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Upper Broadcast Domination Number of Caterpillars with no Trunks

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Abstract

A broadcast on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, \ldots, \mathrm{diam}(G)\}$ such that $f(v) \leq e_G(v)$ for every vertex $v \in V$, where $\mathrm{diam}(G)$ denotes the diameter of G and $e_G(v)$ the eccentricity of v in G. Such a broadcast f is minimal if there does not exist any broadcast $g \neq f$ on G such that $g(v) \leq f(v)$ for all $v \in V$. The upper broadcast domination number of G is the maximum value of $\sum_{v \in V} f(v)$ among all minimal broadcasts f on G for which each vertex of G is at distance at most $f(v)$ from some vertex v with $f(v) \geq 1$. In this paper, we study the minimal dominating broadcasts of caterpillars and give the exact value of the upper broadcast domination number of caterpillars with no trunks.

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1. Introduction

Let $G = (V, E)$ be a graph of *order* $n = |V|$ and *size* $m = |E|$. The *open neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u : uv \in E\}$ of vertices adjacent to v. Each vertex $u \in N_G(v)$ is a *neighbor* of v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open*

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neighborhood of a set $S \subseteq V$ of vertices is $N_G(S) = \cup_{v \in S} N_G(v)$, while the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex v in G, denoted $\deg_G(v)$, is the size of the open neighborhood of v.

A (u, v)-*geodesic* in a graph G is a shortest path joining u and v. We denote by $d_G(u, v)$ the *distance* between the vertices u and v in G, that is, the length of a (u, v)-geodesic in G. A vertex or an edge of G *lies between* two vertices u and v if that vertex or edge is on some (u, v)-geodesic. The *eccentricity* $e_G(v)$ of a vertex v in G is the maximum distance from v to any other vertex of G. The *radius* $\mathrm{rad}(G)$ and the *diameter* $\mathrm{diam}(G)$ of a graph G are the minimum and the maximum eccentricity among the vertices of G, respectively. A *diametrical path* is a (u, v)-geodesic of length $\mathrm{diam}(G)$, and a *peripheral vertex*, is a vertex v such that $e_G(v) = \mathrm{diam}(G)$.

A function $f : V \longrightarrow \{0, \dots, \mathrm{diam}(G)\}$ is a *broadcast* of G if $f(v) \le e_G(v)$ for every vertex $v \in V$. The value $f(v)$ is called the f-value of v. An f*-broadcast vertex* (or an f*-dominating vertex*) is a vertex v for which $f(v) > 0$. The set of all f-broadcast vertices is denoted $V_f^+(G)$. If $v \in V_f^+(G)$ is an f-broadcast vertex, $u \in V$ and $d_G(u, v) \le f(v)$, then the vertex u *hears* a broadcast from v and v *broadcasts* to (or f*-dominates*) u. Note that, in particular, each vertex $v \in V_f^+$ hears a broadcast from itself and f-dominates itself.
The f*-broadcast neighborhood* of a vertex $v \in V_f^+$ is the set of vertices that hear v, that is

$$N_f(v) = \{u \in V : d_G(u, v) \le f(v)\}$$

and the f*-broadcast neighborhood* of f is the set

$$N_f(V_f^+) = \cup_{v \in V^+} N_f(v).$$

The f*-broadcast boundary* of a vertex $v \in V_f^+$ is the set

$$B_f(v) = \{u \in V : d_G(u, v) = f(v)\}.$$

The set of f-broadcast vertices that a vertex $u \in V$ can hear is the set

$$H_f(u) = \{v \in V_f^+ : d_G(u, v) \le f(v)\}.$$

For a vertex $v \in V_f^+$, the *private* f*-neighborhood* of v is the set of vertices that hear only v, that is

$$PN_f(v) = \{u \in V : H_f(u) = \{v\}\},$$

and every vertex $u \in PN_f(v)$ is a *private* f*-neighbor* of v. Moreover, the *private* f*-border* of v is either the set of private f-neighbors of v that are at distance $f(v)$ from v, or the singleton $\{v\}$ if $f(v) = 1$ and $PN_f(v) = \{v\}$, that is

$$PB_f(v) = \begin{cases} \{v\}, & \text{if } f(v) = 1 \text{ and } PN_f(v) = \{v\}, \\ \Big\{u \in PN_f(v) : d_G(u, v) = f(v)\Big\}, & \text{otherwise.} \end{cases}$$

Every vertex in $PB_f(v)$ is a *bordering private* f*-neighbor* of v. In particular, if $f(v) = 1$ and $PN_f(v) = \{v\}$, then v is its own bordering private f-neighbor.

The *cost* of a broadcast f on a graph G is

$$\sigma(f) = \sum_{v \in V_f^+} f(v).$$

A broadcast f on G is a *dominating broadcast* if every vertex in G is f-dominated by some vertex in V_f^+, and f is a *minimal dominating broadcast* if there does not exist a dominating broadcast $g \neq f$ on G such that $g(u) \leq f(u)$ for all $u \in V$.

The *broadcast domination number* of G is

$$\gamma_b(G) = \min\{\sigma(f) : \ f \text{ is a dominating broadcast on } G\},$$

and the *upper broadcast domination number* of G is

$$\Gamma_b(G) = \max\{\sigma(f) : \ f \text{ is a minimal dominating broadcast on } G\}.$$

A minimal dominating broadcast f on a graph G such that $\sigma(f) = \Gamma_b(G)$ (resp. $\sigma(f) = \gamma_b(G)$) is a Γ_b*-broadcast* (resp. γ_b*-broadcast*). If f is a minimal dominating broadcast on G such that $f(v) = 1$ for each $v \in V^+$, then V^+ is a *minimal dominating set* in G, and the minimum (resp. maximum) cost of such a broadcast is the *domination number* $\gamma(G)$ (resp. *upper domination number* $\Gamma(G)$) of G.

The function $f_u : V \longrightarrow \{0, \dots, \mathrm{diam}(G)\}$, defined by $f_u(u) = e(u)$ and $f_u(v) = 0$ for every $v \neq u$, is a minimal dominating broadcast with cost $e(u)$. Such a broadcast f_u is a *radius broadcast* if $e(u) = \mathrm{rad}(G)$ and f_u is a *diameter broadcast* if $e(u) = \mathrm{diam}(G)$. We then immediately have the chain of inequalities

Observation 1 (Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [6])**.** *For any graph G,*

$$\gamma_b(G) \leq \min\{\gamma(G), \mathrm{rad}(G)\} \leq \max\{\Gamma(G), \mathrm{diam}(G)\} \leq \Gamma_b(G). \tag{1}$$

A graph G is *radial* if $\gamma_b(G) = \mathrm{rad}(G)$ and is *diametrical* if $\Gamma_b(G) = \mathrm{diam}(G)$.

Broadcast domination has been discussed first in [7, 8]. Many of these results appeared later in [6] and since then several works followed (see the references of [5] for details). Regarding the upper broadcast domination, the exact value of the parameter Γ_b is given for grids graphs [4], paths and cycles [5] and some very specific classes of trees [12]. In [9], the determination of sufficient conditions for a tree to be non-diametrical as well as the characterization of diametrical caterpillars are given. Other studies of upper broadcast domination such as the relationships between Γ_b and other parameters of broadcast domination can be found in [1, 6, 13]. For a survey of broadcast in graphs, see the chapter by Henning, MacGillivray and Yang [10].

In this paper, we are interested in the upper broadcast domination number of caterpillars. Determining this invariant appears to be a difficult problem in general, and that is why we restrict to caterpillars with no trunks.

Recall that a *caterpillar CT of length* $n \geq 0$ is a tree such that removing all leaves gives a path of length n, called the *spine*. A non-leaf vertex is called a *spine vertex* and, more precisely, a *stem* if it is adjacent to a leaf and a *trunk* otherwise. A leaf adjacent to a stem v is a *pendent neighbor* of v.

2. Preliminaries

We now review some results on the upper broadcast domination. The characterization of minimal dominating broadcasts was first given by Erwin in [8], and then restated in terms of private borders[1] by Mynhardt and Roux in [12].

Proposition 2.1 (Erwin [8], restated in [12]). *A dominating broadcast f is a minimal dominating broadcast if and only if $PB_f(v) \neq \emptyset$ for each $v \in V_f^+$.*

Dunbar *et al.* proved in [6] the following bound on the upper broadcast domination number of graphs.

Theorem 2.1 (Dunbar *et al.* [6]). *For every graph G with size m, $\Gamma_b(G) \leq m$. Moreover, $\Gamma_b(G) = m$ if and only if G is a nontrivial star or path.*

This upper bound was later improved in [4].

Theorem 2.2 (Bouchemakh and Fergani [4]). *If G is a graph of order n with minimum degree $\delta(G)$, then $\Gamma_b(G) \leq n - \delta(G)$, and this bound is sharp.*

In all what follows, we will denote by $P_n = v_0v_1 \ldots v_n$, $n \geq 1$, the path of length n. Moreover, we assume that subscripts of vertices of $v_0v_1 \ldots v_n$ of P_n are "ordered" from left to right.
Let T be a tree with diameter d and a diametrical path $P_d = v_0v_1 \ldots v_d$. For each $i \in \{0, \ldots, d\}$, let T_i be the subtree of T induced by all vertices that are connected to v_i by paths that are internally disjoint from P.

In the following lemmas, Gemmrich and Mynhardt proved that there exist some sufficient conditions for a tree to be non-diametrical.

Lemma 2.1 (Gemmrich and Mynhardt [9]). *Let T be a tree with diameter $d \geq 3$ and diametrical path $P_d = v_0v_1 \ldots v_d$. If there exists an $i \in \{1, \ldots, d-2\}$ such that each of v_i and v_{i+1} is adjacent to a leaf other than v_0 (if $i = 1$) or v_d (if $i + 1 = d - 1$), then $\Gamma_b(T) > \mathrm{diam}(T)$.*

Lemma 2.2 (Gemmrich and Mynhardt [9]). *If there exists an $i \in \{2, \ldots, d-2\}$ such that T_i has an independent set of cardinality 3 that dominates but does not contain v_i, or if $\max\{deg_T(v_1), deg_T(v_{d-1})\} = 4$, then $\Gamma_b(T) > \mathrm{diam}(T)$.*

Lemma 2.3 (Gemmrich and Mynhardt [9]). *If there exists an $i \in \{2, \ldots, d-2\}$ such that T_i has an independent set of cardinality 2 that does not dominate v_i, then $\Gamma_b(T) > \mathrm{diam}(T)$.*

Lemma 2.4 (Gemmrich and Mynhardt [9]). *If $\mathrm{diam}(T_i) = 4$ for some i, or $\mathrm{diam}(T_i) = 3$ and v_i is a peripheral vertex of T_i, then $\Gamma_b(T) > \mathrm{diam}(T)$.*

[1] In their paper, Mynhardt and Roux used a slightly different definition of the set $PB_f(v)$ when $f(v) = 1$ and $N_f(v) \neq \{v\}$, by including the vertex v in $PB_f(v)$. Moreover, they called the set $PB_f(v)$ the *private f-boundary* of v. We here use the term *private f-border* to avoid confusion between these two definitions. However, it is easy to check that the private f-boundary of v is empty if and only if the private f-border of v is empty, so that Proposition 2.1 is still valid in our setting.

For the particular case of caterpillars, Gemmrich and Mynhardt gave another sufficient condition for a caterpillar to be non-diametrical. Before stating the result, we recall that a *strong stem* is a stem that is adjacent to at least two leaves.

Lemma 2.5 (Gemmrich and Mynhardt [9]). *Let T be a caterpillar with diametrical path $P_d = v_0v_1 \ldots, v_d$. If two vertices v_i and v_{i+2k} are strong stems, for some $i \geq 1$ and some integer k such that $i + 2k \leq d - 1$, and v_{i+2r} is a stem for each $r \in \{1, \ldots, k-1\}$, then $\Gamma_b(T) > d$.*

If T is a diametrical caterpillar, then T does not satisfy the hypothesis of any of Lemmas 2.1 - 2.5. The converse remains true and the negation of these hypotheses, applied to caterpillars, gives the characterization of diametrical caterpillars stated in the following theorem

Theorem 2.3 (Gemmrich and Mynhardt [9]). *A caterpillar T with diametrical path $P_d = v_0v_1 \ldots, v_d$ is diametrical if and only if*

1. *each v_i, $i \in \{1, \ldots, d-1\}$, is adjacent to at most two leaves,*
2. *for any $i \in \{1, \ldots, d-2\}$, $\min\{deg_T(v_i), deg_T(v_{i+1})\} = 2$,*
3. *whenever v_i and v_j, $i < j$, are strong stems, there exists a k, $i < k < j$, such that $deg_T(v_k) = deg_T(v_{k+1}) = 2$.*

Let f be any minimal dominating broadcast on a graph G. In view of Proposition 2.1, each $v \in V^+$ has a bordering private f-neighbor (denoted v^p) such that either v^p is at distance $f(v)$ from v, or $v^p = v$ if $f(v) = 1$ and $PN_f(v) = \{v\}$. Dunbar *et al.* defined in [6] a function ϵ on V^+ as follows: $\epsilon(v) = \{e_v\}$, where e_v is any edge incident with v, if $PB_f(v) = \{v\}$, while $\epsilon(v)$ is the set of all edges that lie between v and v^p if v^p is at distance $f(v)$ from v.

In the proof of Theorem 2.1, Dunbar *et al.* showed that the sets $\epsilon(v)$ are pairwise disjoint.

Lemma 2.6 (Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [6], proof of Theorem 5). *For any two f-broadcast vertices u and v, we have $\epsilon(u) \cap \epsilon(v) = \emptyset$.*

Let f be a Γ_b-broadcast on a caterpillar G with size m. For every f-broadcast vertex v, we denote by P_v^f, according to presented case, a (v, v^p)-geodesic path if v^p is at distance $f(v)$ from v or a path with one edge e_v if $PB_f(v) = \{v\}$. We set $\mathcal{P}^f = \{P_v^f : v \in V_f^+(G)\}$. For brevity, we also denote by E_f and $\overline{E_f}$ the sets $\cup_{v \in V_f^+} E(P_v^f)$ and $E(G) \backslash E_f$, respectively. From Theorem 2.1 and Lemma 2.6, we get

$$\Gamma_b(G) = \sum_{v \in V_f^+} f(v) = |E_f| \leq m.$$

Since $\Gamma_b(G) = m - |\overline{E_f}|$, it suffices to find a lower bound on $|\overline{E_f}|$ to get an upper bound on $\Gamma_b(G)$. Thereafter, we will frequently use this idea to reach a conclusion.

Let CT be a caterpillar. We will always draw caterpillars with the spine on a horizontal line, so that we can say that a spine vertex x_i is to the left (resp. to the right) of a spine vertex x_j of CT, and that a pendent neighbor of x_i is to the left (resp. to the right) of a pendent neighbor of x_j

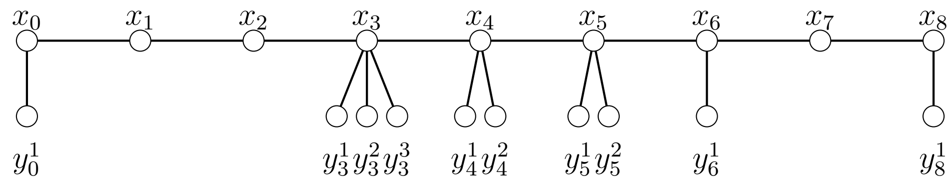


Figure 1: $CT(1, 0, 0, 3, 2, 2, 1, 0, 1)$.

whenever the spine vertex x_i is to the left (resp. to the right) of the spine vertex x_j, that is $i < j$ (resp. $i > j$).

Note that a caterpillar of length 0 is a star $K_{1,k}$ for some $k \geq 1$, and the upper broadcast domination number of a star is determined by Theorem 2.1. Therefore, in the rest of the paper, we will only consider caterpillars with positive length.

Let $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$. Following the terminology of [2] and [14], we denote by $CT(\ell_0, \ldots, \ell_n)$, $n \geq 1$, with $(\ell_0, \ldots, \ell_n) \in \mathbb{N}^\star \times \mathbb{N}^{n-1} \times \mathbb{N}^\star$, the caterpillar of length $n \geq 1$ with spine path $x_0 \ldots x_n$ such that each spine vertex x_i has ℓ_i pendent neighbors. For every i such that $\ell_i > 0$, $i = 0, \ldots, n$, we denote by $L(x_i) = \{y_i^1, \ldots, y_i^{\ell_i}\}$ the set of pendent neighbors of x_i. The caterpillar $CT(1, 0, 0, 3, 2, 2, 1, 0, 1)$ is depicted in Figure 1.

We denote by $CT[i, j]$, the sub-caterpillar of CT induced by vertices $x_i, \ldots, x_j$ and their pendent neighbors if $0 \leq i \leq j \leq n$, and $CT[i, j] = \emptyset$ if $i > j$.

We say that a pattern of length $p + 1$, $\Pi = \pi_0 \ldots \pi_p$, $p \geq 0$, $\pi_i \in \mathbb{N}$ for every i, $0 \leq i \leq p$, *occurs* in a caterpillar $CT = CT(\ell_0, \ldots, \ell_n)$ if there exists an index i_0, $0 \leq i_0 \leq n - p$, such that $CT[i_0, i_0 + p] = CT(\pi_0, \ldots, \pi_p)$, that is, $\ell_{i_0+j} = \pi_j$ for every j, $0 \leq j \leq p$. We will also say that the caterpillar CT *contains* the pattern Π and that the sub-caterpillar $CT(\ell_{i_0}, \ldots, \ell_{i_0+p})$ of CT is an *occurrence* of the pattern Π.
We can extend the notation for patterns by setting π_i^+ to mean a spine vertex having at least π_i pendent neighbors.

We first prove a property of optimal dominating broadcasts of caterpillars.

Lemma 2.7. *For any caterpillar CT, there exists a Γ_b-broadcast such that each broadcast vertex is either a leaf or a trunk.*

Proof. Let f be a Γ_b-broadcast of CT. Assume that there exists an f-broadcast vertex $x_i \in V_f^+, i \in \{1, \ldots, n\}$ such that x_i is a stem. If $f(x_i) > 1$, then the minimality of the dominating broadcast f implies that x_i has a bordering private f-neighbor s such that $d(x_i, s) = f(x_i)$ and $f(y_i^j) = 0$ for every j, $j = 1, \ldots, \ell_i$. Consider the mapping g obtained from f by replacing the f-values of x_i and y_i^1 by $g(x_i) = 0$ and $g(y_i^1) = f(x_i) + 1$. The mapping g is a minimal dominating broadcast with cost $\sigma(g) = \sigma(f) + 1 > \Gamma_b(CT)$, contradicting the optimality of f. Hence, $f(x_i) = 1$. Moreover, $PB_f(x_i)$ contains no trunk, for otherwise the mapping h obtained

from f by replacing the f-values of x_i and y_i^1 by $h(x_i) = 0$ and $h(y_i^1) = 2$ would be a minimal dominating broadcast with cost $\sigma(g) = \sigma(f) + 1 > \Gamma_b(CT) + 1$, contradicting the optimality of f. Now, the mapping k obtained from f by replacing the f-values of x_i and $y_i^1, \dots, y_i^{\ell_i}$ by $k(x_i) = 0$ and $k(y_i^j) = 1$ for every j, $j = 1, \dots, \ell_i$, is a minimal dominating broadcast with cost $\sigma(k) = \sigma(f) + \ell_i - 1$. The optimality of f then implies $\ell_i = 1$, so that we have $\sigma(k) = \sigma(f)$.
We can repeat the previous transformation on f until we get a Γ_b-broadcast where each broadcast vertex is not a stem vertex. This completes the proof. □

3. Caterpillars with no trunks

Let $CT = CT(\ell_0, \dots, \ell_n)$ be a caterpillar of length $n \geq 1$. For any minimal dominating broadcast f on CT, we assume that $f(y_i^1) \geq \dots \geq f(y_i^{\ell_i})$ for every $i = 0, \dots, n$.

We say that CT is *with no trunks* if $\ell_i \geq 1$ for every i, $i = 0, \dots, n$.

In what follows, the *unitary dominating broadcast* is the dominating broadcast μ defined by $\mu(u) = 1$ if u is a leaf and $\mu(u) = 0$ otherwise. Since each stem is μ-dominated by one leaf and $PB_\mu(v) \neq \emptyset$ for each $v \in V_\mu^+$, then μ is a minimal dominating broadcast of cost $\sigma(u) = \sum_{i=0}^{n} \ell_i$.

In order to simplify the reading of this paper, the proofs of the lemmas which are quite technical are given in the appendix.

Lemma 3.1. *If CT is a caterpillar with no trunks, of length $n \geq 1$ and f is a Γ_b-broadcast on CT, then, every f-broadcast vertex v is a leaf and the private f-neighbor of v is also a leaf if $f(v) \geq 2$.*

Proof. By the proof of Lemma 2.7, we already know that every f-broadcast vertex is a leaf. Assume to the contrary that there exists some stem x_i which is a private f-neighbor of some f-broadcast vertex v. Since $f(v) \geq 2$, then we necessarily have, $v \neq y_i^j$, and more than that, $y_i^j \notin V_f^+$ for every $j = 1, \dots, \ell_i$, so that y_i^j cannot be f-dominated, a contradiction. This completes the proof. □

We first determine the upper broadcast domination number of all caterpillars with no trunks of length at most 2.

Lemma 3.2. *If CT is a caterpillar with no trunks, of length $n \leq 2$ and size m, then*

$$\Gamma_b(CT) = \begin{cases} m, & \text{if } n = 1 \text{ and } m = 3, \\ m - 1, & \text{if } n = 1 \text{ and } m \geq 4, \text{ or } n = 2 \text{ and } \ell_0 = \ell_1 = 1, \\ m - 2, & \text{otherwise.} \end{cases}$$

Lemma 3.3. *If CT be a caterpillar with no trunks, of length $n \geq 1$, then $\Gamma_b(CT) \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor$.*

Corollary 3.1. *If $CT = CT(\ell_0, \dots, \ell_n)$ is a caterpillar with no trunks, of length $n \geq 1$, then CT is diametrical if and only if one of the following conditions is satisfied :*

1. *$n = 1$, $\ell_0 + \ell_1 \in \{2, 3\}$.*
2. *$n = 2$, $\ell_0 = \ell_2 = 1$ and $\ell_1 \in \{1, 2\}$.*

Proof. Let $CT = CT(\ell_0, \ldots, \ell_n)$ be a caterpillar with no trunks of length $n \geq 1$, and size m. We know by Lemma 3.3 that $\Gamma_b(CT) \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor$. Since $\operatorname{diam}(CT) = n + 2$, we deduce that $\Gamma_b(CT) \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor > \operatorname{diam}(CT)$, whenever $n \geq 3$.
If $n = 1$, then $\operatorname{diam}(CT) = 3$. From Lemma 3.2, we have $\Gamma_b(CT) = m$ if $m = 3$, and $\Gamma_b(CT) = m - 1$ if $m \geq 4$. It follows, $\Gamma_b(CT) = \operatorname{diam}(CT)$ if and only if, $(\ell_0, \ell_1) \in \{(1,1), (1,2), (2,1)\}$. If $n = 2$, then $\operatorname{diam}(CT) = 4$, and from the same lemma, we also have $\Gamma_b(CT) = m - 1$, if $\ell_0 = \ell_1 = 1$ (or $\ell_1 = \ell_2 = 1$, by symmetry), and $\Gamma_b(CT) = m - 2$ otherwise. Hence, we get $\Gamma_b(CT) = \operatorname{diam}(CT)$ if and only if $(\ell_0, \ell_1, \ell_2) \in \{(1,1,1), (1,2,1)\}$. This completes the proof. □

Thanks to Corollary 3.1, we can only consider in the rest of the paper caterpillars CT with length $n \geq 3$. Hence, each such caterpillar CT is not diametrical and each Γ_b-broadcast f on CT satisfies $|V_f^+| \geq 2$.

Proposition 3.1. *If CT is a caterpillar of length $n \geq 3$, with $\ell_i \geq 2$ for every $i = 0, \ldots, n$, then* $\Gamma_b(CT) = \sum_{i=0}^{n} \ell_i$

Proof. Since the cost of the (minimal) unitary dominating broadcast is $\sum_{i=0}^{n} \ell_i$, we get $\Gamma_b(CT) \geq \sum_{i=0}^{n} \ell_i$. Conversely, let f be a Γ_b-broadcast on CT, such that each f-broadcast vertex is a leaf (such a broadcast exists by Lemma 2.7). We first prove that $|\overline{E_f}| \geq n$. For that, consider any edge $x_i x_{i+1}$, $i \in \{0, \ldots, n-1\}$, of the spine $P_n = x_0 x_1 \ldots x_n$. If $x_i x_{i+1}$ is an edge of some $P_v^f \in \mathcal{P}^f$, then by Lemma 3.1, v^p is also a leaf non-adjacent to x_i. Thus, the set $\overline{E_f}$ contains $\ell_i \geq 2$ or $\ell_i - 1 \geq 1$ edges incidents to x_i depending on whether $x_{i-1}x_i$ is an edge of P_v^f, or not. If none of the paths of $\mathcal{P}^f$ has $x_i x_{i+1}$ as an edge, then $x_i x_{i+1} \in \overline{E_f}$. It follows, $|\overline{E_f}| \geq n$, and thus $\Gamma_b(CT) = |E(CT)| - |\overline{E_f}| \leq |E(CT)| - n = \sum_{i=0}^{n} \ell_i$. This completes the proof. □

Lemma 3.4. *If CT is a caterpillar of length $n \geq 3$, with $\ell_i = 1$ for every $i = 0, \ldots, n$, and f is a Γ_b-broadcast on CT, then $f(u) \neq 2$ for every f-broadcast vertex u.*

Proof. Let f be a Γ_b-broadcast on CT. Assume, to the contrary, that $f(u) = 2$ for some $u \in V_f^+$. By Lemma 3.1, u and its private neighbor u^p are leaves. Since $f(u) = 2$, then u and u^p are adjacent to the same stem, a contradiction with the type of caterpillar, where $\ell_i = 1$ for every $i = 0, \ldots, n$. This completes the proof. □

Theorem 3.1. *If CT is a caterpillar of length $n \geq 3$, with $\ell_i = 1$ for every $i = 0, \ldots, n$, then* $\Gamma_b(CT) = \left\lfloor \frac{3(n+1)}{2} \right\rfloor$.

Proof. By Lemma 3.3, we already have $\Gamma_b(CT) \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor$. For the converse, let f be a Γ_b-broadcast on CT, such that each f-broadcast vertex is a leaf with an f-value different from 2. Thanks to Lemma 2.7 and Lemma 3.4, such a broadcast exists. Let $V_f^+ = \{v_1, \ldots, v_s\}$ be the set of f-broadcast vertices, ordered so that, for every $i, j = 0, \ldots, n-1$, the stem adjacent to v_i, in the spine $P_n = x_0 x_1 \ldots x_n$, lies left to the stem adjacent to v_j whenever $i < j$, and let $v_k \in V_f^+$, $k = 1, \ldots, s$. Since v_k is a leaf, we have $v_k = y_i^1$ for some $i \in \{0, \ldots, n\}$. In what follows, we denote by e_j the pendent edge $y_j^1 x_j$, $j \in \{0, \ldots, n\}$.
To prove the statement, we consider two cases.

1. $f(v_k) \geq 3$.
   By Lemma 3.1, we know that the private neighbor v_k^p is a leaf. Hence, the (v_k, v_k^p)-geodesic P_{v_k} is the path $v_k x_i x_{i+1} \dots x_{i+f(v_k)-2} v_k^p$ or $v_k x_i x_{i-1} \dots x_{i-f(u_k)+2} v_k^p$.
   Therefore, $\{e_{i+1}, \dots, e_{i+f(v_k)-3}\} \subset \overline{E_f}$ or $\{e_{i-1}, \dots, e_{i-f(v_k)+3}\} \subset \overline{E_f}$. In the case where $0 \leq k < s$, $\overline{E_f}$ contains another edge, which is either $x_{i+f(v_k)-2} x_{i+f(v_k)-1}$ or $x_i x_{i+1}$, depending on whether v_k is to the left or to the right of v_k^p. It follows, $|\overline{E_f}| \geq f(v_k) - 3$ if $k = s$, and $|\overline{E_f}| \geq f(v_k) - 2$ otherwise.

2. $f(v_k) = 1$.
   Since, $P_{v_k} = y_i^1 x_i$ (recall that $v_k = y_i^1$), we infer that $x_i x_{i+1} \in \overline{E_f}$, and thus $|\overline{E_f}| \geq 1$, if $0 \leq k < s$.

Note that if an edge $x_j x_{j+1}$, $j = 0, \dots, n-1$, of the spine P_n, appears in $\overline{E_f}$, then x_j is adjacent to the last pendent vertex, namely y_j^1, of some path of $\mathcal{P}^f$, and since the paths of $\mathcal{P}^f$ are pairwise disjoint by Lemma 2.6, we can say that

$$|\overline{E_f}| = \sum_{\substack{k=1 \\ f(v_k) \geq 3}}^{s-1} (f(v_k) - 2) + \sum_{\substack{k=1 \\ f(v_k)=1}}^{s-1} 1 + \begin{cases} f(v_s) - 3, & \text{if } f(v_s) \geq 3, \\ 0, & \text{if } f(v_s) = 1. \end{cases}$$

Hence,

$$|\overline{E_f}| = \left(\sum_{\substack{k=1 \\ f(v_k) \geq 3}}^{s} (f(v_k) - 2) \right) + \sum_{\substack{k=1 \\ f(v_k)=1}}^{s} 1 - 1.$$

It follows,

$$|\overline{E_f}| \geq \Gamma_b(CT) - 2|\{v_k : f(v_k) \geq 3\}| - 1.$$

Since $\Gamma_b(CT) = |E(CT)| - |\overline{E_f}|$ and the size of the caterpillar CT is $2n + 1$, we infer

$$2\Gamma_b(CT) \leq |E(CT)| + 2|\{v_k : f(v_k) \geq 3\}| + 1 = (2n + 2) + 2|\{v_k : f(v_k) \geq 3\}|,$$

which leads to

$$\Gamma_b(CT) \leq n + 1 + |\{v_k : f(v_k) \geq 3\}|.$$

It is not difficult to see that, in each sub-caterpillar $CT[i, i + 3]$, $i = 0, \dots, n - 3$, the number of f-broadcast vertices v with an f-value $f(v) \geq 3$ cannot exceed 2. Then $|\{v_k : f(v_k) \geq 3\}| \leq \frac{n+1}{2}$ and $\Gamma_b(CT) \leq \frac{3(n+1)}{2}$. This completes the proof. □

Lemma 3.5. *If CT is a caterpillar CT with no trunks, of length $n \geq 3$, then CT admits a Γ_b-broadcast f with $f(u) \neq 2$ for every $u \in V_f^+$.*

Proof. Let g be a Γ_b-broadcast on the caterpillar CT and let $u \in V_g^+$, with $g(u) = 2$. By Lemma 3.1, u and its private neighbor u^p are leaves. Since $g(u) = 2$, then $u = y_i^1$ for some $i \in \{1, \dots, n\}$, and u^p are adjacent to the same stem x_i. Consider the mapping f obtained from g by replacing the g-values of y_i^j, $j = 1, \dots, \ell_i$, by $f(y_i^j) = 1$, $j = 1, \dots, \ell_i$. The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) + \ell_i - 2$. The optimality of g implies

$\ell_i = 2$, so that we have $\sigma(f) = \sigma(g)$. We then repeat this transformation on each g-broadcast vertex with a value equal to 2 until we obtain a mapping with the required condition. This completes the proof. □

Lemma 3.6. *If CT is a caterpillar with no trunks, of length $n \geq 3$, then CT admits a Γ_b-broadcast f with $f(u) \leq 3$ for every $u \in V_f^+$.*

Lemma 3.7. *If CT is a caterpillar with no trunks, of length $n \geq 3$, then CT admits a Γ_b-broadcast f, such that*

1. *If $\ell_0 + \ell_1 \geq 3$, then $f(y_0^j) \neq 3$ for every j, $j = 1, \ldots, \ell_0$ (or, if $\ell_{n-1} + \ell_n \geq 3$, then $f(y_n^j) \neq 3$ for every j, $j = 1, \ldots, \ell_n$).*
2. *If y_i^1 is a f-broadcast vertex for some $i = 1, \ldots, n$, with $f(y_i^1) = 3$, then $PB_f(y_i^1)$ is equal to either $L(x_{i-1})$ or $L(x_{i+1})$ (in that case, y_i^1 is said to have only one private side).*
3. *If there exists a pendent vertex f-dominated by two f-broadcast vertices u et u', then $d(u, u') = 3$.*

Let CT_5^4 be a caterpillar with no trunks of length 3, and having five pendent edges. Then CT_5^4 must be one of the caterpillars $CT(2,1,1,1)$, $CT(1,2,1,1)$, $CT(1,1,2,1)$, or $CT(1,1,1,2)$. We say that a caterpillar CT is *CT_5^4-free* if CT contains none of the patterns 2111, 1211, 1121 or 1112. Further, in the following, we say that a mapping g on a caterpillar CT is a *good Γ_b-broadcast* if g is a Γ_b-broadcast satisfying the conditions of Lemmas 3.1, 3.5, 3.6 and 3.7.

Lemma 3.8. *If CT is a caterpillar with no trunks, of length $n \geq 3$, then CT admits a Γ_b-broadcast f such that $f(y_i^j) = 1$ for every $j = 1, \ldots, \ell_i$, whenever $\ell_i \geq 3$, or $\ell_i = 2$ if CT is a CT_5^4-free caterpillar.*

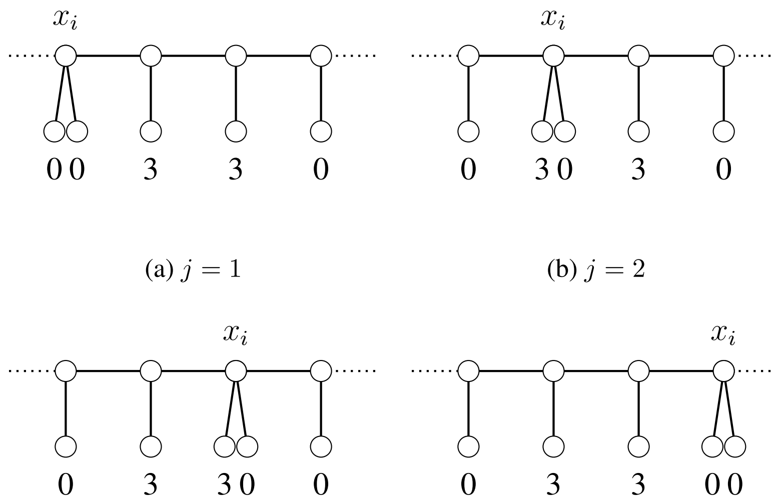
Let CT be a caterpillar with no trunks, of order $n \geq 3$, and let f be a Γ_b-broadcast on CT. For any stem x_i, $i = 0, \ldots, n$, with $\ell_i = 2$, we denote by $F_i^j = CT[i-j+1, i-j+4]$, $j = 1, \ldots, 4$, a caterpillar *of type* CT_5^4. On F_i^j, we consider a mapping θ_i^j, defined by $\theta_i^j(y_{i-j+2}^1) = \theta_i^j(y_{i-j+3}^1) = 3$ and $\theta_i^j(v) = 0$ otherwise (see Figure 2).

Lemma 3.9. *If CT is a caterpillar of length $n \geq 3$ and x_i is a stem with $\ell_i = 2$ for some $i \in \{0, \ldots, n\}$, then CT admits a Γ_b-broadcast f such that*

1. *If x_i does not appear in any F_i^j, $j = 1, \ldots, 4$, then $f(y_i^1) = f(y_i^2) = 1$.*
2. *If x_i is a stem of a sub-caterpillar CT' of CT, of type CT_5^4, then either $f(y_i^1) = f(y_i^2) = 1$, or $f(y_i^1) = \theta_i^j(y_i^1)$ and $f(y_i^2) = \theta_i^j(y_i^2)$ for some $j \in \{1, \ldots, 4\}$, in which case $CT' = F_i^j$ and the restriction of f on CT' is θ_i^j.*

Let CT_1 and CT_2 be two caterpillars of lengths n_1 and n_2 respectively. The *concatenation* of CT_1 and CT_2 is the caterpillar $CT_1 + CT_2$, of length $n_1 + n_2 + 1$, where

$$(CT_1 + CT_2)[0, n_1] = CT_1,$$
$$(CT_1 + CT_2)[n_1 + 1, n_1 + n_2 + 1] = CT_2,$$
$$CT_1 + \emptyset = CT_1, \text{ and, } \emptyset + CT_2 = CT_2.$$



(a) $j = 1$ (b) $j = 2$

(c) $j = 3$ (d) $j = 4$

Figure 2: The function θ_i^j, for some value of j.

Using the concatenation operation, we can define some transformations on any caterpillar CT of length n. For an integer i, $i = 0, \ldots, n - n_1$, let

- $CT[CT_1/\emptyset, i]$ be the caterpillar obtained from CT by removing $CT_1 = CT[i, i + n_1]$,

$$CT[CT_1/\emptyset, i] = \begin{cases} CT[n_1 + 1, n], & \text{if } i = 0, \\ CT[0, n - n_1 - 1], & \text{if } i = n - n_1, \\ CT[0, i - 1] + CT[i + n_1 + 1, n], & \text{if } i = 1, \ldots, n - n_1 - 1, \end{cases}$$

- $CT[\emptyset/CT_2, i]$ be the caterpillar obtained from CT by inserting CT_2 between the stems x_{i-1} and x_i of CT if $i \neq 0$, and the concatenation of CT_2 with CT otherwise,

$$CT[\emptyset/CT_2, i] = \begin{cases} CT_2 + CT, & \text{if } i = 0, \\ CT[0, i - 1] + CT_2 + CT[i, n], & \text{if } i = 1, \ldots, n - n_1, \end{cases}$$

- $CT[CT_1/CT_2, i]$ be the caterpillar obtained from CT by removing $CT_1 = CT[i, i + n_1]$ and by inserting CT_2 between the stems x_{i-1} and x_i of CT,

$$CT[CT_1/CT_2, i] = \begin{cases} CT_2 + CT[n_1 + 1, n], & \text{if } i = 0, \\ CT[0, n - n_1 - 1] + CT_2, & \text{if } i = n - n_1, \\ CT[0, i - 1] + CT_2 + CT[i + n_1 + 1, n], & \text{if } i = 1, \ldots, n - n_1 - 1. \end{cases}$$

Lemma 3.10. *Let CT be a caterpillar with no trunks, of length $n \geq 4$, and containing the patterns 1 and 2^+. If $M = CT(1, 1, 1, 1)$ is a sub-caterpillar of CT, then*

$$\Gamma_b(CT) = \Gamma_b(CT[M/\emptyset, i]) + 6.$$

For any caterpillar CT with no trunks and containing the patterns 1 and 2^+, if the pattern $\Pi = 1 \dots 1$, of length $p+1$, $p \geq 3$, occurs in CT, we can iteratively remove all sub-caterpillars isomorphic to M. The resulting caterpillar, denoted by CT^r, is called the *reduced caterpillar* of CT. We denote by $z_0 \dots z_k$ the spines vertices of CT^r and by $L(z_i) = \{t_i^1, \dots, t_i^{m_i}\}$ the set of pendent neighbors of z_i.

In view of Lemma 3.10, the following result is immediate.

Proposition 3.2. *If CT is a caterpillar with no trunks, of length $n \geq 4$, containing the patterns 1 and 2^+, and CT^r is a caterpillar of length k, then*

$$\Gamma_b(CT) = \Gamma_b(CT^r) + 6n_M,$$

where $n_M = \frac{n+1-k}{4}$ is the number of steps required to transform CT into CT^r.

Thanks to Proposition 3.1, if the length of CT^r is k and each spine z_i of CT^r has m_i pendent neighbors, with $m_i \geq 2$, then

$$\Gamma_b(CT) = \Gamma_b(CT^r) + 6n_M = \sum_{i:m_i \geq 2} m_i + 6n_M,$$

so we henceforth assume that CT^r is a caterpillar with a pattern 1 and 2^+, and the pattern $1 \dots 1$, of length $p+1$, occurs in CT^r only if $0 \leq p \leq 2$.

Let H be one of the three sub-caterpillars $CT(1)$, $CT(1,1)$ or $CT(1,1,1)$, of CT. In order to prove the next proposition, we introduce a new definition. A dominating broadcast h on H is *H-pendent restricted* if the pendent vertices of CT, different from those of H, are not h-dominated by some h-broadcast vertex of V_h^+.
Denote

$$\widetilde{F}_H = \{h : h \text{ is a minimal } H\text{-pendent restricted dominating broadcast on } H\},$$

and let $\widetilde{h}_H$ be a minimal H-pendent restricted dominating broadcast on H with maximum cost

$$\sigma(\widetilde{h}_H) = \max\{\sigma(h) : h \in \widetilde{F}_H\}.$$

Since $\widetilde{h}_H$ is a minimal dominating broadcast on H, we get

$$\sigma(\widetilde{h}_H) \leq \Gamma_b(H).$$

Proposition 3.3. *Let CT be a caterpillar with no trunks, of length $n \geq 4$, and let $H = [i_0, i_1]$ be one of the three sub-caterpillars $CT(1)$, $CT(1,1)$ or $CT(1,1,1)$, of CT. If f is a Γ_b-broadcast on CT, then*

$$\sigma(\widetilde{h}_H) = \begin{cases} \Gamma_b(H), & \textit{if } x_0 \in H \textit{ or } x_n \in H, \textit{ or } p = 0 \textit{ and } x_0, x_n \notin H, \\ p+1, & \textit{if } p = 1, 2 \textit{ and } x_0, x_n \notin H. \end{cases}$$

Proof. Let $H = [i_0, i_1]$, with $1 \leq i_1 - i_0 + 1 \leq 3$, and let h be a minimal H-pendent restricted dominating broadcast on H. We distinguish two cases.

1. $x_0 \in H$ or $x_n \in H$, or $p = 0$ and $x_0, x_n \notin H$.
   By symmetry, it suffices to consider the case $x_n \in H$ or, $p = 0$ and $x_0, x_n \notin H$.
   The mapping defined in Lemma 3.3 is a minimal H-pendent restricted dominating broadcast on H with cost $\left\lfloor \frac{3(n+1)}{2} \right\rfloor$. Then,

$$\left\lfloor \frac{3(n+1)}{2} \right\rfloor \leq \sigma(\widetilde{h}_H) \leq \Gamma_b(H)$$

   Since $\Gamma_b(H) = \left\lfloor \frac{3(n+1)}{2} \right\rfloor$, we get $\sigma(\widetilde{h}_H) = \Gamma_b(H) = \left\lfloor \frac{3(n+1)}{2} \right\rfloor$.

2. $p = 1, 2$ and $x_1, x_n \notin H$.
   If $p = 1$, then $i_1 = i_0 + 1$ and only these possibilities can occur:

$$\begin{aligned}
&h(x_{i_0}) = h(x_{i_1}) = 0 \text{ and } h(y^1_{i_0}) = h(y^1_{i_1}) = 1, \text{ or}\\
&h(x_{i_0}) = h(x_{i_1}) = 1 \text{ and } h(y^1_{i_0}) = h(y^1_{i_1}) = 0, \text{ or}\\
&h(x_{i_0}) = h(y^1_{i_1}) = 0 \text{ and } h(y^1_{i_0}) = h(x_{i_1}) = 1, \text{ or}\\
&h(x_{i_0}) = h(y^1_{i_1}) = 1 \text{ and } h(x_{i_1}) = h(y^1_{i_0}) = 0.
\end{aligned}$$

   Since in each case, $\sigma(h) = 2$, we get $\sigma(\widetilde{h}_H) = 2 = p + 1$.
   If $p = 2$, then $i_1 = i_0 + 2$ and only these possibilities can occur:

$$\begin{aligned}
&h(y^1_{i_0}) = h(y^1_{i_0+1}) = h(y^1_{i_0+2}) = 1 \text{ and } h(x_{i_0}) = h(x_{i_0+1}) = h(x_{i_0+2}) = 0, \text{ or}\\
&h(x_{i_0+1}) = h(y^1_{i_0}) = h(y^1_{i_0+2}) = 1 \text{ and } h(x_{i_0}) = h(x_{i_0+2}) = h(y^1_{i_0+1}) = 0, \text{ or}\\
&h(x_{i_0}) = h(x_{i_0+1}) = h(x_{i_0+2}) = 1 \text{ and } h(y^1_{i_0}) = h(y^1_{i_0+1}) = h(y^1_{i_0+2}) = 0, \text{ or}\\
&h(x_{i_0}) = h(x_{i_0+2}) = h(y^1_{i_0+1}) = 1 \text{ and } h(y^1_{i_0}) = h(y^1_{i_0+2}) = h(x_{i_0+1}) = 0, \text{ or}\\
&h(x_{i_0}) = h(x_{i_0+2}) = h(y^1_{i_0}) = h(y^1_{i_0+1}) = h(y^1_{i_0+2}) = 0 \text{ and } h(x_{i_0+1}) = 2, \text{ or}\\
&h(x_{i_0}) = h(x_{i_0+1}) = h(x_{i_0+2}) = h(y^1_{i_0}) = h(y^1_{i_0+2}) = 0 \text{ and } h(y^1_{i_0+1}) = 3.
\end{aligned}$$

   Since in each case, $\sigma(h)$ is equal to 2 or 3, we get $\sigma(\widetilde{h}_H) = 3 = p + 1$.

This completes the proof. □

Let $H_1,\ldots,H_s$ be the sequence of all maximal sub-caterpillars $CT(1)$, $CT(1,1)$ and $CT(1,1,1)$ in CT^r. In view of the previous results (Lemmas 1, 8-12,15 and 16), we can at this step, give the exact value of $\Gamma_b(CT^r)$ when the reduced caterpillar CT^r of CT contains the patterns 1 and 2^+, and is CT^4_5*-free*.

Lemma 3.11. *If CT is a caterpillar with no trunks of length $n \geq 3$ and let CT^r be the reduced caterpillar of CT containing the patterns 1 and 2^+. If CT^r is and CT^4_5*-free*, then*

$$\Gamma_b(CT^r) = \sum_{i=1}^{s} \sigma(\widetilde{h}_{H_i}) + \sum_{i:m_i \geq 2} m_i.$$

From Proposition 3.2, and Lemma 3.11, we deduce the following formula.

Theorem 3.2. *If CT is a caterpillar with no trunks, of length $n \geq 3$, containing the patterns 1 and 2^+, and CT_5^4-free, then*

$$\Gamma_b(CT) = 6 \times n_M + \sum_{i=1}^{s} \sigma(\widetilde{h}_{H_i}) + \sum_{i:m_i \geq 2} m_i.$$

Concerning reduced caterpillars CT^r of length k, the formula of $\Gamma_b(CT^r)$ cannot be deduced so simply when CT_5^4 is an induced subgraph of CT^r, we need to prove some results beforehand. For that, we introduce a new mapping which gives, for a given dominating broadcast f, the f-values of the pendent neighbors of a stem z_i, with $m_i = 2$, $i = 0, \dots, k$, where all possibilities of these f-values are known thanks to Lemma 3.9.

Let $D = \{d_1, d_2, \dots, d_{s'}\}$ be the set of stems in CT^r which are adjacent to exactly two leaves. We assume that the sequence D is ordered according to CT^r, that is d_i occurs before d_j in D if $i < j$.

For $d_i \in D$ and $j = 1, \dots, 4$, let P_f be the function from D to $\{\theta_i^j, j = 1, \dots, 5\}$, defined as follows

$$P_f(d_i) = \begin{cases} \theta_i^j, & \text{if } CT[i-j+1, i-j+4] \text{ is a caterpillar of type } CT_5^4 \\ & \text{and } (f(t_i^1), f(t_i^2)) = (\theta_i^j(t_i^1), \theta_i^j(t_i^2)), \\ \theta_i^5, & \text{if } f(t_i^1) = f(t_i^2) = 1. \end{cases}$$

We use the notation CT_f^i to denote either the caterpillar $F_i^j = CT[i-j+1, i-j+4]$ or $CT[i,i]$

$$CT_f^i = \begin{cases} F_i^j, & \text{if } P_f(d_i) = \theta_i^j, j = 1, \dots, 4, \\ CT[i,i], & \text{if } P_f(d_i) = \theta_i^5. \end{cases}$$

Using previous results and applying them on the reduced caterpillar CT^r with CT_5^4, we obtain the following theorem.

Theorem 3.3. *Let CT be a caterpillar with no trunks such that the reduced caterpillar CT^r has length $k \geq 3$. If CT^r contains CT_5^4, then CT^r admits a Γ_b-broadcast f such that*

1. *V_f^+ contains no stems.*
2. *For every f-broadcast vertex u, $f(u) \in \{1, 3\}$.*
3. *For every pendent vertex t_i^j, with $m_i \geq 3$ and $j = 1, \dots, m_i$, $f(t_i^j) = 1$.*
4. *For every f-broadcast vertex t_i^1 with $f(t_i^1) = 3$,*
   (a) *If $i = 0$ (resp. $i = k$), then $m_0 + m_1 = 2$ (resp. $m_{k-1} + m_k = 2$).*
   (b) *If $i \notin \{0, k\}$, then $z_i \in CT_5^4$ and $P_f(z_i) \in \{\theta_i^1, \theta_i^2, \theta_i^3, \theta_i^4\}$.*

Proof. From Lemmas 1, 8-11, CT^r admits a Γ_b-broadcast f satisfying Items 1, 2, 3 and 4(a). We have to prove Item 4(b).
Let z_i be a stem of CT^r, $i \notin \{0, k\}$. The caterpillar CT^r contains CT_5^4 and thus CT^r contains the patterns 1 and 2^+. From Lemma 3.7(2), we have either $PB_f(t_i^1) = L(z_{i-1})$ or $PB_f(t_i^1) = L(z_{i+1})$, and if there exists a pendent vertex f-dominated by two f-broadcast vertices u and u', then $d(u, u') = 3$. Hence, the f-values of the pendent vertices of the sub-caterpillar $CT[i-1, i+2]$ (or, similarly $CT[i-2, i+1]$) of CT^r, are zero except for t_i^1 and t_{i+1}^1 in $CT[i-1, i+2]$, where $f(t_i^1) = f(t_{i+1}^1) = 3$. Since f satisfies the item 3 and CT^r contains no pattern 1111, we get $m_j \leq 2$ for every $j = i-1, \ldots, i+2$ in $CT[i-1, i+2]$, and more precisely $m_{i-1} + m_i + m_{i+1} + m_{i+2} \leq 6$, for otherwise we could define a mapping on CT^r by modifying to 1 the f-values of each leaf of $CT[i-1, i+2]$, giving a minimal dominating broadcast on CT^r with cost greater than $\Gamma_b(CT)$, a contradiction. On the other hand, if $m_{i-1} + m_i + m_{i+1} + m_{i+2} = 6$, we use the previous mapping, in order to have each leaf with an f-value different from 3, without modifying the cost of f. Therefore, $m_{i-1} + m_i + m_{i+1} + m_{i+2} = 5$ and we are done. □

Lemma 3.12. *Let CT be a caterpillar with no trunks such that the reduced caterpillar CT^r has length $k \geq 3$. If CT^r contains CT_5^4, then CT^r admits a Γ_b-broadcast f such that, for every stem $d_i \in D$, we have*

1. *If $P_f(d_i) = \theta_i^j$ for some $j \in \{1, \ldots, 4\}$, then $\Gamma_b(CT^r) = \Gamma_b(CT^r[CT_f^i/K_{1,6}, i-j+1])$*
2. *If $P_f(d_i) = \theta_i^5$, then $\Gamma_b(CT^r) = \Gamma_b(CT^r[CT_f^i/K_{1,6}, i]) - 4$.*

Using Lemma 3.12 $|D|$ times, we can infer the value of $\Gamma_b(CT^r)$ as a function of $\Gamma_b(CT^r_{\overline{D_2}})$, where $CT^r_{\overline{D_2}}$ is the reduced caterpillar of a caterpillar CT with no pattern 2.

Theorem 3.4. *If CT is a caterpillar with no trunks such that the reduced caterpillar CT^r has length $k \geq 3$, then*

$$\Gamma_b(CT^r) = \Gamma_b(CT^r_{\overline{D_2}}) - 4n_{P_2},$$

where n_{P_2} is the number of stems in D, for which $P_f(d_i) = \theta_i^5$.

It should be noted that the exact value of $\Gamma_b(CT^r_{\overline{D_2}})$ is completely defined by Proposition 3.1 or Lemma 3.11 depending on whether $CT^r_{\overline{D_2}}$ contains the pattern 1 or not.

To use Lemma 3.12, we need to know, for a given Γ_b-broadcast f, the values of $P_f(d_i)$, for every stem d_i of CT^r adjacent to two leaves. Lemmas 3.13 and 3.14 provide a response to this need. For this, let us recall some notations previously introduced.
Let $CT^r = CT(m_0, \ldots, m_k)$ be the reduced caterpillar of CT, $z_0, \ldots, z_k$ the spines vertices of CT^r, $L(z_i) = \{t_i^1, \ldots, t_i^{m_i}\}$ the set of pendent neighbors of z_i, for every $i = 0, \ldots, k$, and $D = \{d_1, d_2, \ldots, d_{s'}\}$ the set of stems in CT^r adjacent to two leaves. Denote by z_{i_0} and z_{i_1}, the first and the last stems of CT^r respectively, with $m_{i_0}, m_{i_1} \geq 2$.

We first study, in Lemma 3.13, the case where $m_{i_0}, m_{i_1} \geq 3$ by proving that CT^r admits a Γ_b-broadcast f such that if $d_1 = z_i$ for some index i, does not appear in any F_i^j (of type CT_5^4), $j = 1, \ldots, 4$, then $P_f(d_1) = \theta_i^5$. Otherwise, $P_f(d_1) = \theta_i^j$, where j is the smallest integer for which $F_i^j = CT[i-j+1, i-j+4]$.

Lemma 3.13. *Let CT be a caterpillar with no trunks such that the reduced caterpillar CT^r has length $k \geq 3$, and satisfying $m_{i_0}, m_{i_1} \geq 3$. If CT^r contains CT_5^4 and $d_1 = z_i$ for some index i, then CT^r admits a Γ_b-broadcast f such that*

1. *If $m_{i-3} = m_{i-2} = m_{i-1} = 1$, then $P_f(d_1) = \theta_i^4$.*
2. *If $m_{i-2} = m_{i-1} = 1$, $m_{i+1} = 1$ and $m_{i-3} \neq 1$, then $P_f(d_1) = \theta_i^3$.*
3. *If $m_{i-1} = 1$, $m_{i+1} = m_{i+2} = 1$ and $m_{i-2} \neq 1$, then $P_f(d_1) = \theta_i^2$.*
4. *If $m_{i+1} = m_{i+2} = m_{i+3} = 1$ and $m_{i-1} \neq 1$, then $P_f(d_1) = \theta_i^1$.*
5. *If d_1 does not appear in any sub-caterpillar F_i^j, $j = 1, \ldots, 4$, then $P_f(d_1) = \theta_i^5$.*

Thanks to Lemma 3.13, we are able to determine $P_f(d_1)$. Afterwards, we consider the caterpillar $CT^r[CT_f^i/K_{1,6}, i-j+1]$ or $CT^r[CT_f^i/K_{1,6}, i]$, according to $P_f(d_1) = \theta_i^j$ for some $j \in \{1, \ldots, 4\}$ or $P_f(d_1) = \theta_i^5$. We use again Lemma 3.13 for the concerned caterpillar, with $|D| - 1$ stems adjacent to two leaves. Repeating this procedure $|D|$ times, we obtain a caterpillar without pattern 2 (that is, a CT_5^4-free caterpillar) and $P_f(d_i)$ is determined for every $i = 1, \ldots, s'$. The value of $\Gamma_b(CT^r)$ is deduced from Lemma 3.11 and Theorem 3.4.

Lemma 3.14. *Let CT be a caterpillar with no trunks such that the reduced caterpillar CT^r has length $k \geq 3$. If CT^r contains CT_5^4 and $d_1 = z_{i_0}$, then CT^r admits a Γ_b-broadcast f such that*

1. *$P_f(d_1) \notin \{\theta_{i_0}^3, \theta_{i_0}^4\}$.*
2. *If $i_0 \in \{1, 3\}$ and $d_1 \in F_{i_0}^2$, then $P_f(d_1) = \theta_{i_0}^2$.*
3. *If $i_0 \in \{0, 2\}$ and $d_1 \in F_{i_0}^1$, then $P_f(d_1) = \theta_{i_0}^1$.*
4. *If d_1 does not appear in any sub-caterpillar $F_{i_0}^j$, $j \in \{1, 2\}$, then $P_f(d_1) = \theta_{i_0}^5$.*

For any reduced caterpillar with $m_{i_0} = 2$ (or $m_{i_1} = 2$ by symmetry), we are able to determine $P_f(d_1)$ (and $P_f(d_{s'})$ when $m_{i_1} = 2$), from Lemma 3.14. Similarly to what was discussed previously (case $m_{i_0} > 2$ and $m_{i_1} > 2$), we consider the caterpillar CT_1 representing $CT^r[CT_f^{i_0}/K_{1,6}, i_0 - j + 1]$ or $CT^r[CT_f^{i_0}/K_{1,6}, i_0]$, according to $P_f(d_1) = \theta_{i_0}^j$ for some $j \in \{1, \ldots, 4\}$ or $P_f(d_1) = \theta_{i_0}^5$. By symmetry, we do the same thing again on CT_1 when $m_{i_1} = 2$. Then, we use Lemma 3.13 for the resulting caterpillar, with $|D| - 1$ (or $|D| - 2$ when $m_{i_1} = 2$) stems adjacent to two leaves. Repeating this procedure $|D|$ times, we obtain a caterpillar without pattern 2 (that is, a CT_5^4-free caterpillar) and for every $i = 1, \ldots, s'$, $P_f(d_i)$ is determined. The value of $\Gamma_b(CT^r)$ is deduced from Lemma 3.11 and Theorem 3.4.

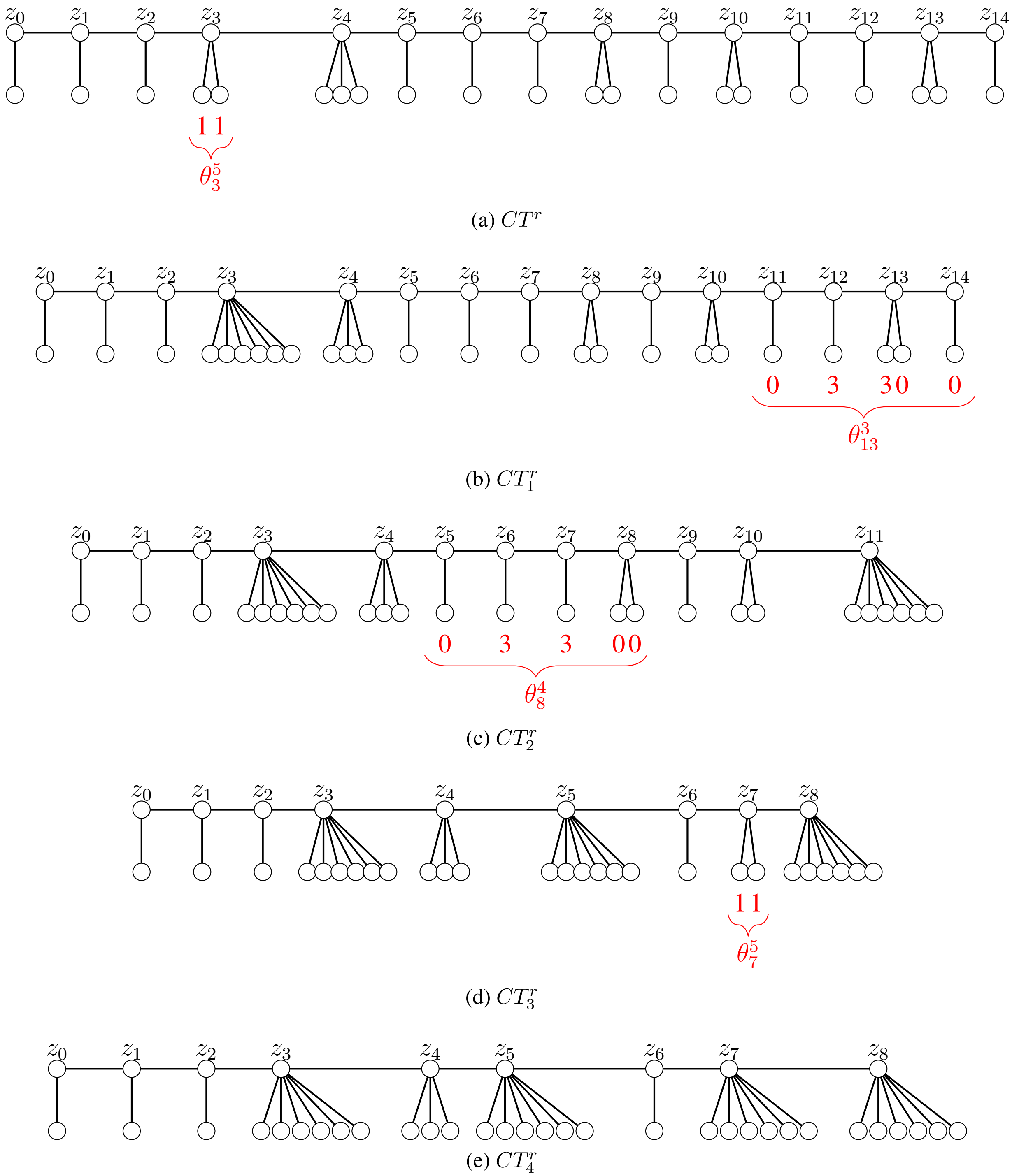


Figure 3: Determination of CT_4^r.

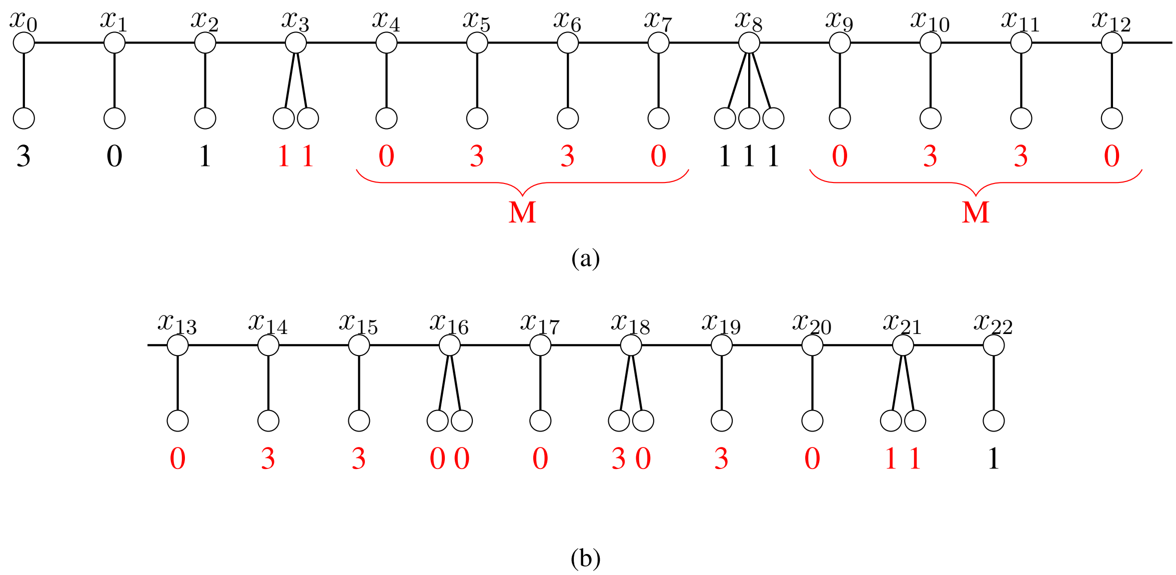


Figure 4: Γ_b-broadcast on CT.

4. Example

We illustrate through an example how we can find a Γ_b-broadcast for caterpillars CT which contains the patterns 1 and 2^+, and containing CT_5^4. For this, we consider the following caterpillar $CT[(1)^3, 2, (1)^4, 3, (1)^7, 2, 1, 2, (1)^2, 2, 1]$.

Step 1. We delete the two occurrences of M in CT, that is $CT[4:7]$ and $CT[9:12]$.
Let $CT^r = [(1)^3, 2, 3, (1)^3, 2, 1, 2, (1)^2, 2, 1]$ (see Figure 3.(a)) and $n_M = 2$.
We have $\Gamma_b(CT) = \Gamma_b(CT^r) + 6 \times n_M = \Gamma_b(CT^r) + 12$.

Step 2. We determine θ_i^j for each pattern 2.

1. In CT^r, $i_0 = 3$, $d_1 = z_3$ and $m_3 = 2$. According to Lemma 3.14, we have $P_f(d_1) = \theta_3^5$. We consider $CT_1^r = [(1)^3, 6, 3, (1)^3, 2, 1, 2, (1)^2, 2, 1]$(see Figure 3.(b)).
2. In CT_1^r, $m_{i_1} = 2$, $d_{|D_2|} = z_{13}$, and $i_0 = n - 1$. According to Lemma 3.14, $P_f(d_{|D_2|}) = \theta_{13}^3$. We consider $CT_2^r = [(1)^3, 6, 3, (1)^3, 2, 1, 2, 6]$(see Figure 3.(e)).
3. In CT_2^r, $m_{i_0} \geq 3$, $d_1 = z_8$, $m_5 = m_6 = m_7 = 1$ and $m_4 = 3 \neq 1$. According to Lemma 3.13, $P_f(d_1) = \theta_8^4$. We consider $CT_3^r = [(1)^3, 6, 3, 6, 1, 2, 6]$(see Figure 3.(c)).
4. In CT_3^r, $m_{i_0} \geq 3$, $d_1 = z_7$, and $d_1 \notin F_7^j, \forall j \in \{1, ..., 4\}$. According to Lemma 3.13, $P_f(d_1) = \theta_7^5$. We consider $CT_4^r = [(1)^3, 6, 3, 6, 1, 6, 6]$(see Figure 3.(d)).

The last reduced caterpillar $CT_4^r = [(1)^3, 6, 3, 6, 6, 6, 1]$ is a caterpillar without pattern 2 and $n_{P_2} = 2$.

Step 3. Calculation of $\Gamma_b(CT)$.

Thanks to Proposition 3.2 and Theorem 3.4, we have
$\Gamma_b(CT) = \Gamma_b(CT_4^r) + 6 \times n_M - 4 \times n_{P_2} = \Gamma_b(CT_4^r) + 4$.
The cost of Γ_b on caterpillar $CT_4^r[(1)^3, 6, 3, 6, 6, 6, 1]$ is calculate from the formula givin by Lemma 3.11. It follows, $\Gamma_b(CT) = 36$ and the Γ_b-broadcast on CT is depicted in Figure 4.

5. Conclusion

In this paper, we gave the exact value of Γ_b for any caterpillar without trunks. The study of caterpillars containing trunks seems more complicated in general. For future research, several problems seem interesting.

- Determine the value of $\Gamma_b(CT)$ for more general caterpillar classes, such that the class of caterpillars with no k consecutive trunks, $k \geq 2$.
- Let m and n be two positive integers. The value of $\Gamma_b(P_m \Box P_n)$, where $\Box$ stands for the Cartesian product of graphs, has been determined in [4]. Determine the value of $\Gamma_b(P_m \circ P_n)$, for any other operation $\circ$, as it was done for the variant γ_b in [15].
- Determine the ratio between Γ_b and any other broadcast invariant (to our knowledge, this question has been studied in the literature only for boundary independence numbers in [13]).

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6. Appendix

Proof of Lemma 3.2. Let CT be a caterpillar with no trunks, of length $n \leq 2$ and size m, and let f be a Γ_b-broadcast on CT.

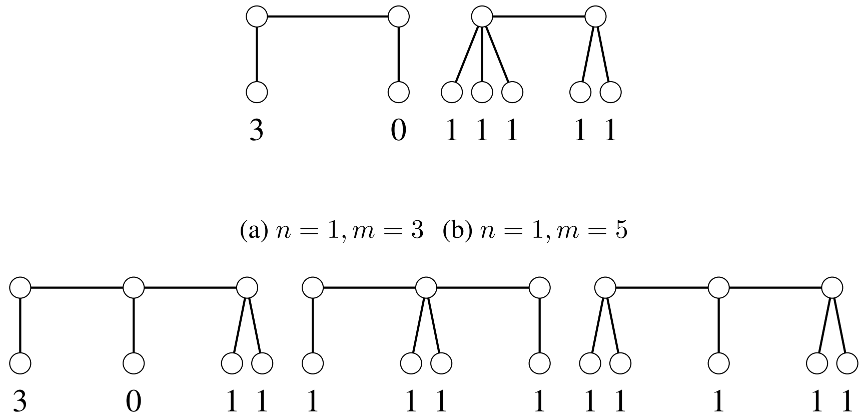
If $n = 1$ and $m = 3$, then CT is a path and $\Gamma_b(CT) = m$ (see Figure 5 (a)).
If $n \geq 2$ or $m \geq 4$, then CT is neither a path nor a star. By Theorem 2.1, we get $\Gamma_b(CT) \leq m - 1$. For the converse, we have to define a minimal dominating broadcast on CT with cost $m - 1$ or $m - 2$, according to the studied case.
Let μ be the unitary dominating broadcast on CT. Since μ is a minimal dominating broadcast with cost $m - n$, we infer $\Gamma_b(CT) \geq m - n$. For $n = 1$ and $m \geq 4$, we immediately get $\Gamma_b(CT) \geq m - 1$, and thus $\Gamma_b(CT) = m - 1$ (see Figure 5 (b)).
If $n = 2$ and $\ell_0 = \ell_1 = 1$ (the case $\ell_1 = \ell_2 = 1$ is similar, by symmetry), then the mapping g defined by $g(y_2^j) = 1$ for every j, $j = 1, \dots, \ell_2$, $g(y_0^1) = 3$, and $g(x) = 0$ otherwise is a minimal dominating broadcast with cost $m - 1$. Hence, $\Gamma_b(CT) \geq m - 1$, and thus $\Gamma_b(CT) = m - 1$ (see Figure 5 (c)).
If $n = 2$ and $\ell_1 \geq 2$, then $f(y_1^1) \leq 2$. Indeed, since the f-value for each vertex of CT does not exceed its eccentricity, we have $f(y_1^j) \leq 3$ for every $j = 1, \dots, \ell_1$. On the other hand $f(y_1^1) = 3$ cannot hold (recall that we assumed $f(y_i^1) \geq \dots \geq f(y_i^{\ell_i})$ for every $i = 0, \dots, n$), since otherwise $V_f^+ = \{y_1^1\}$ and we could set $g(x) = 1$ for every leaf x, giving a minimal dominating broadcast with cost $\sigma(g) \geq 4 \geq \sigma(f) + 1$, contradicting the optimality of f.
According to the f-values of pendent vertices y_1^j, $j = 1, \dots, \ell_1$, we discuss three cases. In each case, we prove the existence of at least two elements in $\overline{E_f}$, which allows us to get $\Gamma_b(CT) \leq m-2$.

1. $f(y_1^j) = 1$ for every $j = 1, \dots, \ell_1$.
   We have $PB_f(y_1^j) = \{y_1^j\}$ and then, $P_{y_1^j} = y_1^j x_1$ for every $j = 1, \dots, \ell_1$ and x_1 does not lie to any path P_v^f, where v is an f-broadcast vertex of CT, $v \neq y_1^j$. Thus, the edges x_0x_1 and x_1x_2 belong to $\overline{E_f}$.

2. $f(y_1^j) = 0$ for every $j = 1, \dots, \ell_1$.
   By Lemma 2.7, y_1^j is f-dominated by y_0^1 or y_2^1. By Lemma 2.6, we have either $PB_f(y_0^1) = L(x_1)$ or $PB_f(y_2^1) = L(x_1)$. Therefore, we have either $P_{y_0^1} = y_0^1 x_0 x_1 y_1^j$ or $P_{y_2^1} = y_2^1 x_2 x_1 y_1^j$, for some $j \in \{1, \dots, \ell_1\}$, and the set $\overline{E_f}$ contains $\ell_1 - 1 \geq 1$ pendent edges and one of the edges x_0x_1 or x_1x_2.

3. $f(y_1^1) = 2$.
   We have $PB_f(y_1^1) = \{y_1^2, \dots, y_1^{\ell_1}\}$, for otherwise the leaves adjacent to x_0 or to x_2 would not be dominated. Hence, $P_{y_1^1} = y_1^1 y_1^j$ for some $j \in \{2, \dots, \ell_1\}$ and x_1 cannot lie on some path P_v^f, where v is a broadcast vertex different from y_1^1. Therefore, the edges x_0x_1 and x_1x_2 belong to $\overline{E_f}$.

If $n = 2$, $\ell_0 \geq 2$, $\ell_1 = 1$ and $\ell_2 \geq 2$, then, by the same arguments as above, the f-values of the leaves cannot exceed 3. We distinguish six cases.



(a) $n = 1, m = 3$ (b) $n = 1, m = 5$

(c) $n = 2, \ell_0 = 1, \ell_1 = 1$ (d) $n = 2, \ell_0 = 1, \ell_1 = 2$ (e) $n = 2, \ell_0 = 2, \ell_1 = 1$

Figure 5: Examples of Γ_b-broadcasts for $n = 1, 2$.

1. $f(y_0^j) = 0$ for every $j = 1, \dots, \ell_0$.
   The vertex y_0^j is f-dominated by y_2^1, for otherwise $\sigma(f) = f(y_1^1) = 3$, contradicting the optimality of f. Therefore, $V_f^+ = \{y_2^1\}$ and $P_{y_2^1} = y_2^1 x_2 x_1 x_0 y_0^j$ for some $j \in \{1, \dots, \ell_0\}$. Hence, $|\overline{E_f}| \geq (\ell_0 - 1) + \ell_1 + (\ell_2 - 1) = \ell_0 + \ell_2 - 1 \geq 3$.

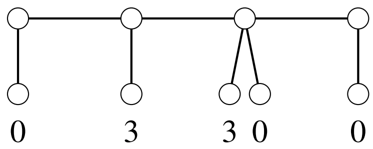
2. $f(y_0^j) = 1$ for every $j = 1, \dots, \ell_0$, and $f(y_2^l) = 1$ for every $l = 1, \dots, \ell_2$.
   We have $PB_f(y_0^j) = \{y_0^j\}$ and $PB_f(y_2^l) = \{y_2^l\}$, and then $P_{y_0^j} = y_0^j x_0$ and $P_{y_2^l} = y_2^l x_2$. Therefore, both edges $x_0 x_1$ and $x_1 x_2$ are in the set $\overline{E_f}$.

3. $f(y_0^j) = 1$ for every $j = 1, \dots, \ell_0$, and $f(y_2^1) = 2$ (the case $f(y_0^1) = 2$ and $f(y_2^l) = 1$ for every $l = 1, \dots, \ell_2$ is similar, by symmetry).
   We have $PB_f(y_0^j) = y_0^j$ and $PB_f(y_2^1) = \{y_2^2, \dots, y_2^{\ell_2}\}$, and then $P_{y_0^j} = y_0^j x_0$ and $P_{y_2^1} = y_2^1 y_2^l$ for some $l \in \{2, \dots, \ell_2\}$. We have again both edges $x_0 x_1$ and $x_1 x_2$ in the set $\overline{E_f}$.

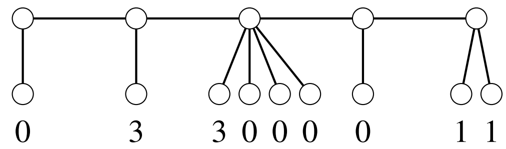
4. $f(y_0^j) = 1$ for every $j = 1, \dots, \ell_0$, and $f(y_2^1) = 3$ (the case $f(y_0^1) = 3$ and $f(y_2^l) = 1$ for every $l = 1, \dots, \ell_2$ is similar, by symmetry).
   We have $PB_f(y_0^j) = \{y_0^j\}$ for every $j = 1, \dots, \ell_0$, and $PB_f(y_2^1) = y_1^1$, and then $P_{y_0^j} = y_0^j x_0$ and $P_{y_2^1} = y_2^1 x_2 x_1 y_1^k$ for some $k \in \{1, \dots, \ell_1\}$. Thus, the edges $x_0 x_1$ and the $\ell_2 - 1 \geq 1$ leaves $y_2^l x_2$, $l = 2, \dots, \ell_2$ belong to $\overline{E_f}$.

5. $f(y_0^1) = 2$ and $f(y_2^1) = 2$.
   We have $PB_f(y_0^1) = \{y_0^2, \dots, y_0^{\ell_0}\}$ and $PB_f(y_2^1) = \{y_2^2, \dots, y_2^{\ell_2}\}$, and then $P_{y_0^1} = y_0^1 y_0^j$ for some $j \in \{2, \dots, \ell_0\}$, and $P_{y_2^1} = y_2^1 y_2^2$ for some $l \in \{2, \dots, \ell_2\}$. It follows, $f(y_1^1) = 1$ and $PB_f(y_1^1) = \{x_1\}$. Thus, both edges $x_0 x_1$ and $x_1 x_2$ belong to $\overline{E_f}$.

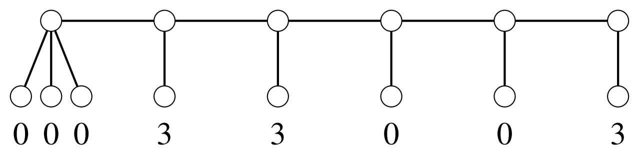
6. $f(y_0^1) = 2$ and $f(y_2^1) = 3$ (the case $f(y_0^1) = 3$ and $f(y_2^l) = 2$ is similar, by symmetry).
   We have $PB_f(y_0^1) = \{y_0^2, \dots, y_0^{\ell_0}\}$ and $PB_f(y_2^1) = \{y_1^1\}$, and then $P_{y_0^1} = y_0^1 y_0^j$ for some



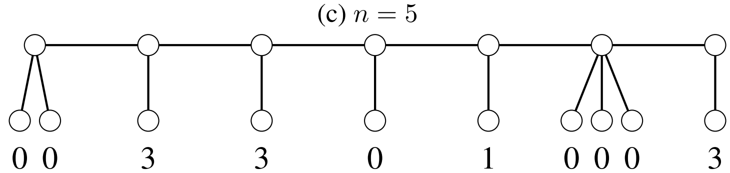
(a) $n = 3$



(b) $n = 4$



(c) $n = 5$



(d) $n = 6$

Figure 6: Examples of the broadcast f defined in Lemma 3.3.

$j \in \{2, \ldots, \ell_0\}$, and $P_{y_2^1} = y_2^1 x_2 x_1 y_1^l$. Hence, the edges x_0x_1 and the $\ell_2 - 1 \geq 1$ leaves $y_2^l x_2$, $l = 2, \ldots, \ell_2$ belong to $\overline{E_f}$.

In each case, we proved that $\Gamma_b(CT) \leq m - 2$. Since $\Gamma_b(CT) \geq m - n \geq m - 2$, we get $\Gamma_b(CT) = m - 2$ (see Figure 5 (d) and (e)). This completes the proof. □

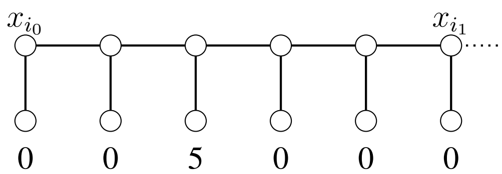
Proof of Lemma 3.3. Let $CT = CT(\ell_0, \ldots, \ell_n)$ be a caterpillar with no trunks, where $n + 1 = 4q + r$, $q \in \mathbb{N}^*$ and $r = 0, \ldots, 3$. We define a mapping f (see Figure 6), by setting, for $i = 0, \ldots, n - r$

$$\begin{cases} f(y_i^1) = 3 & \text{if } i \equiv 1, 2[4] \\ f(y_n^j) = 1 \text{ for every } j = 1, \ldots, \ell_n, & \text{if } r = 1 \\ f(y_n^1) = 3, & \text{if } r = 2 \\ f(y_n^1) = 3 \text{ and } f(y_{n-2}^j) = 1 \text{ for every } j = 1, \ldots, \ell_{n-2}, & \text{if } r = 3 \\ f(u) = 0, & \text{otherwise.} \end{cases}$$

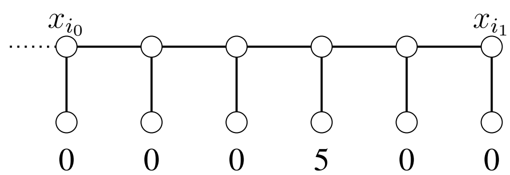
For all other vertex u of CT, we set $f(u) = 0$. The mapping f is clearly a minimal dominating broadcast, with cost

$$\sigma(f) = \begin{cases} \frac{3(n+1)}{2}, & \text{if } r = 0, 2, \\ \frac{3n}{2} + \ell_n, & \text{if } r = 1, \\ \frac{3n}{2} + \ell_{n-2}, & \text{if } r = 3. \end{cases}$$

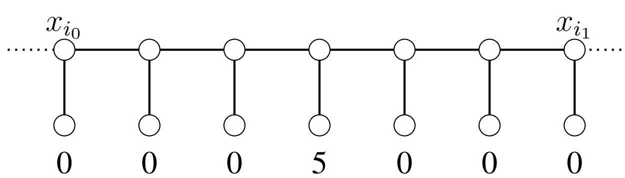
It follows, $\sigma(f) \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor$, and then, $\Gamma_b(CT) \geq \left\lfloor \frac{3(n+1)}{2} \right\rfloor$. This completes the proof. □



(a) $i = 2, g(u) = 5$ and $i - g(u) + 2 < 0$



(b) $i - g(u) + 2 \geq 0$ and $i + g(u) - 2 > n$



(c) $i - g(u) + 2 \geq 0$ and $i + g(u) - 2 \leq n$

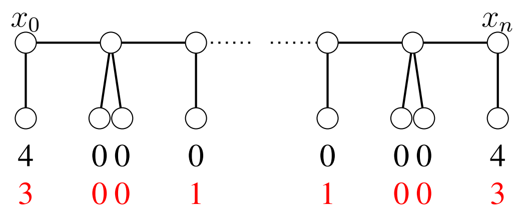
Figure 7: Illustration for the proof of Lemma 3.6, Case 1.

Proof of Lemma 3.6. Let g be a Γ_b-broadcast of CT. Assume that there exists a g-broadcast vertex $u = y_i^1$ for some $i \in \{0, \ldots, n\}$, with $g(u) \geq 4$ and u is the leftmost g-broadcast vertex with this property. By Lemma 3.1, u and its private neighbor u^p are leaves.
We will consider the sub-caterpillar $CT^* = CT[i_0, i_1]$, where i_0 and i_1 will be defined depending on the two following cases.

1. Every pendent vertex in $B_g(u)$ belongs to $PB_g(u)$.
   In that case, we set

$$\begin{cases} i_0 = 0 \text{ and } i_1 = i + g(u) - 2, & \text{if } \ i - g(u) + 2 < 0, \\ i_0 = i - g(u) + 2 \text{ and } i_1 = n, & \text{if } \ i + g(u) - 2 > n, \\ i_0 = i - g(u) + 2 \text{ and } i_1 = i + g(u) - 2, & \text{otherwise.} \end{cases} \qquad \text{(see Figure 7)}$$

   Obviously, we have $i_0 < i_1$. Moreover, $i_1 - i_0 + 1 \leq 3$ holds if and only if $i = 0$ and $g(u) = 4$ (or, $i = n$ and $g(u) = 4$, by symmetry). Indeed,
   If $i = 0$ and $g(u) = 4$, then $i - g(u) + 2 = -2 < 0$ and $i_1 - i_0 + 1 = 3 \leq 3$.
   Conversely, assume that $i_1 - i_0 + 1 \leq 3$ and $g(u) \geq 4$. If $i_1 - i_0 + 1 = i + g(u) - 1 \leq 3$, then $i + 3 \leq 3$, that is $i = 0$, and $i - g(u) + 2 < 0$. If $i_1 - i_0 + 1 = n - i + g(u) - 1 \leq 3$, then



(a) $i_0 = 0$ (b) $i_1 = n$

Figure 8: Illustration for the proof of Lemma 3.6, Case 1.

$n - i + 3 \leq 3$, that is $i = n$, and $i + g(u) - 2 > n$. If $0 \leq i - g(u) + 2 < i + g(u) - 2 \leq n$, then $i_1 - i_0 + 1 = 2g(u) - 3 \leq 3$ leads to $g(u) \leq 3$, a contradiction.

2. There exists a pendent vertex v, such that $v \in B_g(u)$ and $v \notin PB_g(u)$.
   In that case, there exists a broadcast vertex u', $u' \neq u$, such that v is g-dominated by u and by u' with $g(u') \geq 3$. Since u' is a leaf, let $u' = y_j^1$ for some $j > i$. The bordering private g-neighbors of u and u' are $PB_g(u) = \{y^1_{i-g(u)+2}, \ldots, y^{\ell_{i-g(u)+2}}_{i-g(u)+2}\}$ and $PB_g(u') = L(x^1_{j+g(u')-2})$, respectively.

   We set $i_0 = i - g(u) + 2$ and $i_1 = j + g(u') - 2$. The equality $i_1 - i_0 + 1 \geq 4$ must hold in this case since $i_1 - i_0 + 1 = j - i + g(u) + g(u') - 4 + 1 \geq 5$, so we can write $i_1 - i_0 + 1 = 4q + r$, where $q \in \mathbb{N}^*$ and $0 \leq r \leq 3$.

We define a mapping h, obtained from g by modifying only the g-values of the leaves between $y^1_{i_0}$ and $y^{\ell_{i_1}}_{i_1}$ (we already know that the stems must have h-value 0), according to the value of $i_1 - i_0 + 1$. We have two cases to consider.

1. $i_1 - i_0 + 1 \leq 3$.
   In that case, every pendent vertex in $B_g(u)$ belongs to $PB_g(u)$, $i = 0$ and $g(u) = 4$ (the case $i = n$ and $g(u) = 4$ is similar, by symmetry).

   If $i = 0$, we set $h(y_0^1) = 3$, $h(y_2^j) = 1$ for every $j = 1, \ldots, \ell_2$, and $h(z) = 0$ for every $z \in \{y_0^2, \ldots, y_0^{\ell_0}, y_1^1, \ldots, y_1^{\ell_1}\}$(see Figure 8). The mapping h is a minimal dominating broadcast with cost $\sigma(h) = \sigma(g) + 3 + \ell_2 - g(u) = \sigma(g) + \ell_2 - 1$. The optimality of g then implies $\ell_2 = 1$, so that $\sigma(h) = \sigma(g)$.

2. $i_1 - i_0 + 1 \geq 4$.
   For $t = i_0, \ldots, i_1 - r$, we set $h(y_t^j) = 0$ for every $j = 2, \ldots, \ell_t$ with $\ell_t \geq 2$, and

$$h(y_t^1) = \begin{cases} 0, & \text{if } t - i_0 + 1 \equiv 0, 1[4], \\ 3, & \text{if } t - i_0 + 1 \equiv 2, 3[4]. \end{cases}$$

   For the case $r = 0$, all the vertices have a h-value. We can thus now assume $r \neq 0$. We consider two sub-cases depending on $i_0 = 0$ or not.

(a) $i_0 \neq 0$.
We set $h(y_t^j) = 1$ for every $t = i_1 - r + 1, \dots, i_1$ and $j = 1, \dots, \ell_t$,

(b) $i_0 = 0$.
We set

$$\begin{cases} h(y_{i_1}^j) = 1 \text{ for every } j = 1, \dots, \ell_{i_1}, & \text{if } r = 1 \\ h(y_{i_1-1}^j) = 0 \text{ for every } j = 1, \dots, \ell_{i_1-1}, h(y_{i_1}^1) = 3 \text{ and} & \\ h(y_{i_1}^j) = 0 \text{ for every } j = 2, \dots, \ell_{i_1}, & \text{if } r = 2 \\ h(y_{i_1-2}^j) = 0 \text{ for every } j = 1, \dots, \ell_{i_1-2}, & \\ h(y_{i_1-1}^j) = 0 \text{ for every } j = 1, \dots, \ell_{i_1-1}, & \\ h(y_{i_1}^1) = 3 \text{ and } h(y_{i_1}^j) = 0 \text{ for every } j = 2, \dots, \ell_{i_1}, & \text{if } r = 3. \end{cases}$$

We now determine the cost of the minimal dominating broadcast h. We distinguish three cases.

(i) Every pendent vertex in $B_g(u)$ belongs to $PB_g(u)$ and $i - g(u) + 2 < 0$.
(the case $i + g(u) - 2 > n$ is similar by symmetry).

In that case, $4 \leq i_1 - i_0 + 1 = i + h(u) - 1$, that is $i + h(u) \geq 5$. We get

$$\sigma(h) = \sigma(g) - g(u) + \begin{cases} \frac{3(i_1-i_0+1)}{2}, & \text{if } r = 0, \\ \frac{3(i_1-i_0)}{2} + 1, & \text{if } r = 1, \\ \frac{3(i_1-i_0-1)}{2} + 3, & \text{if } r = 2, \\ \frac{3(i_1-i_0-2)}{2} + 4, & \text{if } r = 3, \end{cases}$$

that is,

$$\sigma(h) = \sigma(g) + \begin{cases} i + \frac{i+g(u)-3}{2}, & \text{if } r = 0, 2, \\ i + \frac{i+g(u)-4}{2}, & \text{if } r = 1, 3. \end{cases} \qquad \text{(see Figure 9)}$$

Since, $i + h(u) \geq 5$, we obtain $\sigma(h) \geq \sigma(g) + i + 1$ if $r = 0, 2$ and $\sigma(h) \geq \sigma(g) + i + \frac{1}{2}$, otherwise, contradicting the optimality of g.

(ii) Every pendent vertex in $B_g(u)$ belongs to $PB_g(u)$ and $0 \leq i - g(u) + 2 < i + g(u) - 2 \leq n$.
In that case, $4 \leq i_1 - i_0 + 1 = 2h(u) - 3$ is odd.

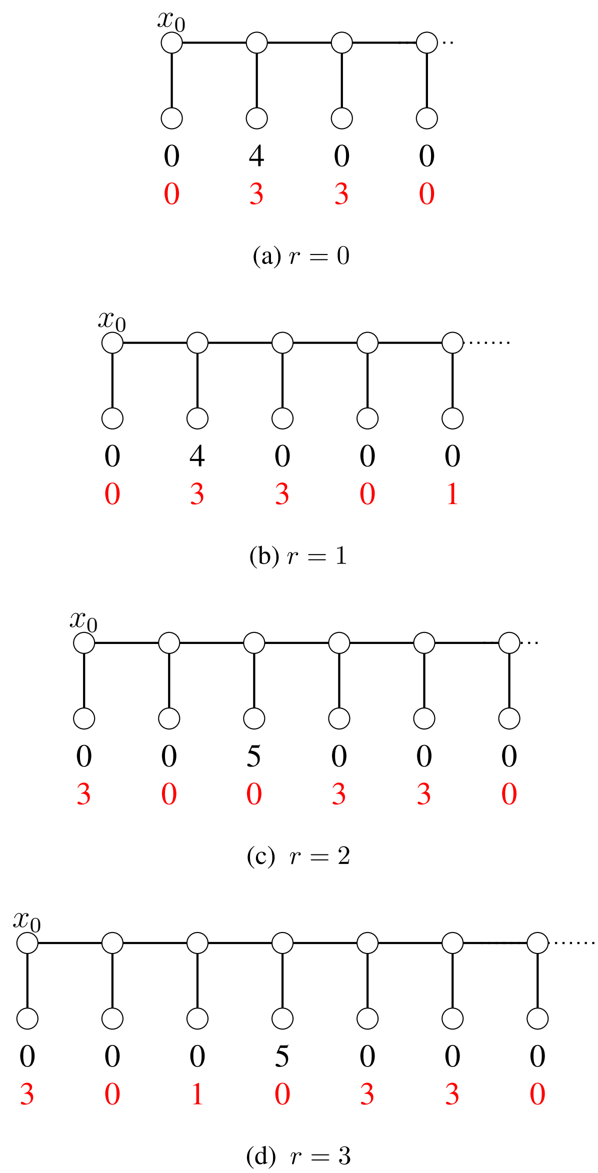
We get

$$\sigma(h) = \sigma(g) - g(u) + \begin{cases} \frac{3(2g(u)-4)}{2} + 1, & \text{if } r = 1, \\ \frac{3(2g(u)-6)}{2} + 4, & \text{if } r = 3, \end{cases}$$

and then $\sigma(h) = \sigma(g) + 2g(u) - 5 \geq \sigma(g) + 3$, contradicting the optimality of g (see Figure 10).

(iii) Items (i) and (ii) are not satisfied.
In that case, we have $i_1 - i_0 + 1 = j - i + g(u') + g(u) - 3 \geq 6$. Indeed, we have $g(u) \geq 4$, $g(u') \geq 3$, $j - i \geq 1$ and if $j - i = 1$, then $g(u') = g(u) \geq 4$, for otherwise u' g-dominates u^p.

For $i_0 = 0$, we get



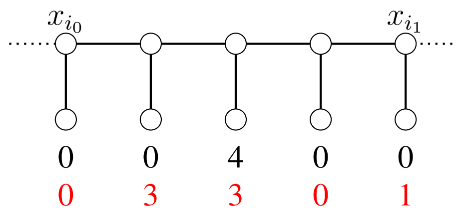
(a) $r = 0$

(b) $r = 1$

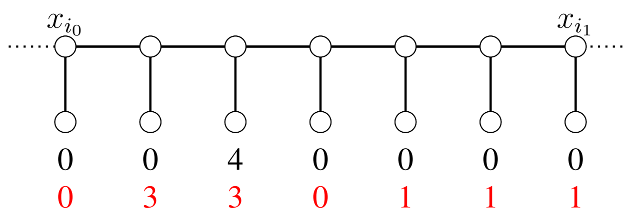
(c) $r = 2$

(d) $r = 3$

Figure 9: Illustration for the proof of Lemma 3.6, Case 2.(i).



(a) $r = 1$



(b) $r = 3$

Figure 10: Illustration for the proof of Lemma 3.6, Case 2.(ii).

$$\sigma(h) = \sigma(g) - g(u) - g(u') + \begin{cases} \frac{3(i_1-i_0+1)}{2}, & \text{if } r = 0, \\ \frac{3(i_1-i_0)}{2} + 1, & \text{if } r = 1, \\ \frac{3(i_1-i_0-1)}{2} + 3, & \text{if } r = 2, \\ \frac{3(i_1-i_0-2)}{2} + 4, & \text{if } r = 3, \end{cases}$$

that is,

$$\sigma(h) = \sigma(g) + \begin{cases} j - i + \frac{j-i+g(u')+g(u)-9}{2}, & \text{if } r = 0, 2, \\ j - i + \frac{j-i+g(u')+g(u)-10}{2}, & \text{if } r = 1, 3. \end{cases}$$

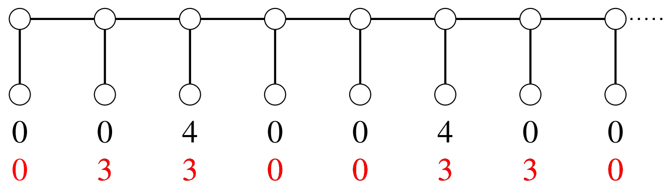
Therefore, $\sigma(h) > \sigma(g)$, contradicting the optimality of g (see Figure 11).

For $i_0 > 0$, we get

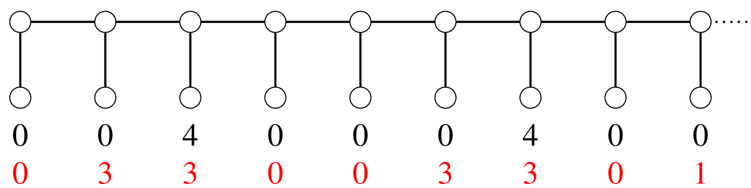
$$\sigma(h) = \sigma(g) - g(u) - g(u') + \begin{cases} \frac{3(i_1-i_0+1)}{2}, & \text{if } r = 0, \\ \frac{3(i_1-i_0)}{2} + \ell_{i_1}, & \text{if } r = 1, \\ \frac{3(i_1-i_0-1)}{2} + \ell_{i_1-1} + \ell_{i_1}, & \text{if } r = 2, \\ \frac{3(i_1-i_0-2)}{2} + \ell_{i_1-2} + \ell_{i_1-1} + \ell_{i_1}, & \text{if } r = 3, \end{cases}$$

that is,

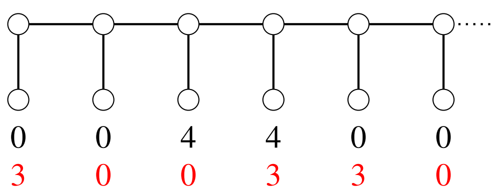
$$\sigma(h) = \sigma(g) + \begin{cases} j - i + \frac{j-i+g(u')+g(u)-9}{2}, & \text{if } r = 0, \\ j - i + \frac{j-i+g(u')+g(u)-12}{2} + \ell_{i_1}, & \text{if } r = 1, \\ j - i + \frac{j-i+g(u')+g(u)-15}{2} + \ell_{i_1-1} + \ell_{i_1}, & \text{if } r = 2, \\ j - i + \frac{j-i+g(u')+g(u)-18}{2} + \ell_{i_1-2} + \ell_{i_1-1} + \ell_{i_1}, & \text{if } r = 3. \end{cases} \quad (see\ Figure\ 12)$$



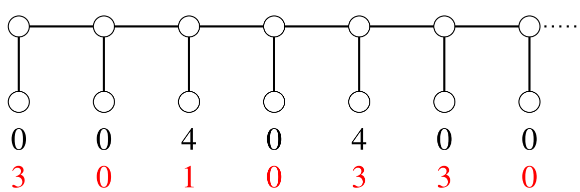
(a) $r = 0$



(b) $r = 1$

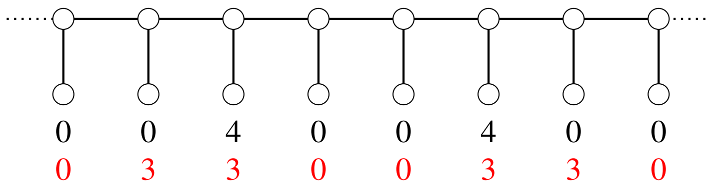


(c) $r = 2$

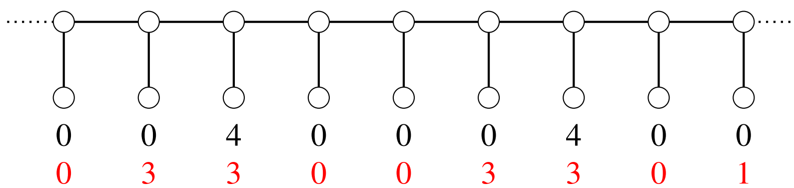


(d) $r = 3$

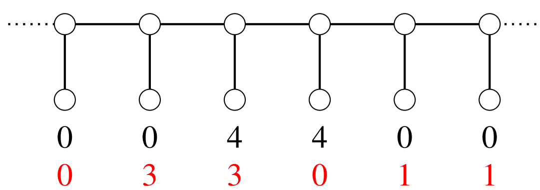
Figure 11: Illustration for the proof of Lemma 3.6, Case 2.(iii) and $i_0 = 0$.



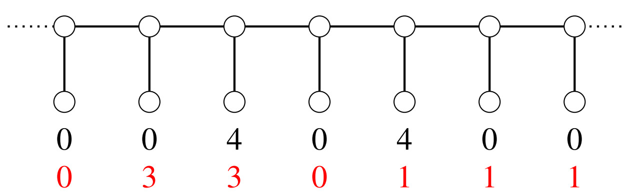
(a) $r = 0$



(b) $r = 1$



(c) $r = 2$



(d) $r = 3$

Figure 12: Illustration for the proof of Lemma 3.6, Case 2.(iii) and $i_0 > 0$.

If $r = 0$ or $r = 1$, we immediately obtain $\sigma(h) > \sigma(g)$, contradicting the optimality of g. If $r = 2$, then $\sigma(h) = \sigma(g) + j - i + \frac{j-i+g(u')+g(u)-15}{2} + \ell_{i_1-1} + \ell_{i_1} \geq \sigma(g) - 2 + \ell_{i_1-1} + \ell_{i_1}$. The optimality of g then implies $\ell_{i_1-1} = \ell_{i_1} = 1$, in which case $\sigma(h) = \sigma(g)$. If $r = 3$, then $\sigma(h) = \sigma(g) + j - i + \frac{j-i+g(u')+g(u)-18}{2} + \ell_{i_1-2} + \ell_{i_1-1} + \ell_{i_1}$ and $j - i + g(u') + g(u)$ must be even. Hence

$$\sigma(h) \geq \sigma(g) + (j-i) - 4 + \ell_{i_1-2} + \ell_{i_1-1} + \ell_{i_1} \geq \sigma(g) - 3 + \ell_{i_1-1} + \ell_{i_1}.$$

The optimality of g implies $\ell_{i_1-2} = \ell_{i_1-1} = \ell_{i_1} = 1$, in which case $\sigma(h) = \sigma(g)$. We repeat this transformation on each g-broadcast vertex with a value greater than 3 until obtaining a mapping with required condition. This completes the proof. □

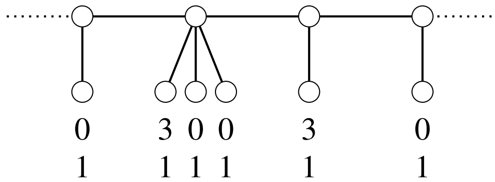
Proof of Lemma 3.7. Let g be a Γ_b-broadcast on the caterpillar CT, satisfying the conditions of Lemmas 2.7, 3.5 and 3.6. Then each g-broadcast vertex u is a leaf and has a g-value $g(u) \in \{1, 3\}$. Since $n \geq 3$, $|V_g^+| \geq 2$ by Corollary 3.1.

1. $\ell_0 + \ell_1 \geq 3$ and $g(y_0^1) = 3$.
   In that case, we consider the mapping f obtained from g by replacing the g-values of the leaves of $CT[x_0, x_1]$ by the value 1 . The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) - 3 + \ell_0 + \ell_1 \geq \Gamma_b(CT)$. The optimality of g implies $\ell_0 + \ell_1 = 3$, so that we have $\sigma(f) = \sigma(g)$. By symmetry, we also get $f(y_n^j) = 1$ for every j, $j = 1, \ldots, \ell_n$, if $\ell_{n-1} + \ell_n \geq 3$.

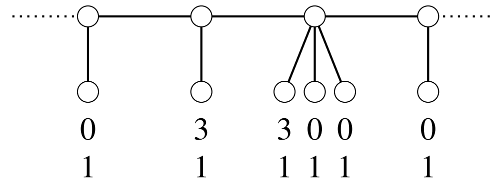
2. y_i^1 is a f-broadcast vertex for some $i = 1, \ldots, n$, with $f(y_i^1) = 3$.
   By the minimality of the dominating broadcast g, $PB_f(y_0^1) = L(x_1)$ (resp. $PB_f(y_n^1) = L(x_{n-1})$) if $g(y_0^1) = 3$ (resp. $g(y_n^1) = 3$). Now, assume to the contrary that there exists a g-broadcast vertex y_i^1, $i = 2, \ldots, n-1$, with $g(y_i^1) = 3$ and $PB_g(y_i^1) = L(x_{i-1}) \cup L(x_{i+1})$. Consider the mapping f obtained from g by replacing the g-values of the leaves of $CT[i-1, i+1]$ by the value 1. The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) - 3 + \ell_{i-1} + \ell_i + \ell_{i+1} \geq \Gamma_b(CT)$. The optimality of g implies $\ell_{i-1} + \ell_i + \ell_{i+1} = 3$, so that we have $\sigma(f) = \sigma(g)$. By symmetry, we also get $f(y_n^j) = 1$ for every j, $j = 1, \ldots, \ell_n$, if $\ell_{n-1} + \ell_n \geq 3$.

3. There exists a pendent vertex f-dominated by two f-broadcast vertices u et u'.
   Let u and u' be two g-broadcast vertices such that $N_f[u] \cap N_f[u']$ contains some leaf, say y_i^1, and assume that u is to the left of u'. Then, we have $g(u) = g(u') = 3$. If $d(u, u') \neq 3$ then necessarily $d(u, u') = 4$, $PB_f(u) = L(x_{i-2})$ and $PB_f(u') = L(x_{i+2})$. Consider a mapping f defined by $f(y_{i-2}^j) = 1$ for every $j = 1, \ldots, y_{i-2}^{\ell_{i-2}}$, $f(y_i^1) = f(y_{i+1}^1) = 3$, $f(y_{i-1}^j) = f(y_i^k) = f(y_{i+1}^l) = 0$ for every $j = 1, \ldots, y_{i-1}^{\ell_{i-1}}$, $k = 2, \ldots, y_i^{\ell_i}$, $l = 2, \ldots, y_{i+1}^{\ell_{i+1}}$, and $f(v) = g(v)$ otherwise. The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) + \ell_{i-2}$, contradicting the optimality of g. This completes the proof.

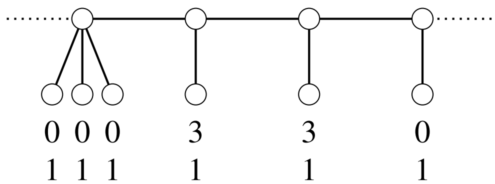
□



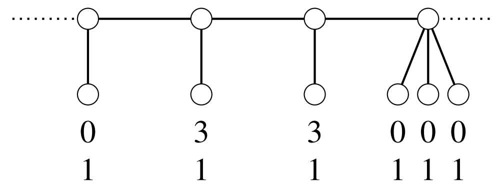
(a) $\ell_{i-1} + \ell_{i+1} + \ell_{i+2} = 3$



(b) $\ell_{i-2} + \ell_{i-1} + \ell_{i+1} = 3$



(c) $\ell_{i+1} + \ell_{i+2} + \ell_{i+3} = 3$



(d) $\ell_{i-3} + \ell_{i-2} + \ell_{i-1} = 3$

Figure 13: Illustration for the proof of Lemma 3.8, Case (1.a) and Case 2.

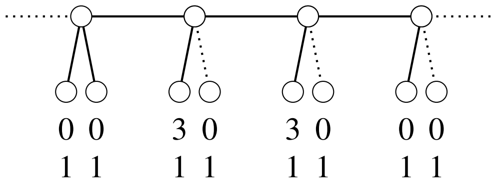
Proof of Lemma 3.8. Let CT be a caterpillar with no trunks, of length $n \geq 3$, and let g be a good Γ_b-broadcast on CT. Assume to the contrary that there exists a stem x_i with $\ell_i \geq 2$ and $g(y_i^1) \neq 1$ (that is, $g(y_i^j) \neq 1$ for every $j = 1, \ldots, \ell_i$).

If $i = 0$ (the case $i = n$ is similar, by symmetry), then $\ell_0 + \ell_1 \geq 3$ and $g(y_0^1) \neq 3$ by Lemma 3.7(1). Hence, $g(y_0^1) = 0$ and y_0^1 is g-dominated by y_1^1 with a g-value $g(y_1^1) = 3$.
By considering the same mapping f as in the proof of Lemma 3.7(1), we are done.
Assume now $0 < i < n$. We have either $g(y_i^1) = 3$, or $g(y_i^1) = 0$.

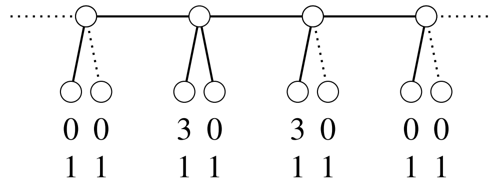
1. $g(y_i^1) = 3$.
   The leaf y_i^1 has only one private side by Lemma 3.7(2), and assume, without loss of generality, that $PB_g(y_i^1) = L(x_{i-1})$, which gives $i+1 \neq n$. By Lemma 3.7(3), we have $g(y_{i+1}^1) = 3$ and by Lemma 3.7(2), we have $PB_g(y_{i+1}^1) = L(x_{i+2})$.
   Consider the mapping f obtained from g by replacing the g-values of the leaves of $CT[x_{i-1}, x_{i+2}]$ by the value 1. The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) - 6 + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}$. According to the value of ℓ_i, we have two subcases to consider.

   (a) $\ell_i \geq 3$.
   In this case, the optimality of g implies $\ell_i = 3$ and $\ell_{i-1} = \ell_{i+1} = \ell_{i+2} = 1$, so that we have $\sigma(f) = \sigma(g)$ (see Figure 13(a)).

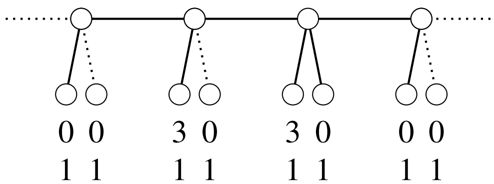
   (b) $\ell_i = 2$ and CT is CT_5^4-free.
   In this case, it must be at least six pendent edges in the sub-caterpillar $CT[i-1, i+2]$, and then $\sigma(f) = \sigma(g) - 6 + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} \geq \sigma(g) = \Gamma_b(CT)$. The optimality of g implies $\ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} = 6$, that is the existence of two stems adjacent to two leaves and both others to one leaf, so that we have $\sigma(f) = \sigma(g)$.



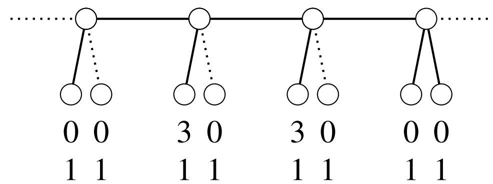
(a) $\ell_{i+1} + \ell_{i+2} + \ell_{i+3} = 4$



(b) $\ell_{i-1} + \ell_{i+1} + \ell_{i+2} = 4$



(c) $\ell_{i-2} + \ell_{i-1} + \ell_{i+1} = 4$



(d) $\ell_{i-1} + \ell_{i-2} + \ell_{i-3} = 4$

Figure 14: Illustration for the proof of Lemma 3.9, Case 1.

2. $g(y_i^1) = 0$.
   In that case, y_i^1 is g-dominated by some g-broadcast vertex, say without loss of generality y_{i+1}^1, of g-value $g(y_{i+1}^1) = 3$, and then y_i^1 is a private g-border of y_{i+1}^1 by Lemma 3.7(3). Since $\ell_i + \ell_{i+1} \geq 3$, then $i + 1 \neq n$, by Lemma 3.7(1). Further, $i + 2 \neq n$, for otherwise $y_n^1, \ldots, y_n^{\ell_n}$ would be in $PB_g(y_{i+1}^1)$, contradicting Lemma 3.7(2). It follows, as in previous case, $PB_g(y_{i+1}^1) = L(x_i)$, $g(y_{i+2}^1) = 3$ and $PB_g(y_{i+2}^1) = L(x_{i+3})$. As before, we consider the mapping f obtained from g by replacing the g-values of the leaves of $CT[x_i, x_{i+3}]$ by the value 1 (see Figure 13 (c) and (d)). The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) - 6 + \ell_i + \ell_{i+1} + \ell_{i+2} + \ell_{i+3}$ and we conclude as previously. This completes the proof.

□

Proof of Lemma 3.9. Let g be a good Γ_b-broadcast on the caterpillar CT satisfying Lemma 3.8. If $g(y_i^1) = g(y_i^2) = 1$, we are done. Assume now $g(y_i^1) \neq 1$, that is $(g(y_i^1), g(y_i^2)) \in \{(0,0),(3,0)\}$. The vertices y_i^1 and y_i^2 are g-dominated by some g-broadcast vertex u ($u = y_i^1$ can occur), with $g(u) = 3$ (observe that, by Lemma 3.7(1), $i \neq 0$). By Lemma 3.7(2), u has only one private side, and by Lemma 3.7(3), there exists a g-broadcast vertex u', such that $g(u') = 3$ and $d(u, u') = 3$. Let $X = CT[i_0, i_0+3]$ be the sub-caterpillar of CT, whose leaves are those which are g-dominated by u or u' in CT. We consider two cases according to whether x_i appears in F_i^j or not.

1. x_i does not appear in any F_i^j, $j = 1, \ldots, 4$.
   In that case, X must have at least six pendent edges. Consider the mapping f obtained from g by replacing the g-values of the leaves of X by the value 1. The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g) - 6 + \ell_{i_0} + \ell_{i_0+1} + \ell_{i_0+2} + \ell_{i_0+3} \geq \Gamma_b(CT)$. The optimality of g implies $\ell_{i_0} + \ell_{i_0+1} + \ell_{i_0+2} + \ell_{i_0+3} = 6$, so that we have $\sigma(f) = \sigma(g)$ and f satisfies the property (item 1) of the lemma, as required (see Figure 14).

2. x_i is a stem of a sub-caterpillar CT' of CT, of type CT_5^4.
In that case, $\ell_{i_0} + \ell_{i_0+1} + \ell_{i_0+2} + \ell_{i_0+3} \leq 6$, for otherwise we could replace the g-values of every leaf of X by the value 1, and would get a minimal dominating broadcast on CT, with cost $\sigma(g) > \Gamma_b(CT)$, a contradiction with the optimality of g. On the other hand, if the equality $\ell_{i_0} + \ell_{i_0+1} + \ell_{i_0+2} + \ell_{i_0+3} = 6$ holds, then we consider the mapping f obtained from g by replacing the g-values of the leaves of $CT[i_0, i_0+3]$ by the value 1. The mapping f is a minimal dominating broadcast on CT with cost $\sigma(f) = \sigma(g)$ and satisfies $f(y_i^1) = f(y_i^2) = 1$. Hence, we assume in what follows, $\ell_{i_0} + \ell_{i_0+1} + \ell_{i_0+2} + \ell_{i_0+3} = 5$, and we distinguish two cases depending on the value of $g(y_i^1)$ and $g(y_i^2)$.

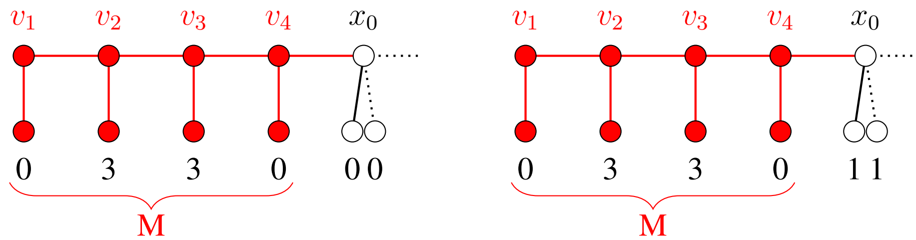
   (a) $g(y_i^1) = g(y_i^2) = 0$.
   In that case, $X = CT[i-3, i]$ with $u = y_{i-1}^1$ and $u' = y_{i-2}^1$, or $X = CT[i, i+3]$ with $u = y_{i+1}^1$ and $u' = y_{i+2}^1$. In the first case, and since $\ell_{i-3} + \ell_{i-2} + \ell_{i-1} + \ell_i = 5$ holds, we deduce that $CT[i-3, i]$ is of type CT_5^4, $g(y_i^1) = \theta_i^4(y_i^1)$ and $g(y_i^2) = \theta_i^4(y_i^2)$, in which case $CT' = X = F_i^4$ and the restriction of g on CT' is θ_i^4. In the second case, and since $\ell_i + \ell_{i+1} + \ell_{i+2} + \ell_{i+3} = 5$ holds, we also deduce that $CT[i, i+3]$ is of type CT_5^4, $g(y_i^1) = \theta_i^1(y_i^1)$ and $g(y_i^2) = \theta_i^1(y_i^2)$, in which case $CT' = X = F_i^1$ and the restriction of g on CT' is θ_i^1.

   (b) $g(y_i^1) = 3$ and $g(y_i^2) = 0$.
   In that case, $u = y_i^1$ and $u' \in \{y_{i-1}^1, y_{i+1}^1\}$. The case $u' = y_{i-1}^1$, leads to $PB(y_i^1) = L(x_{i+1})$ and $PB(y_{i-1}^1) = L(x_{i-2})$, that is $X = CT[i-2, i+1]$. Since $\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} = 5$ holds, $CT[i-2, i+1]$ is of type CT_5^4, $g(y_i^1) = \theta_i^3(y_i^1)$ and $g(y_i^2) = \theta_i^3(y_i^2)$, in which case $CT' = X = F_i^3$ and the restriction of g on CT' is θ_i^3. The case $u' = y_{i+1}^1$, implies $PB(y_i^1) = L(x_{i-1})$ and $PB(y_{i+1}^1) = L(x_{i+2})$, that is $X = CT[i-1, i+2]$. Since $\ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} = 5$ holds, $CT[i-1, i+2]$ is of type CT_5^4, $g(y_i^1) = \theta_i^2(y_i^1)$ and $g(y_i^2) = \theta_i^2(y_i^2)$, in which case $CT' = X = F_i^2$ and the restriction of g on CT' is θ_i^2.

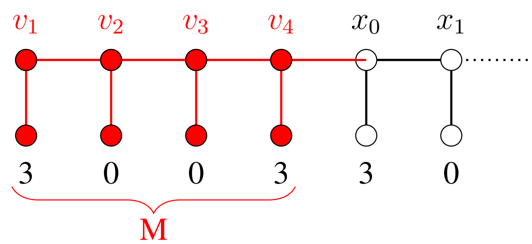
This completes the proof. □

Proof of Lemma 3.10. Let CT be a caterpillar of length $n \geq 4$, with no trunks and containing the patterns 1 and 2^+, and let $v_0v_1v_2v_3$ be the spine of the sub-caterpillar M, where w_i is the leaf adjacent to v_i for $i = 0, \ldots, 3$. Proving the equality $\Gamma_b(CT) = \Gamma_b(CT[M/\emptyset, i]) + 6$, is equivalent to proving both inequalities: (1) $\Gamma_b(CT) + 6 \leq \Gamma_b(CT[\emptyset/M, i])$ and (2) $\Gamma_b(CT) - 6 \leq \Gamma_b(CT[M/\emptyset, i])$.

1. Let f be a good Γ_b-broadcast on the caterpillar CT satisfying Lemmas 3.8 and 3.9. To prove (1), it is enough to find a minimal dominating broadcast g on $CT[\emptyset/M, i]$ with cost $\Gamma_b(CT) + 6$.
If $i = 0$, then either $f(y_0^j) \in \{0, 1\}$ for every $j = 1, \ldots, \ell_0$ (that is, $f(y_0^j) = 0$ for every $j = 1, \ldots, \ell_0$ or $f(y_0^j) = 1$ for every $j = 1, \ldots, \ell_0$), or $f(y_0^1) = 3$ (and then $f(y_0^j) = 0$ for every $j = 2, \ldots, \ell_0$). We distinguish two cases depending on the value of $f(y_0^j), \forall j \in \{1, \ldots, \ell_0\}$.



(a)



(b)

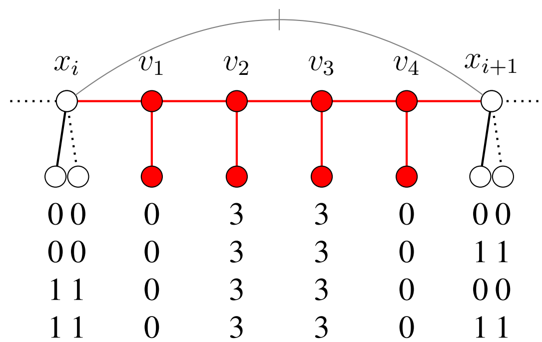
Figure 15: Illustration for the proof of Lemma 3.10, Case 1 $i = 0$, Cases (a) and (b).

(a) $f(y_0^j) = 0$ (resp. $f(y_0^j) = 1$) for every $j = 1, \ldots, \ell_0$.
In that case, $PB_f(y_1^1) = L(x_0)$ (resp. $PB_f(y_0^j) = \{y_0^j\}$ for every $j = 1, \ldots, \ell_0$ when $\ell_0 > 1$, or $PB_f(y_0^1) = \{x_0\}$ when $\ell_0 = 1$). We consider the mapping g defined by $g(w_1) = g(w_2) = 3$, $g(w_0) = g(w_3) = g(v_i) = 0$ for $i = 0, 1, 2, 3$, and $g(u) = f(u)$ otherwise (see Figure 15.(a)). We have $PB_g(w_1) = \{w_0\}$ and $PB_g(w_2) = \{w_3\}$, which implies that g is a minimal dominating broadcast on $CT[\emptyset/M, i]$ with cost $\Gamma_b(CT) + 6$.

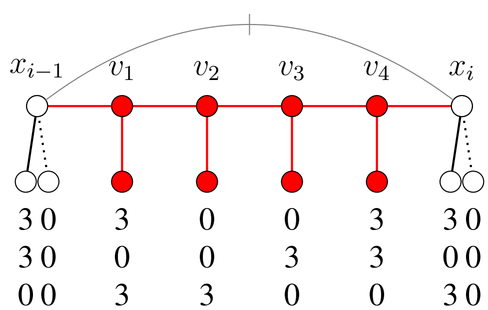
(b) $f(y_0^1) = 3$.
In that case, $PB_f(y_0^1) = L(x_1)$ in CT and we consider the mapping g defined by $g(w_0) = g(w_3) = 3$, $g(w_1) = g(w_2) = g(v_i) = 0$ for $i = 0, 1, 2, 3$, and $g(u) = f(u)$ otherwise (see Figure 15.(b)). We have $PB_g(w_0) = \{w_1\}$ and $PB_g(w_3) = \{w_2\}$, which implies that g is a minimal dominating broadcast on $CT[\emptyset/M, i]$ with cost $\Gamma_b(CT) + 6$.

Let $i \in \{1, \ldots, n\}$. We distinguish four cases :

(a) $f(y_{i-1}^j)$ and $f(y_i^k) \in \{0, 1\}$ for every $j = 1, \ldots, \ell_{i-1}$ and $k = 1, \ldots, \ell_i$.
In that case, every leaf y_{i-1}^j (resp. y_i^k) is either its own private neighbor or is a private neighbor of y_{i-2}^1 (resp. y_{i+1}^1). We consider the mapping g defined as in Case 1a (see Figure 16.(a)).



(a) Case (a)



(b) Cases (b)-(d)

Figure 16: Illustration for the proof of Lemma 3.10, Case 1 $i \neq 0$, Cases (a)-(d).

(b) $f(y^1_{i-1}) = f(y^1_{y_i}) = 3$.
In that case, $PB_f(y^1_{i-1}) = L(x_{i-2})$ and $PB_f(y^1_i) = L(x_{i+1})$ in CT. We consider the mapping g defined as in Case 1b (see Figure 16.(b)).

(c) $f(y^1_{i-1}) = 3$ and $f(y^k_i) \in \{0, 1\}$ for every $k = 1, \dots, \ell_i$.
In that case, $PB_f(y^1_{i-1}) = L(x_i)$ in CT. We consider the mapping g defined by $g(w_2) = g(w_3) = 3$, $g(w_0) = g(w_1) = g(v_i) = 0$, for $i = 0, 1, 2, 3$, and $g(u) = f(u)$ otherwise (see Figure 16.(b)). We have $PB_g(y^1_{i-1}) = \{w_0\}$, $PB_g(w_2) = \{w_1\}$ and $PB_g(w_3) = L(x_i)$. Therefore, g is a minimal dominating broadcast on $CT[\emptyset/M, i]$ with cost $\Gamma_b(CT) + 6$.

(d) $f(y^j_{i-1}) \in \{0, 1\}$ for every $j = 1, \dots, \ell_i$ and $f(y^1_i) = 3$.
In that case, $PB_f(y^1_i) = L(x_{i-1})$ in CT. We consider the mapping g defined by $g(w_0) = g(w_1) = 3$, $g(w_2) = g(w_3) = g(v_i) = 0$ for $i = 0, 1, 2, 3$, and $g(u) = f(u)$ otherwise (see Figure 16.(b)). We have $PB_g(w_0) = L(x_{i-1})$, $PB_g(w_1) = \{w_2\}$ and $PB_g(y^1_i) = \{w_3\}$. Therefore, g is a minimal dominating broadcast on $CT[\emptyset/M, i]$ with cost $\Gamma_b(CT) + 6$.

2. Let f be a good Γ_b-broadcast on the caterpillar CT satisfying Lemmas 3.8 and 3.9. We prove the existence of a minimal dominating broadcast g on $CT[M/\emptyset, 0]$ with cost $\sigma(g) \geq \Gamma_b(CT) - 6$.
We distinguish two cases, depending on whether $i \in \{0, n-4\}$ or not.
Assume first $i = 0$ (the case $i = n - 4$ is similar by symmetry). We consider two subcases.

(a) $f(y^1_0) = f(y^1_3) = 0$ and $f(y^1_1) = f(y^1_2) = 3$.
In that case, $PB_f(y^1_1) = \{y^1_0\}$ and $PB_f(y^1_2) = \{y^1_3\}$. The mapping g, defined as the restriction of f on $CT[M/\emptyset, 0]$ remains a minimal dominating broadcast on $CT[M/\emptyset, 0]$ with cost $\Gamma_b(CT) - 6$.
Similarly, if $f(y^1_0) = f(y^1_3) = 3$ and $f(y^1_1) = f(y^1_2) = 0$, then $PB_f(y^1_0) = \{y^1_1\}$ and $PB_f(y^1_3) = \{y^1_1\}$. The previous broadcast g remains available.

(b) $f(y^1_0) = 3$, $f(y^1_2) = 1$ and $f(y^1_1) = f(y^1_3) = 0$.
In that case, $PB_f(y^1_0) = \{y^1_1\}$, and $PB_f(y^1_4) = \{y^1_3\}$ and and $PB_f(y^1_2) = \{y^1_2\}$, where $f(y^1_4) = 3$. If $n = 4$, then $CT[M/\emptyset, 0] = CT[4, 4]$ and by Theorem 2.1, $\Gamma_b(CT[M/\emptyset, 0]) = \ell_4$. The relation $\ell_4 = 1$ must be held, for otherwise we could set $h(y^1_1) = h(y^1_2) = 3$, $h(y^j_4) = 1$ for every $j = 1, \dots, \ell_4$ and $h(u) = 0$ otherwise which would be a minimal dominating broadcast with cost $6 + \ell_4$, contradicting the optimality of f when $\ell_4 > 1$. Thus, $\Gamma_b(CT) - 6 = 1 = \Gamma_b(CT[M/\emptyset, 0])$.
Since y^1_4 has one private side by Lemma 3.7(2), we have $n \neq 5$. Let then $n \geq 6$. We have $CT[3, 6] = CT(1, 1, 1, 1)$ or $CT[3, 6]$ is a caterpillar of type CT^4_5, different from F^1_i, by Lemmas 3.8 and 3.9 and by the fact that $\ell_3 = 1$. It follows, $f(y^1_5) = 3$ and $f(u) = 0$ for every other vertex of $CT[3, 6]$. On $CT[M/\emptyset, 0]$, consider a mapping g, obtained from f by replacing the f-values of y^1_5 and y^1_6 by $g(y^1_5) = 0$ and $g(y^j_6) = 1$ for every $j = 1, \dots, \ell_6$. So we have $PB_g(y^1_4) = L(x_5)$ and $PB_g(y^j_6) = \{y^j_6\}$ for every $j = 1, \dots, \ell_6$, which allows to say that g is a minimal dominating broadcast on $CT[M/\emptyset, 0]$ with cost $\sigma(g) = \Gamma_b(CT) + \ell_6 - 7 \geq \Gamma_b(CT) - 6$.

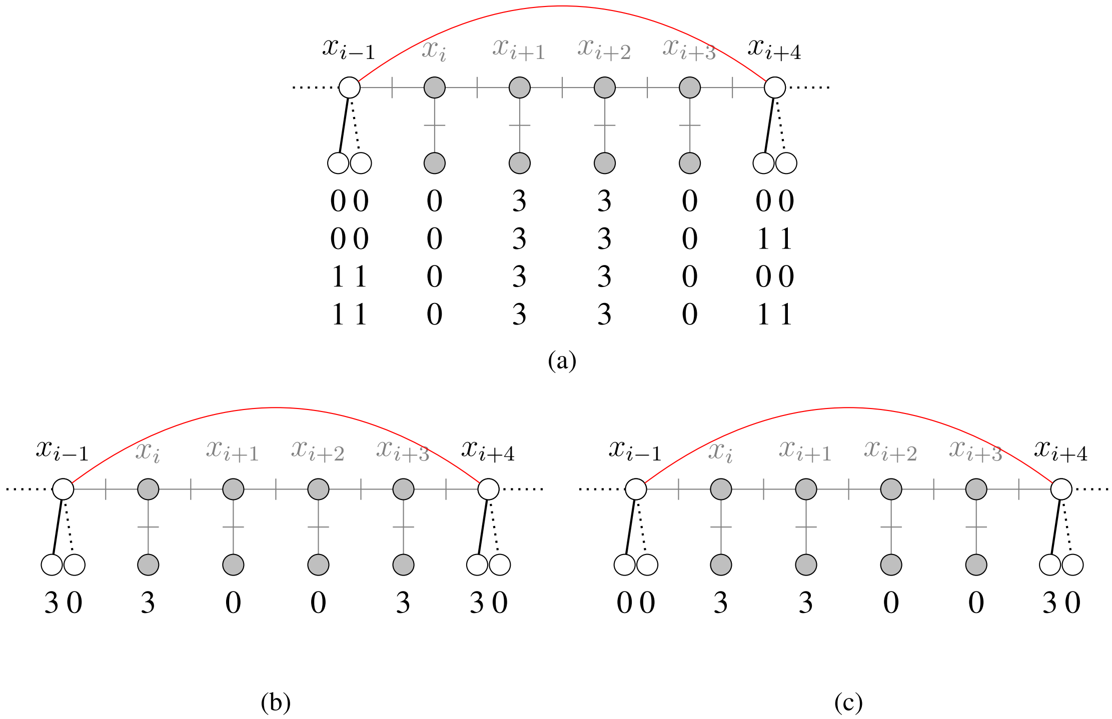


Figure 17: Illustration for the proof of Lemma 3.10, Case 2 $i \neq 0$, Case (a)

Let now $i \in \{1, \dots, n-1\}$. We distinguish five sub-cases.

(a) $f(y_i^1) = f(y_{i+3}^1) = 0$ and $f(y_{i+1}^1) = f(y_{i+2}^1) = 3$.
In that case, $PB_f(y_{i+1}^1) = \{y_i^1\}$ and $PB_f(y_{i+2}^1) = \{y_{i+3}^1\}$. The mapping g defined as the restriction of f on $CT[M/\emptyset, i]$ remains a minimal dominating broadcast on $CT[M/\emptyset, i]$ with cost $\Gamma_b(CT) - 6$ (see Figure 17.(a)).
Similarly, if $f(y_i^1) = f(y_{i+3}^1) = 3$ and $f(y_{i+1}^1) = f(y_{i+2}^1) = 0$, then $PB_f(y_i^1) = \{y_{i+1}^1\}$ and $PB_f(y_{i+3}^1) = \{y_{i+2}^1\}$. The previous broadcast g remains available (see Figure 17.(b)).
If $f(y_i^1) = f(y_{i+1}^1) = 3$ and $f(y_{i+2}^1) = f(y_{i+3}^1) = 0$, then $PB_f(y_{i+1}^1) = \{y_{i+2}^1\}$, $PB_f(y_i^1) = L(x_{i-1})$ and $PB_f(y_{i+4}^1) = \{y_{i+3}^1\}$, with $f(y_{i+4}) = 3$. By considering again the same mapping g, we obtain $PB_g(y_{i+4}^1) = L(x_{i-1})$. Hence, g is a minimal dominating broadcast on $CT[M/\emptyset, 0]$ with cost $\sigma(g) = \Gamma_b(CT) - 6$ (see Figure 17.(c)).

(b) $f(y_i^1) = f(y_{i+1}^1) = 3$, $f(y_{i+2}^1) = 0$ and $f(y_{i+3}^1) = 1$.
In that case, $PB_f(y_i^1) = L(x_{i-1})$, $PB_f(y_{i+1}^1) = \{y_{i+2}^1\}$ and $PB_f(y_{i+3}^1) = \{y_{i+3}^1\}$. Consider the mapping g on $CT[M/\emptyset, 0]$, obtained from f by replacing, for every $j = 1, \dots, \ell_{i-1}$, the f-values of y_{i-1}^j by 1 (see Figure 18.(a)). We have $PB_g(y_{i-1}^j) = \{x_{i-1}\}$ or $PB_g(y_{i-1}^j) = \{y_{i-1}^j\}$ for every $j = 1, \dots, \ell_{i-1}$. The mapping g is then a minimal dominating broadcast with cost $\sigma(g) = \Gamma_b(CT) - 7 + \ell_{i-1} \geq \Gamma_b(CT) - 6$.

(c) $f(y_i^1) = 3$, $f(y_{i+1}^1) = f(y_{i+3}^1) = 0$ and $f(y_{i+2}^1) = 1$.

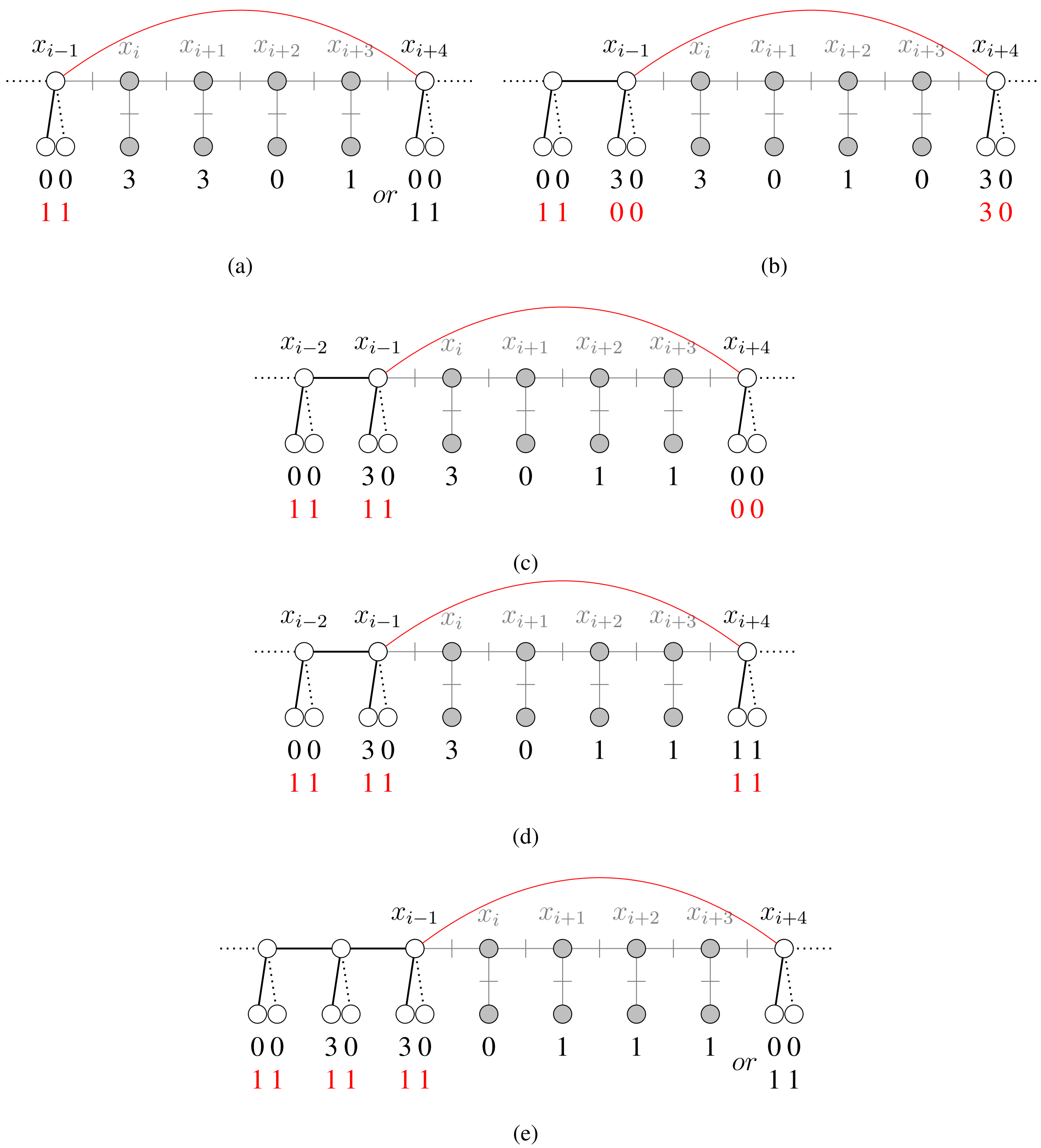


Figure 18: Illustration for the proof of Lemma 3.10, Case 2 $i \neq 0$, Cases (b)-(e).

In that case, by Lemma 3.7(3), $f(y^1_{i-1}) = 3$ which gives $f(y^j_{i-2}) = 0$ for every $j = 1, \dots, \ell_{i-2}$. Hence, $PB_f(y^1_{i-1}) = \{y^1_{i-2}\}$, $PB_f(y^1_i) = \{y^1_{i+1}\}$, $PB_f(y^1_{i+2}) = \{y^1_{i+2}\}$ and $PB_f(y^1_{i+4}) = \{y^1_{i+3}\}$, with $f(y^1_{i+4}) = 3$. Consider the mapping g on $CT[M/\emptyset, 0]$, obtained from f by replacing, for every $j = 1, \dots, \ell_{i-2}$, the f-values of y^j_{i-2} by 1 and the f-value of y^1_{i-1} by 0 (see Figure 18.(b)). We have $PB_g(y^j_{i+4}) = L(x_{i-1})$ and $PB_g(y^j_{i-2}) = \{y^j_{i-2}\}$ for every $j = 1, \dots, \ell_{i-2}$. The mapping g is then a minimal dominating broadcast with cost $\sigma(g) = \Gamma_b(CT) - 7 + \ell_{i-2} \geq \Gamma_b(CT) - 6$.

(d) $f(y^1_i) = 3$, $f(y^1_{i+1}) = 0$ and $f(y^1_{i+2}) = f(y^1_{i+3}) = 1$.
In that case, by Lemma 3.7(3), $f(y^1_{i-1}) = 3$ and thus $f(y^j_{i-2}) = 0$ for every $j = 1, \dots, \ell_{i-2}$. Hence, $PB_f(y^1_{i-1}) = L(x_{i-2})$, $PB_f(y^1_i) = \{y^1_{i+1}\}$, $PB_f(y^1_{i+2}) = \{y^1_{i+2}\}$, $PB_f(y^1_{i+3}) = \{y^1_{i+3}\}$ and $f(y^1_{i+4}) \neq 3$. Consider the mapping g on $CT[M/\emptyset, 0]$, obtained from f by replacing, for every $j = 1, \dots, \ell_{i-2}$, the f-values of y^j_{i-2} by 1 and for every $k = 1, \dots, \ell_{i-1}$ the f-value of y^k_{i-1} by 1 (see Figure 18.[(c) and (d)]). We infer $PB_g(y^j_{i-2}) = \{y^j_{i-2}\}$, $j = 1, \dots, \ell_{i-2}$ and $PB_g(y^k_{i-1}) = \{y^k_{i-1}\}$ for every $k = 1, \dots, \ell_{i-1}$. The mapping g is then a minimal dominating broadcast with cost $\sigma(g) = \Gamma_b(CT) - 8 + \ell_{i-1} + \ell_{i-2} \geq \Gamma_b(CT) - 6$.

(e) $f(y^1_i) = 0$, $f(y^1_{i+1}) = f(y^1_{i+2}) = f(y^1_{i+3}) = 1$.
In that case, $f(y^1_{i-1}) = f(y^1_{i-2}) = 3$, $f(y^j_{i-3}) = 0$ for every $j = 1, \dots, \ell_{i-3}$, and $f(y^1_{i+4}) \neq 3$. Moreover, we have $PB_f(y^1_{i-2}) = L(x_{i-3})$ and $PB_f(y^1_{i-1}) = \{y^1_i\}$. Consider the mapping g on $CT[M/\emptyset, 0]$, obtained from f by replacing, the f-values of y^j_{i-3}, y^k_{i-2} and y^l_{i-1} by 1 for every $j = 1, \dots, \ell_{i-3}$, $k = 1, \dots, \ell_{i-2}$, $l = 1, \dots, \ell_{i-1}$ (see Figure 18.(e)). The mapping g is a minimal dominating broadcast with cost $\sigma(g) = \Gamma_b(CT) - 9 + \ell_{i-3} + \ell_{i-2} + \ell_{i-1} \geq \Gamma_b(CT) - 6$.

In each case, we proved the existence of a minimal dominating broadcast g on $CT[M/\emptyset, 0]$ with cost $\sigma(g) \geq \Gamma_b(CT) - 6$. Therefore, $\Gamma_b(CT) - 6 \leq \Gamma_b(CT[M/\emptyset, 0])$, as required. This completes the proof. □

Proof of Lemma 3.12. Let CT^r be the reduced caterpillar of CT and let d_i be a stem of CT^r with $m_i = 2$. Consider a Γ_b-broadcast f on CT^r satisfying the properties of Theorem 3.3.

1. $P_f(d_i) = \theta^j_i$ for some $j \in \{1, \dots, 4\}$.
In that case, $CT^i_f = F^j_i$ and in the sub-caterpillar $F^j_i = CT^r[i-j+1, i-j+4]$ of type CT^4_5, we have by Theorem 3.3(4.b), the only f-broadcast vertices are t^1_{i-j+2} and t^1_{i-j+3}, with $f(t^1_{i-j+2}) = f(t^1_{i-j+3}) = 3$. Therefore,

$$\sigma(f) = \sum_{v \in V(CT^r[0,i-j])} f(v) + 6 + \sum_{v \in V(CT^r[i-j+5,n])} f(v).$$

Consider now a Γ_b-broadcast g on $CT^r[CT^i_f/K_{1,6}, i-j+1]$. Thanks to Theorem 3.3(3), $g(t^s_{i-j+1}) = 1$ for every $s = 1, \dots, 6$. Then,

$$\sigma(g) = \sum_{v \in V(CT^r[0,i-j])} g(v) + 6 + \sum_{v \in V(CT^r[i-j+2,n-3])} g(v).$$

We have $\sum_{v\in V(CT^r[0,i-j])} f(v) = \sum_{v\in V(CT^r[0,i-j])} g(v)$. Indeed, assume first

$$\sum_{v\in V(CT^r[0,i-j])} f(v) > \sum_{v\in V(CT^r[0,i-j])} g(v).$$

In CT^r, the private f-borders of the f-broadcast vertices t^1_{i-j+2} and t^1_{i-j+3} lie in F^j_i, and apart from these f-private borders, F^j_i does not contain any other f-private borders. Then the mapping h defined by $h(v) = f(v)$ if $v \in V(CT^r[0, i-j])$ and $h(v) = g(v)$ otherwise, would be a minimal dominating broadcast on $CT^r[CT^i_f/K_{1,6}, i-j+1]$ with cost $\sigma(h) > \sigma(g)$, a contradiction with the optimality of g. Now if

$$\sum_{v\in V(CT^r[0,i-j])} f(v) < \sum_{v\in V(CT^r[0,i-j])} g(v)$$

then, the mapping k defined by $k(v) = g(v)$ if $v \in V(CT^r[0, i-j])$, and $k(v) = f(v)$ otherwise, would be a minimal dominating broadcast on CT^r with cost $\sigma(k) > \sigma(f)$, again a contradiction with the optimality of f.
By the same arguments as above, we can prove that

$$\sum_{v\in V(CT^r[i-j+5,n])} f(v) = \sum_{v\in V(CT^r[i-j+2,n-3])} g(v).$$

It follows, $\sigma(f) = \sigma(g)$.

2. $P_f(d_i) = \theta^5_i$.
In that case, $CT^i_f = CT[i,i]$ and $f(t^1_i) = f(t^2_i) = 1$. Moreover, each of these f-broadcast vertices is its own bordering private f-neighbor and apart these two f-private borders, $CT[i,i]$ does not contain any other f-private borders. Let g be a Γ_b-broadcast on $CT^r[CT^i_f/K_{1,6}, i]$ as defined in Item 1, that is, $g(t^s_i) = 1$ for every $s = 1, \ldots, 6$. Again, each of these six g-broadcast vertices is its own bordering private g-neighbor and $CT[i,i]$ does not contain any other private g-neighbor. We have,

$$\sigma(f) = \sum_{v\in V(CT^r[0,i-1])} f(v) + 2 + \sum_{v\in V(CT^r[i+1,n])} f(v),$$

and

$$\sigma(g) = \sum_{v\in V(CT^r[0,i-1])} g(v) + 6 + \sum_{v\in V(CT^r[i+1,n])} g(v).$$

By the same arguments as in the proof of Item 1, we get

$$\sum_{v\in V(CT^r[0,i-1])} f(v) = \sum_{v\in V(CT^r[0,i-1])} g(v)$$

and

$$\sum_{v\in V(CT^r[i+1,n])} f(v) = \sum_{v\in V(CT^r[i+1,n])} g(v).$$

Hence, $\sigma(f) = \sigma(g) - 4$.

This completes the proof. □

Proof of Lemma 3.13. Let g be a Γ_b-broadcast on CT^r satisfying the properties of Theorem 3.3 and let $d_1 = z_i$ for some index $i \in \{0, \dots, k\}$.

1. Assume that $m_{i-3} = m_{i-2} = m_{i-1} = 1$. Since the pattern 1111 does not occur in CT^r, we have $m_{i-4} \geq 3$ and then $g(t^j_{i-4}) = 1$ for every $j = 1, \dots, m_{i-4}$. Moreover, $P_f(d_1) = \theta^5_i$ cannot hold, because otherwise $g(t^1_{i-3}) = g(t^1_{i-2}) = g(t^1_{i-1}) = g(t^1_i) = g(t^2_i) = 1$ and the mapping h obtained from g by setting $h(t^1_{i-3}) = h(t^1_i) = h(t^2_i) = 0$, $h(t^1_{i-2}) = h(t^1_{i-1}) = 3$ and $h(u) = g(u)$, otherwise, the mapping h would be a minimal dominating broadcast on CT^r with cost $\sigma(h) = \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with the optimality of g.
   If $P_g(d_1) = \theta^1_i$, then $g(t^1_i) = g(t^2_i) = g(t^1_{i+3}) = 0$, $g(t^1_{i-3}) = g(t^1_{i-2}) = g(t^1_{i-1}) = 1$ and $g(t^1_{i+1}) = g(t^1_{i+2}) = 3$. We define a mapping f, obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-3, i+3]$ as follows. We set $f(t^1_{i-3}) = 0$, $f(t^1_{i-2}) = f(t^1_{i-1}) = 3$, $f(t^1_{i+1}) = f(t^1_{i+2}) = f(t^1_{i+3}) = 1$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g)$.
   If $P_g(d_1) = \theta^2_i$, then $g(t^1_{i-1}) = g(t^2_i) = g(t^1_{i+2}) = 0$, $g(t^1_{i-3}) = g(t^1_{i-2}) = 1$ and $g(t^1_i) = g(t^1_{i+1}) = 3$. We define a mapping f, obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-3, i+2]$ as follows. We set $f(t^1_{i-3}) = f(t^1_i) = 0$, $f(t^1_{i+1}) = f(t^1_{i+2}) = 1$, $f(t^1_{i-2}) = f(t^1_{i-1}) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g)$.
   If $P_g(d_1) = \theta^3_i$, then $g(t^1_{i-2}) = g(t^2_i) = g(t^2_{i+1}) = 0$, $g(t^1_{i-3}) = 1$ and $g(t^1_{i-1}) = g(t^1_i) = 3$. We define a mapping f, obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-3, i+1]$ as follows. We set $f(t^1_{i-3}) = f(t^1_i) = 0$, $f(t^1_{i+1}) = 1$, $f(t^1_{i-2}) = f(t^1_{i-1}) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g)$. Hence, CT^r admits a Γ_b-broadcast f such that $P_f(d_1) = \theta^4_i$.

2. Assume that $m_{i-2} = m_{i-1} = 1$ and $m_{i+1} = 1$. Since $m_{i-3} \geq 3$, we have $P_g(d_1) \neq \theta^4_i$. We also have $P_g(d_1) \neq \theta^5_i$, because otherwise $g(t^1_{i-2}) = g(t^1_{i-1}) = g(t^1_i) = g(t^2_i) = 1$, $g(t^1_{i+1}) \in \{0, 1\}$ and the mapping h obtained from g by setting $h(t^1_{i-2}) = h(t^2_i) = h(t^1_{i+1}) = 0$, $h(t^1_{i-1}) = h(t^1_i) = 3$, and $h(u) = g(u)$ otherwise, the mapping h would be a minimal dominating broadcast on CT^r with cost $\sigma(h) \geq \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with the optimality of g.
   If $P_g(d_1) = \theta^1_i$, then $g(t^1_i) = g(t^2_i) = g(t^1_{i+3}) = 0$, $g(t^1_{i-2}) = g(t^1_{i-1}) = 1$ and $g(t^1_{i+1}) = g(t^1_{i+2}) = 3$. We define a mapping f, obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-2, i+3]$ as follows. We set $f(t^1_{i-2}) = f(t^1_{i+1}) = 0$, $f(t^1_{i+2}) = f(t^1_{i+3}) = 1$, $f(t^1_{i-1}) = f(t^1_i) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g)$.
   If $P_g(d_1) = \theta^2_i$, then $g(t^1_{i-1}) = g(t^2_i) = g(t^1_{i+2}) = 0$, $g(t^1_{i-2}) = 1$ and $g(t^1_i) = g(t^1_{i+1}) = 3$. We define a mapping f, obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-2, i+2]$ as follows. We set $f(t^1_{i-2}) = f(t^1_{i+1}) = 0$, $f(t^1_{i+2}) = 1$, $f(t^1_{i-1}) = f(t^1_i) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating

broadcast on CT^r with cost $\sigma(f) = \sigma(g)$. Hence, CT^r admits a Γ_b-broadcast f such that $P_f(d_1) = \theta_i^3$.

3. Assume that $m_{i-1} = 1$, $m_{i+1} = m_{i+2} = 1$ and $m_{i-2} \neq 1$. Since $m_{i-2} \geq 3$, we have $P_g(d_1) \notin \{\theta_i^3, \theta_i^4\}$.
   If $P_g(d_1) = \theta_i^1$, and since the pattern 1111 does not occur in CT^r, then $m_{i+3} = 1$, $m_{i+4} \geq 2$, $g(t_i^1) = g(t_i^2) = g(t_{i+3}^1) = 0$, $g(t_{i-1}^1) = g(t_{i+4}^j) = 1$ for every $j \in \{1, \dots, m_{i+4}\}$, and $g(t_{i+1}^1) = g(t_{i+2}^1) = 3$. We define a mapping f, obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-1, i+3]$ as follows. We set $f(t_{i-1}^1) = f(t_{i+2}^1) = 0$, $f(t_{i+3}^1) = 1$, $f(t_i^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g)$.
   If $P_g(d_1) = \theta_i^5$, then $g(t_{i-1}^1) = g(t_i^1) = g(t_i^2) = 1$, but $g(t_{i+1}^1) \neq 1$ and $g(t_{i+2}^1) \neq 1$, because otherwise the mapping h obtained from g by setting $h(t_{i-1}^1) = h(t_i^2) = h(t_{i+2}^1) = 0$, $h(t_i^1) = h(t_{i+1}^1) = 3$, and $h(u) = g(u)$ otherwise, the mapping h would be a minimal dominating broadcast on CT^r with cost $\sigma(h) = \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with the optimality of g. Therefore, $(g(t_{i+1}^1), g(t_{i+2}^1)) \in \{(0, 3), (1, 0)\}$. Assume first $(g(t_{i+1}^1), g(t_{i+2}^1)) = (0, 3)$. Thanks to Theorem 3.3, we must have $g(t_{i+3}^1) = 3$ and $g(t_{i+4}^1) = 0$, and since the pattern 1111 does not occur in CT^r, we also have $m_{i+3} + m_{i+4} \geq 3$. We now define a mapping f obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-1, i+4]$ as follows. We set $f(t_{i-1}^1) = f(t_i^2) = f(t_{i+2}^1) = 0$, $f(t_{i+3}^j) = f(t_{i+4}^k) = 1$ for every $j \in \{1, \dots, m_{i+3}\}$, $k \in \{1, \dots, m_{i+4}\}$, $f(t_i^1) = f(t_{i+1}^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g) - 9 + 6 + m_{i+3} + m_{i+4} = \sigma(g) + m_{i+3} + m_{i+4} - 3$. The optimality of g implies $m_{i+3} + m_{i+4} = 3$, and thus $\sigma(f) = \sigma(g)$.
   For the case $(g(t_{i+1}^1), g(t_{i+2}^1)) = (1, 0)$, we have, $g(t_{i+3}^1) = g(t_{i+4}^1) = 3$ and $g(t_{i+5}^j) = 0$ for every $j \in \{1, \dots, m_{i+5}\}$. We again define a mapping f obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i-1, i+5]$ as follows. We set $f(t_{i-1}^1) = f(t_i^2) = f(t_{i+2}^1) = 0$, $f(t_{i+3}^j) = f(t_{i+4}^k) = f(t_{i+5}^\ell) = 1$ for every $j \in \{1, \dots, m_{i+3}\}$, $k \in \{1, \dots, m_{i+4}\}$, $\ell \in \{1, \dots, m_{i+5}\}$, $f(t_i^1) = f(t_{i+1}^1) = 3$, and $f(u) = g(u)$ otherwise. As previously, we have, $m_{i+3} + m_{i+4} = 3$ and the mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g) - 10 + 6 + m_{i+3} + m_{i+4} + m_{i+5} \geq \sigma(g) - 4 + 3 + m_{i+5}$. The optimality of g implies $m_{i+5} = 1$, and thus $\sigma(f) = \sigma(g)$. Hence, CT^r admits a Γ_b-broadcast f such that $P_f(d_1) = \theta_i^2$.

4. Assume that $m_{i+1} = m_{i+2} = m_{i+3} = 1$ and $m_{i-1} \neq 1$. Since the pattern 1111 does not occur in CT^r, we have $m_{i+4} \geq 2$ et since $m_{i-1} \geq 3$, we also have $P_g(d_1) \notin \{\theta_i^2, \theta_i^3, \theta_i^4\}$.
   If $P_g(d_1) = \theta_i^5$, then $g(t_i^1) = g(t_i^2) = 1$ and equalities $g(t_{i+1}^1) = g(t_{i+2}^1) = g(t_{i+3}^1) = 1$ cannot hold, because otherwise the mapping h obtained from g by setting $h(t_i^1) = h(t_i^2) = h(t_{i+3}^1) = 0$, $h(t_{i+1}^1) = h(t_{i+2}^1) = 3$, and $h(u) = g(u)$ otherwise, would be a minimal dominating broadcast on CT^r with cost $\sigma(h) = \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with the optimality of g. The case $g(t_{i+1}^1) = 0$ and $g(t_{i+2}^1) = 3$ leads to $g(t_{i+3}^1) = 3$ and $g(t_{i+4}^1) = 0$, and then we can define a mapping f obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i, i + 4]$ as follows. We set $f(t_i^1) = f(t_i^2) =$

$f(t^1_{i+3}) = 0$, $f(t^j_{i+4}) = 1$ for every $j \in \{1, \dots, m_{i+4}\}$, $f(t^1_{i+1}) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g) - 5 + 3 + m_{i+4} = \sigma(g) + m_{i+4} - 2$. The optimality of g implies $m_{i+4} = 2$, and thus $\sigma(f) = \sigma(g)$.
The case $g(t^1_{i+1}) = 1$ and $g(t^1_{i+2}) = 0$ leads to $g(t^1_{i+3}) = g(t^1_{i+4}) = 3$ and $g(t^1_{i+5}) = 0$, and then we can define a mapping f obtained from g by modifying the g-values of the leaves of the sub-caterpillar $CT[i, i+5]$ as follows. We set $f(t^1_i) = f(t^2_i) = f(t^1_{i+3}) = 0$, $f(t^j_{i+4}) = f(t^k_{i+5}) = 1$ for every $j \in \{1, \dots, m_{i+4}\}$ and $k \in \{1, \dots, m_{i+5}\}$, $f(t^1_{i+1}) = f(t^1_{i+2}) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g) - 9 + 6 + m_{i+4} + m_{i+5} = \sigma(g) + m_{i+4} + m_{i+5} - 3$. The optimality of g implies $m_{i+4} = 2$ and $m_{i+5} = 1$, and thus $\sigma(f) = \sigma(g)$.
The case $g(t^1_{i+1}) = g(t^1_{i+2}) = 1$ and $g(t^1_{i+3}) = 0$ leads to $g(t^1_{i+4}) = g(t^1_{i+5}) = 3$ and $g(t^1_{i+6}) = 0$, and then we can again define a mapping f obtained from g by modifying some g-values of the leaves of the sub-caterpillar $CT[i, i+6]$ as follows. We set $f(t^1_i) = f(t^2_i) = 0$, $f(t^j_{i+4}) = f(t^k_{i+5}) = f(t^\ell_{i+6}) = 1$ for every $j \in \{1, \dots, m_{i+4}\}$, $k \in \{1, \dots, m_{i+5}\}$ and $\ell \in \{1, \dots, m_{i+6}\}$, $f(t^1_{i+1}) = f(t^1_{i+2}) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g) - 10 + 6 + m_{i+4} + m_{i+5} + m_{i+6} = \sigma(g) + m_{i+4} + m_{i+5} + m_{i+6} - 4$. The optimality of g implies $m_{i+4} = 2$ and $m_{i+5} = m_{i+6} = 1$, and thus $\sigma(f) = \sigma(g)$. Hence CT^r admits a Γ_b-broadcast f such that $P_f(d_1) = \theta^1_i$.

5. This result is immediate from Lemma 3.9.

This completes the proof. □

Proof of Lemma 3.14. Let g be a Γ_b-broadcast on CT^r satisfying the properties of Theorem 3.3 and let $d_1 = z_{i_0}$ for some index $i \in \{0, \dots, k\}$.

1. If $P_g(d_1) = \theta^3_{i_0}$, then $g(t^1_{i_0-2}) = g(t^1_{i_0+1}) = 0$ and $g(t^1_{i_0-1}) = g(t^1_{i_0}) = 3$. Since $i_0 \in \{2, 3\}$, we can define, in the case $i_0 = 2$, a mapping f by setting $f(t^1_{i_0-1}) = 0$, $f(t^1_{i_0}) = f(t^2_{i_0}) = f(t^1_{i_0+1}) = 1$, $f(t^1_{i_0-2}) = 3$, and $f(u) = g(u)$ otherwise, and in the case $i_0 = 3$, $f(t^1_{i_0-1}) = f(t^1_{i_0}) = f(t^2_{i_0}) = f(t^1_{i_0+1}) = 1$, $f(t^1_{i_0-3}) = 3$, and $f(u) = g(u)$ otherwise. In both cases, f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g)$ and $P_f(d_1) \neq \theta^3_{i_0}$.
If $P_g(d_1) = \theta^4_{i_0}$, then $g(t^1_{i_0-3}) = g(t^1_{i_0}) = 0$ and $g(t^1_{i_0-2}) = g(t^1_{i_0-1}) = 3$. We define a mapping f by setting $f(t^1_{i_0-2}) = 0$, $f(t^1_{i_0-1}) = f(t^1_{i_0}) = f(t^2_{i_0}) = 1$, $f(t^1_{i_0-3}) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g)$, and $P_f(d_1) \neq \theta^4_{i_0}$.

2. From Item 1, we can assume without loss of generality that $P_g(d_1) \in \{\theta^1_{i_0}, \theta^2_{i_0}, \theta^5_{i_0}\}$.

   (a) Let $i_0 = 1$ and $d_1 \in F^2_1 = CT[0, 3]$. We have then $m_0 = m_2 = m_3 = 1$ and $m_1 = 2$. If $P_g(d_1) = \theta^1_1$, then $m_0 = m_2 = m_3 = m_4 = 1$, $m_1 = 2$, $g(t^1_1) = g(t^1_2) = g(t^1_4) = 0$, $g(t^1_0) = 1$ and $g(t^1_2) = g(t^1_3) = 3$. We define a mapping f by setting $f(t^1_0) = f(t^1_3) = 0$, $f(t^1_4) = 1$, $f(t^1_1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 4 + 4 = \sigma(g)$, and $P_f(d_1) = \theta^2_1$.

If $P_g(d_1) = \theta_1^5$, then $g(t_1^1) = g(t_1^2) = 1$ and equalities $g(t_2^1) = g(t_3^1) = 1$ cannot hold, because otherwise the mapping h obtained from g by setting $h(t_0^1) = h(t_1^2) = h(t_3^1) = 0$, $h(t_1^1) = h(t_2^1) = 3$, and $h(u) = g(u)$, otherwise the mapping h would be a minimal dominating broadcast on CT^r with cost $\sigma(h) = \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with the optimality of g. Hence, we get $(g(t_2^1), g(t_3^1)) \in \{(1,0),(0,3)\}$. The case $g(t_2^1) = 1$ and $g(t_3^1) = 0$ implies $m_4 + m_5 = 3$ and $m_6 = 1$, $g(t_4^1) = g(t_5^1) = 3$ and $g(t_6^1) = 0$. We define a mapping f by setting $f(t_0^1) = f(t_1^2) = 0$, $f(t_4^j) = f(t_5^k) = f(t_6^1) = 1$ for every $j = 1, \ldots, m_4$, $k = 1, \ldots, m_5$, $f(t_1^1) = f(t_2^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 10 + 7 + m_4 + m_5 = \sigma(g)$. The case $g(t_2^1) = 0$ and $g(t_3^1) = 3$ implies again $m_4 + m_5 = 3$, $g(t_4^1) = 3$ and $g(t_5^1) = 0$. We define a mapping f by setting $f(t_0^1) = f(t_1^2) = f(t_3^1) = 0$, $f(t_4^j) = f(t_5^k) = 1$ for every $j = 1, \ldots, m_4$, $k = 1, \ldots, m_5$, $f(t_1^1) = f(t_2^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r with cost $\sigma(f) = \sigma(g) - 9 + 6 + m_4 + m_5 = \sigma(g)$. Hence, in both cases, we get $P_f(d_1) = \theta_1^2$.

(b) Let $i_0 = 3$ and $d_1 \in F_3^2 = CT[2,5]$. We have then $m_0 = m_1 = m_2 = m_4 = m_5 = 1$ and $m_3 = 2$. If $P_g(d_1) = \theta_3^1$, then $m_6 = 1$, $g(t_1^1) = g(t_3^1) = g(t_6^1) = 0$, $g(t_2^1) = 1$ and $g(t_0^1) = g(t_4^1) = g(t_5^1) = 3$. We define a mapping f by setting $f(t_2^1) = f(t_5^1) = 0$, $f(t_6^1) = 1$, $f(t_3^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 4 + 4 = \sigma(g)$, and $P_f(d_1) = \theta_3^2$. If $P_g(d_1) = \theta_3^5$, then $g(t_1^1) = 0$, $g(t_2^1) = g(t_3^1) = g(t_3^2) = 1$ and $g(t_0^1) = 3$. Moreover, equalities $g(t_4^1) = g(t_5^1) = 1$ cannot hold, because otherwise the mapping h obtained from g by setting $h(t_1^1) = h(t_2^1) = h(t_3^2) = h(t_5^1) = 0$, $h(t_0^1) = h(t_3^1) = h(t_4^1) = 3$ and $h(u) = g(u)$, otherwise, the mapping h would be a minimal dominating broadcast on CT^r with cost $\sigma(h) = \sigma(g) - 8 + 9 = \Gamma_b(CT^r) + 1$, a contradiction with optimality of g. Therefore, $(g(t_4^1), g(t_5^1)) \in \{(1,0),(0,3)\}$. The case $g(t_4^1) = 1$ and $g(t_5^1) = 0$ implies $m_6 + m_7 = 3$, $m_8 = 1$, $g(t_6^1) = g(t_7^1) = 3$ and $g(t_8^1) = 0$. We define a mapping f by setting $f(t_2^1) = f(t_3^2) = 0$, $f(t_6^j) = f(t_7^k) = f(t_8^1) = 1$ for every $j = 1, \ldots, m_6$, $k = 1, \ldots, m_7$, $f(t_0^1) = f(t_3^1) = f(t_4^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 10 + 7 + m_6 + m_7 = \sigma(g)$. The case $g(t_4^1) = 0$ and $g(t_5^1) = 3$ implies $m_6 + m_7 = 3$, $g(t_6^1) = 3$ and $g(t_7^1) = 0$. We define a mapping f by setting $f(t_2^1) = f(t_3^2) = f(t_5^1) = 0$, $f(t_6^j) = f(t_7^k) = 1$ for every $j = 1, \ldots, m_6$, $k = 1, \ldots, m_7$, $f(t_3^2) = f(t_4^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 9 + 6 + m_6 + m_7 = \sigma(g)$. Hence, in both cases, we get $P_f(d_1) = \theta_3^2$.

3. As previously, we can assume that $P_g(d_1) \in \{\theta_{i_0}^1, \theta_{i_0}^2, \theta_{i_0}^5\}$.

(a) Let $i_0 = 0$ and $d_1 \in F_0^1 = CT[0,3]$. We have then $m_1 = m_2 = m_3 = 1$, $m_0 = 2$, and $P_g(d_1) \neq \theta_0^2$. If $P_g(d_1) = \theta_0^5$, then $g(t_2^1) = g(t_3^1) = 1$ cannot hold, because otherwise $g(t_0^1) = g(t_0^2) = g(t_1^1) = 1$, and the mapping h obtained from g by setting $h(t_0^1) = h(t_0^2) = h(t_3^1) = 0$, $h(t_1^1) = h(t_2^1) = 3$ and $h(u) = g(u)$, otherwise, would be a minimal dominating broadcast on CT^r with cost $\sigma(h) = \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with optimality of g. Therefore, $(g(t_2^1), g(t_3^1)) \in \{(1,0),(0,3),(3,3)\}$. The case $g(t_2^1) = 1$ and $g(t_3^1) = 0$ implies $m_4 = 2$, $m_5 = m_6 = 1$, $g(t_6^1) = 0$, $g(t_1^1) = 1$, and $g(t_4^1) = g(t_5^1) = 3$. We define a mapping f by setting $f(t_0^1) = f(t_0^2) = 0$, $f(t_4^1) = f(t_4^2) = f(t_5^1) = f(t_6^1) = 1$, $f(t_1^1) = f(t_2^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 10 + 10 = \sigma(g)$. The case $g(t_2^1) = 0$ and $g(t_3^1) = 3$ implies $m_4 = 2$, $m_5 = 1$, $g(t_5^1) = 0$, $g(t_1^1) = 1$, and $g(t_4^1) = 3$. We define a mapping f by setting $f(t_0^1) = f(t_0^2) = f(t_3^1) = 0$, $f(t_4^1) = f(t_4^2) = f(t_5^1) = 1$, $f(t_1^1) = f(t_2^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 9 + 9 = \sigma(g)$. The case $g(t_2^1) = g(t_3^1) = 3$ implies $m_4 = 2$ and $g(t_1^1) = g(t_4^1) = 0$. We define a mapping f by setting $f(t_0^1) = f(t_0^2) = f(t_3^1) = 0$, $f(t_4^1) = f(t_4^2) = 1$, $f(t_1^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 8 + 8 = \sigma(g)$. Hence, in all three cases, we get $P_f(d_1) = \theta_0^1$.

(b) Let $i_0 = 2$ and $d_1 \in F_2^1 = CT[2,5]$. We have then $m_0 = m_1 = m_3 = m_4 = m5 = 1$, $m_2 = 2$, and $P_g(d_1) \neq \theta_2^2$. Indeed, if $P_g(d_1) = \theta_2^2$, then $g(t_1^1) = g(t_4^1) = 0$, $g(t_0^1) = 1$, $g(t_5^1) \in \{0,1\}1$ and $g(t_2^1) = g(t_3^1) = 3$, and the mapping h obtained from g by setting $h(t_2^1) = h(t_2^2) = h(t_5^1 = 0$, $h(t_0^1) = h(t_4^1) = 3$ and $h(u) = g(u)$, otherwise, would be a minimal dominating broadcast on CT^r with cost $\sigma(h) \geq \sigma(g) - 5 + 6 = \Gamma_b(CT^r) + 1$, a contradiction with optimality of g. Assume now $P_g(d_1) = \theta_2^5$. We then have $g(t_1^1) = 0$, $g(t_2^1) = g(t_2^2) = 1$ and $g(t_0^1) = 3$ and, either $g(t_3^1) = 1$ or $g(t_3^1) = 0$. For the case $g(t_3^1) = 1$, we define a mapping f by setting $f(t_0^1) = f(t_2^2) = f(t_3^1) = 0$, $f(t_1^1) = f(t_2^1) = 3$ and, $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 6 + 6 = \sigma(g)$. For the case $g(t_3^1) = 0$, we get $m_6 = 2$, $g(t_4^1) = g(t_5^1) = 3$, and thus, we define again a mapping f by setting $f(t_2^1) = f(t_2^2) = f(t_5^1) = 0$, $f(t_6^1) = f(t_6^2) = 1$, $f(t_3^1) = 3$, and $f(u) = g(u)$ otherwise. The mapping f is a minimal dominating broadcast on CT^r, with cost $\sigma(f) = \sigma(g) - 5 + 5 = \sigma(g)$. Hence, in both cases, we get $P_f(d_1) = \theta_2^1$.

4. This result is immediate from Lemma 3.9.

This completes the proof. □