = St(x) \text{ and } Z(\chi_0,\chi_1) = Z(x)\}.$$

Clearly, $\{C(x) : x \in P\}$ is a partition of C. For every $x \in P$ and $(\chi_0,\chi_1) \in C(x)$, $[(\chi_0,\chi_1)]_e = \{(\chi_0,\chi_1)a^{-1} : a \in Z(x)\}$, $St((\chi_0,\chi_1)a^{-1}) = aSt(x)a^{-1}$, and $Z((\chi_0,\chi_1)a^{-1}) = Z(x)a^{-1}$, so in general, $[(\chi_0,\chi_1)]_e \nsubseteq C(x)$. To correct this situation, define the equivalence $\equiv$ on P by

$$x \equiv y \Leftrightarrow St(y) = aSt(x)a^{-1} \text{ and } Z(y) = Z(x)a^{-1} \text{ for some } a \in Z(x).$$

For every $x \in P$, let $\bar{x}$ denote the $\equiv$-class containing x and let $C(\bar{x}) = \bigcup_{y \in \bar{x}} C(y)$. Then whenever $y \in \bar{x}$ and $(\chi_0,\chi_1) \in C(y)$,

$$[(\chi_0,\chi_1)]_e \subseteq C(\bar{x}),\ |[(\chi_0,\chi_1)]_e| = \frac{|Z(x)|}{|St(x)|} \text{ and } |[(\chi_0,\chi_1)]| = \frac{|G|}{|St(x)|}.$$

It follows that

$$|C(\bar{x})/\sim| = \frac{|St(x)||C(\bar{x})|}{|Z(x)|} = \sum_{y \in \bar{x}} \frac{|St(y)||C(y)|}{|Z(y)|}$$

and the number of pairs equivalent to pairs from $C(\bar{x})$ is

$$|C(\bar{x})/\sim| \cdot \frac{|G|}{|St(x)|} = |G| \sum_{y \in \bar{x}} \frac{|C(y)|}{|Z(y)|}.$$

Consequently, the number of all symmetric pairs of r-colorings of G is

$$|G| \sum_{y \in P} \frac{|C(y)|}{|Z(y)|}.$$

Now to compute $|C(y)|$, note that

$$\sum_{y \le x} |C(x)| = r^{2|y|}.$$

Then applying Möbius inversion (see [1, IV, §2]) gives us that

$$|C(y)| = \sum_{y \le x} \mu(y,x) r^{2|x|}.$$

Finally, we obtain that the number of symmetric pairs of r-colorings of G is

$$|G| \sum_{y \in P} \sum_{y \le x} \frac{\mu(y,x)}{|Z(y)|} r^{2|x|} = |G| \sum_{x \in P} \sum_{y \le x} \frac{\mu(y,x)}{|Z(y)|} r^{2|x|}.$$

□

Theorem 1.1 is immediate from Theorem 3.1 and Theorem 2.1.

Remark 3.1. The proof of Theorem 3.1 shows also that the number $s_r^2(G)$ of equivalence classes of symmetric pairs of r-colorings of G is

$$\sum_{x\in P}\sum_{y\leq x}\frac{\mu(y,x)St(y)}{|Z(y)|}r^{2|x|}.$$

Consequently, if $g(r)$ is the polynomial representing $s_r(G)$, then $s_r^2(G) = g(r^2)$. However, in contrast to Corollary 3.1, $s_r(G\times\mathbb{Z}_2)$ is not equal to $s_r^2(G)$. For example, $s_r(\mathbb{Z}_3\times\mathbb{Z}_2) = \frac{1}{2}r^4 + \frac{1}{2}r^2$, $s_r(\mathbb{Z}_3) = r^2$, and $s_r^2(\mathbb{Z}_3) = r^4$.

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