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Ramsey minimal graphs for a pair of a cycle on four vertices and an arbitrary star

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Abstract

Let F, G and H be simple graphs. The notation $F \rightarrow (G, H)$ means that for any red-blue coloring on the edges of graph F, there exists either a red copy of G or a blue copy of H. A graph F is called a Ramsey (G, H)-minimal graph if it satisfies two conditions: (i) $F \rightarrow (G, H)$ and (ii) $F - e \nrightarrow (G, H)$ for any edge e of F. In this paper, we give some finite and infinite classes of Ramsey $(C_4, K_{1,n})$-minimal graphs for any $n \geq 3$.

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1. Introduction

All graphs in this paper are simple. For any three graphs F, G and H, the notation of $F \rightarrow (G, H)$ to mean that for any red-blue coloring on the edges of F, there exists a red copy of G or a blue copy of H.

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Definition 1.1. A graph F is called a *Ramsey graph* for a pair of graphs (G, H) if F satisfies that $F \rightarrow (G, H)$.

Definition 1.2. A graph F is called a *Ramsey* (G, H)*-minimal* if F satisfies the following conditions:

(i) $F \rightarrow (G, H)$, and
(ii) $F - e \nrightarrow (G, H)$, for any $e \in E(F)$.

The set of all Ramsey (G, H)-minimal graphs will be denoted by $\mathcal{R}(G, H)$.

The pair (G, H) is called a *Ramsey-finite* if $\mathcal{R}(G, H)$ is finite. Otherwise, the pair (G, H) is called *Ramsey-infinite*. The study of Ramsey minimal graphs was initiated by Burr et al. [3]. The problem of characterizing or determining all Ramsey (G, H)-minimal graphs for a certain pair of G and H is a challenging problem.

Burr et al. [2] showed that for an arbitrary graph G, the pair (mK_2, G) is Ramsey-finite. Nešetřil and Rödl proved that if both G and H are 3-connected or if G and H are forest and neither of which is a union of stars, then the pair (G, H) is Ramsey-infinite [7]. Next, Baskoro et al. [1] determined the graphs in $\mathcal{R}(K_{1,2}, C_4)$. In 2015, Mushi and Baskoro [6] gave necessary and sufficient conditions for all members of $\mathcal{R}(3K_2, K_{1,n})$ for each $n \geq 3$. Furthermore, for $3 \leq n \leq 7$ they were able to list all Ramsey $(3K_2, K_{1,n})$-minimal graphs of order at most 10 vertices. In the same year, Wijaya et al. [5] determined all non-isomorphic Ramsey $(2K_2, K_4)$-minimal graphs of order at least 9. Furthermore, they also gave a general class graph which belong to $\mathcal{R}(2K_2, K_n)$, for $n \geq 3$. Nisa et al. [9] gave some graphs in $\mathcal{R}(C_6, K_{1,2})$. In 2021, Nabila and Baskoro [8] gave some Ramsey $(C_n, K_{1,2})$-minimal graphs for $n \in \{5, 6, 8\}$. In the same year, Hadiputra and Silaban [4] studied an infinite family of graphs that belongs to $\mathcal{R}(K_{1,2}, C_4)$. In 2022, Nabila et al. [10] gave some Ramsey $(C_n, K_{1,2})$-minimal graphs for any $n \in [7, 10]$ and construct Ramsey $(C_n, K_{1,2})$-graphs from the well-known Harary graph, for any integer $n \geq 6$.

In this paper, we construct some new finite and infinite classes of graphs that belong to the set $\mathcal{R}(C_4, K_{1,n})$ for any $n \geq 3$.

2. Main Results

Our main results will be divided into two sections. In the first section, we present some finite classes of Ramsey $(C_4, K_{1,n})$-minimal graphs. The second section, we propose some infinite classes of such Ramsey minimal graphs.

For any vertex $x \in V$ and $A \subset V$, let us denote by (x, A) the set of all edges connecting x and all vertices of A. This set can also be denoted by (A, x). Throughout the paper, we define $[a, b] = \{x \in \mathbb{N} | a \leq x \leq b\}$, except in the proof Theorem 2.2, we use the notation for a different thing, but the context is clear.

2.1. *Some finite classes of graph in* $\mathcal{R}(C_4, K_{1,n})$

In this section, we give some finite class of graphs which belongs to $\mathcal{R}(C_4, K_{1,n})$ for any integer $n \geq 3$.

Definition 2.1. For any positive integer $n \geq 3$, $H(n)$ is a graph with the vertex-set and edge-set:

$$\begin{aligned} V &= \{c_i, v_j \mid i \in [1,3],\ j \in [1, 2n-1]\} \quad \text{and} \\ E &= \{c_1 v_i, c_2 v_i, c_3 v_j \mid i \in [1, 2n-1], j \in [1, n+2]\}. \end{aligned}$$

In the following we show that the graph $H(n)$ is a Ramsey $(C_4, K_{1,n})$-minimal graph for any $n \geq 3$.

Theorem 2.1. *For any integer $n \geq 3$, $H(n) \in \mathcal{R}(C_4, K_{1,n})$.*

Proof. Let α be any red-blue coloring of the edges of $H(n)$ with no blue $K_{1,n}$. Let $W = \{v \in V \mid vc_1, vc_2 \in E\}$. Let $A = \{v \in V \mid vc_1, vc_2, vc_3 \in E\}$ and $B = W \backslash A$. Since $d(c_1) = 2n-1$, then there are at most $n-1$ blue edges incident to c_1. Let $S = \{v \in W \mid c_1 v \text{ is red}\}$ and $T = \{v \in W \mid c_1 v \text{ is blue}\}$. Then $|S| \geq n$ and $|T| \leq n-1$.

Now, consider the edges incident to c_2. Since there is no blue $K_{1,n}$, there are at most $n-1$ blue edges connecting c_2 and vertices of W. If there are at most $n-2$ blue edges connecting c_2 to S then it creates a red C_4. Thus, there are exactly $n-1$ blue edges connecting c_2 with the vertices of S and no blue edges connecting T with c_2.

Next, consider the edges incident to c_3. Clearly, there are at most $n-1$ blue edges and at least 3 red edges connecting between A and c_3. If there are two red edges connecting $T \cap A$ and c_3 then a red copy of C_4 occurs (involving c_2, c_3 and T). Similarly, if there are two red edges connecting $S \cap A$ and c_3 then a red copy of C_4 occurs (involving c_1, c_3 and S). Therefore, $H(n) \rightarrow (C_4, K_{1,n})$.

To show the minimality, consider $G \cong H(n) - e$ for any edge $e \in H(n)$. Up to isomorphism, we consider three cases:

(i) Let $e = c_1 v_1 \in (c_1, A)$. Then, consider a red-blue coloring on G with all edges in the set $(c_1, A \backslash \{v_2, v_3\}) \cup (c_2, B \backslash \{v_{2n-1}\}) \cup (c_2, \{v_2, v_3, v_4\}) \cup (c_3, A \backslash \{v_1, v_2, v_3\})$ are blue and the remaining edges are red.

(ii) Let $e = c_2 v_{2n-1} \in (c_2, B)$. Then, consider a red-blue coloring on G with all edges in the set $\{(c_1, A \backslash \{v_1, v_2, v_3\}) \cup (c_2, B \backslash \{v_{2n-1})\} \cup (c_2, \{v_2, v_3, v_4\}) \cup (c_3, A \backslash \{v_3, v_4, v_5\})\}$ are blue and the remaining edges are red.

(iii) Let $e = c_3 v_1 \in (c_3, A)$. Then, consider a red-blue coloring on G with all edges in the set $\{(c_1, A \backslash \{v_1, v_2, v_3\}) \cup (c_2, B \backslash \{v_{2n-1})\} \cup (c_2, \{v_1, v_2, v_4\}) \cup (c_3, A \backslash \{v_2, v_4\})\}$ are blue and the remaining edges are red.

Therefore, in such a coloring, there is neither red copy of C_4 nor blue copy of $K_{1,n}$. Thus, $G \nrightarrow (C_4, K_{1,n})$. As a consequence, $H(n)$ is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

Let t be any natural number, define a theta-path graph $G[a_1, ..., a_t]$ with $a_i \geq 3$ for $i \in [1, t]$ as follows.

Definition 2.2. The *theta-path graph* of *length* t, denoted by $G[a_1, a_2, ..., a_t]$, is the graph with the vertex set and the edge set:

$$\begin{aligned} V &= \{c_1, c_2, ..., c_{t+1}\} \cup A_1 \cup \ldots \cup A_t \ \text{ with } |A_i| = a_i \text{ and } A_i = \{u_{i,1}, ..., u_{i,a_i}\} \\ &\quad \text{for } i \in [1, t] \\ E &= \{(c_i, A_i), (A_i, c_{i+1}) \mid i \in [1, t]\}. \end{aligned}$$

Note that if $t = 1$, then $G[a_1] \cong K_{2,a_1}$.

Let α be any red-blue coloring on the edges of the theta-path graph $G[a_1, a_2, ..., a_t]$. For any $i \in [1, t]$, let b_i^+ be the number of blue edges in (c_i, A_i) under α. For any $i \in [2, t+1]$ let b_i^- be the number of blue edges in (A_{i-1}, c_i) under α. We use the notation $[b_1^+, b_2^- | b_2^+, b_3^- | \ldots | b_{t-1}^+, b_t^- | b_t^+, b_{t+1}^-]$ for the coloring α if there are exactly b_i^+ blue edges in (c_i, A_i) and b_i^- blue edges in (A_{i-1}, c_i) for any i in α. Additionally, if the number of vertices of A_i incident to blue edges is $b_i^+ + b_{i+1}^-$ for every $i \in [1, t]$, then the coloring α is called *maximal*.

For example, Figure 1 represents a red blue coloring $[4, 2|3, 0]$ (left) and a maximal red blue coloring $[5, 2|4, 1]$ (right) in $G[7, 5]$. Note that, in general, the colorings with the notation $[b_1^+, b_2^- | b_2^+, b_3^- | \ldots | b_{t-1}^+, b_t^- | b_t^+, b_{t+1}^-]$ may not be unique.

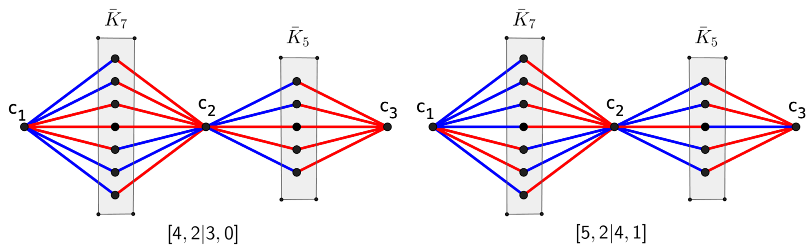

Figure 1. Some red-blue colorings in the theta-path graph $G[7, 5]$.

2.1.1. The theta-path graph of length 1.

In this section, we present the theta-path graph of length one which is in $\mathcal{R}(C_4, K_{1,n})$.

Theorem 2.2. *For any integer $n \geq 3$, the theta-path graph $G[2n] \in \mathcal{R}(C_4, K_{1,n})$.*

Proof. Let $G = G[2n]$ for any fixed integer $n \geq 3$. First, we will show that $G \to (C_4, K_{1,n})$. Consider any red-blue coloring α on the edges of G with containing no blue $K_{1,n}$. We will show that there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^-]$ for some integers b_1^+ and b_2^-. The number of vertices in A_1 incident to blue edges is denoted by n_1. Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_2^- \leq n-1$, and $b_1^+ + b_2^- = n_1 \leq 2n-2$. Thus, there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge e. Let $e \in (c_1, A_1)$ or (A_1, c_2), then consider the maximal red-blue coloring $\alpha_1 \cong [n-1, n-1]$ on G such that $\alpha_1(e)$ is red. By considering the restriction of the coloring α_1 on $G - e$, we obtain that there is neither blue copy of $K_{1,n}$ nor red copy of C_4 in $G - e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

2.1.2. The theta-path graph of length two.

In this section, we construct the theta-path graph of length two which is in $\mathcal{R}(C_4, K_{1,n})$.

Theorem 2.3. *Let n and k be integers, with $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. Then, the theta-path graph $G[a_1, a_2]$ in $\mathcal{R}(C_4, K_{1,n})$, where $a_1 = 2n - k$ and $a_2 = n + k$.*

Proof. Let $G = G[2n-k, n+k]$ for any fixed integers $n \geq 3$ and $k \in [1, \lfloor (n-1)/2 \rfloor]$. First, we will show that $G \rightarrow (C_4, K_{1,n})$. Consider any red-blue coloring α on the edges of G with containing no blue $K_{1,n}$. We will show there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^- | b_2^+, b_3^-]$ for some integers b_1^+, b_2^-, b_2^+ and b_3^-.

For $i = 1, 2$, denote by n_i the number of vertices in A_i incident to blue edges. Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_2^- + b_2^+ \leq n-1$, $b_3^- \leq n-1$, $n_1 \leq 2n-k$, and $n_2 \leq n+k$. However, $n_1 \geq 2n-k-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2, or two vertices in A_2 together with c_2 and c_3. Thus, $2n-k-1 \leq n_1 \leq 2n-k$.

Since $2n-k-1 \leq n_1 \leq 2n-k$, then $b_2^- \geq (2n-k-1)-(n-1) = n-k$. Since $b_2^- + b_2^+ \leq n-1$ then $b_2^+ \leq (n-1)-(n-k) = k-1$. But, since $b_2^+ \leq k-1$ and $b_3^- \leq n-1$ then $n_2 \leq (n-1)+(k-1) = n+k-2$. Therefore, there is a red C_4 in G.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge $e \in G$. If $e \in (c_1, A_1)$ or (A_1, c_2) then consider the maximal red-blue coloring $\alpha_1 \cong [n-1, n-k-1 | k, n-1]$ on G such that $\alpha_1(e)$ is red. By considering the restriction of the coloring α_1 on $G-e$, we obtain that there is neither blue copy of $K_{1,n}$ nor red copy of C_4 in $G-e$.

If $e \in (c_2, A_2)$ or (A_2, c_3) then consider the maximal red-blue coloring $\alpha_2 \cong [n-1, n-k | k-1, n-1]$ on G such that no two blue edges incident to the same vertex of A_i, for $i = 1, 2$, and $\alpha_2(e)$ is red. By restricting α_2 on $G-e$, we obtain that there is neither blue $K_{1,n}$ nor red C_4 in $G-e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

2.1.3. The theta-path graph of length 3*.*

In this section, we give the theta-path graph of length 3 which is in $\mathcal{R}(C_4, K_{1,n})$.

Theorem 2.4. *Let n and k be integers, with $n \geq 3$ and $2 \leq k \leq \lfloor (n-1)/2 \rfloor$. Then, the theta-path graphs $G[n+k-1, 2n-k, n+1]$, $G[2n-k, n+k-1, n+1]$, and $G[2n-k, n+1, n+k-1]$ are in $\mathcal{R}(C_4, K_{1,n})$.*

Proof. Let $G \cong G[n+(k-1), 2n-k, n+1]$ for any fixed integers $n \geq 3$ and $2 \leq k \leq \lfloor (n-1)/2 \rfloor$. First, we will show that $G \rightarrow (C_4, K_{1,n})$. Consider any red-blue coloring α on the edges of G with containing no blue $K_{1,n}$. We will show that there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^- | b_2^+, b_3^- | b_3^+, b_4^-]$ for some integers b_i^+, b_{i+1}^- where $i \in [1, 3]$. For $i \in [1, 3]$, denote by n_i the number of vertices in A_i incident to blue edges. Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_2^- + b_2^+ \leq n-1$, $b_3^- + b_3^+ \leq n-1$, and $b_4^- \leq n-1$. However, $n_1 \geq n+(k-1)-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2, or two vertices in A_2 together with c_2 and c_3. Thus, $n+(k-1)-1 \leq n_1 \leq n+(k-1)$.

Since $n+(k-1)-1 \leq n_1 \leq n+(k-1)$ then $b_2^- \geq (n+k-2)-(n-1) = k-1$. Since $b_2^- + b_2^+ \leq n-1$ then $b_2^+ \leq (n-1)-(k-1) = n-k$. However, $n_2 \geq 2n-k-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_2 together with c_2 and c_3, or two vertices in A_3 together with c_3 and c_4. Thus, $2n-k-1 \leq n_2 \leq 2n-k$. Since $2n-k-1 \leq n_2 \leq 2n-k$, then $b_2^+ \geq (2n-k-1)-(n-k) = n-1$. Since $b_3^- + b_3^+ \leq n-1$ then $b_3^+ \leq (n-1)-(n-1) = 0$. But, since $b_3^+ \leq 0$ and $b_4^- \leq 0+(n-1) = n-1$, then $n_3 \leq n-1$. Therefore, there is a red C_4 in G composed by two vertices in A_3 with c_3 and c_4.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge $e \in G$. If $e \in (c_1, A_1)$ or (A_1, c_2) then consider the maximal red-blue coloring $\alpha_1 \cong [n-1, k-2|n-k+1, n-2|1, n-1]$ on G such that $\alpha_1(e)$ is red. If $e \in (c_2, A_2)$ or (A_2, c_3) then consider the maximal red-blue coloring $\alpha_2 \cong [n-1, k-1|n-k, n-2|1, n-1]$ on G such that $\alpha_2(e)$ is red. If $e \in (c_3, A_3)$ or (A_3, c_4) then consider the maximal red-blue coloring $\alpha_3 \cong [n-1, k-1|n-k, n-1|0, n-1]$ on G such that $\alpha_3(e)$ is red. By considering the restriction of the coloring α_1, α_2, and α_3 on $G - e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph.

If $G \cong G[2n-k, n+k-1, n+1]$ or $G \cong G[2n-k, n+1, n+k-1]$ then the proofs are similar. □

Theorem 2.5. *Let n and k be integers, with $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. Then, the theta-path graph $G[2n-k, n, n+k]$ in $\mathcal{R}(C_4, K_{1,n})$.*

Proof. Let $G = G[2n-k, n, n+k]$ for any fixed integers $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. First, we will show that $G \rightarrow (C_4, K_{1,n})$. Consider any red-blue coloring α on the edges of G with containing no blue $K_{1,n}$. We will show there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^-|b_2^+, b_3^-|b_3^+, b_4^-]$ for some integers b_i^+, b_{i+1}^- where $i \in [1, 3]$.

Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_i^- + b_i^+ \leq n-1$ for $i = 2, 3$, $b_4^- \leq n-1$, $n_1 \leq 2n-k$, $n_2 \leq n$, and $n_3 \leq n+k$. However, $n_1 \geq 2n-k-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2, or two vertices in A_2 together with c_2 and c_3. Thus, $2n-k-1 \leq n_1 \leq 2n-k$.

Since $2n-k-1 \leq n_1 \leq 2n-k$ then $b_2^- \geq (2n-k-1)-(n-1) = n-k$. Since $b_2^- + b_2^+ \leq n-1$ then $b_2^+ \leq (n-1)-(n-k) = k-1$. However, $n_2 \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_2 together with c_2 and c_3, or two vertices in A_3 together with c_3 and c_4. Thus, $n-1 \leq n_2 \leq n$. Since $n-1 \leq n_2 \leq n$, then $b_3^- \geq (n-1)-(k-1) = n-k$. Since $b_3^- + b_3^+ \leq n-1$ then $b_3^+ \leq (n-1)-(n-k) = k-1$. But, since $b_3^+ \leq k-1$ and $b_4^- \leq n-1$ then $n_3 \leq (k-1)+(n-1) = n+k-2$. Therefore, there is a red C_4 in G composed by two vertices in A_3 with c_3 and c_4.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge $e \in G$. If $e \in (c_1, A_1)$ or (A_1, c_2) then consider the maximal red-blue coloring $\alpha_1 \cong [n-1, n-k-1|k, n-k-1|k, n-1]$ on G such that $\alpha_1(e)$ is red. If $e \in (c_2, A_2)$ or (A_2, c_3) then consider the maximal red-blue coloring $\alpha_2 \cong [n-1, n-k|k-1, n-k-1|k, n-1]$ on G such that $\alpha_2(e)$ is red. If $e \in (c_3, A_3)$ or (A_3, c_4) then consider the maximal red-blue coloring $\alpha_3 \cong [n-1, n-k|k-1, n-k|k-1, n-1]$ on G such that $\alpha_3(e)$ is red. By considering the restriction of the coloring α_1, α_2, and α_3 on $G - e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

2.1.4. The theta-path graph with a longer length.

In this section, we present the theta-path graph of length k which is $\mathcal{R}(C_4, K_{1,n})$, with $4 \leq k \leq n+1$.

Theorem 2.6. *Let n and k be integers, with $n \geq 3$ and $3 \leq k \leq n$. Then, the theta-path graph $G[a_1, a_2, ..., a_{k+1}]$ in $\mathcal{R}(C_4, K_{1,n})$, with $a_1 = 2n-k$ and $a_i = n+1$ for $i \in [2, k+1]$.*

Proof. Let $G = G[2n-k, a_2, ..., a_{k+1}]$ for any fixed integers $n \geq 3$, $2 \leq k \leq n$, where $a_i = n+1$ for $i \in [2, k+1]$. First, we will show that $G \rightarrow (C_4, K_{1,n})$. Consider any red-blue coloring α

on the edges of G with containing no blue $K_{1,n}$. We will show there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^-|b_2^+, b_3^-|...|b_k^+, b_{k+1}^-|b_{k+1}^+, b_{k+2}^-]$ for some integers b_i^+, b_{i+1}^- with $i \in [1, k+1]$. For $i \in [1, k+1]$, denote by n_i the number of vertices in A_i incident to blue edges. Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_i^- + b_i^+ \leq n-1$ for $i \in [2, k+1]$, $b_{k+2}^- \leq n-1$, $n_1 \leq 2n-k$, and $n_i \leq n+1$ for $i \in [2, k+1]$. However, $n_1 \geq 2n-k-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2, or two vertices in A_2 together with c_2 and c_3. Thus, $2n-k-1 \leq n_1 \leq 2n-k$.

Since $2n-k-1 \leq n_1 \leq 2n-k$, then $b_2^- \geq (2n-k-1)-(n-1) = n-k$. Since $b_2^- + b_2^+ \leq n-1$ then $b_2^+ \leq (n-1)-(n-k) = k-1$. However, $n_2 \geq n$ since otherwise there exists a red C_4 in G composed by two vertices in A_2 together with c_2 and c_3, or two vertices in A_3 together with c_3 and c_4. Thus, $n \leq n_2 \leq n+1$. Since $n \leq n_2 \leq n+1$, then $b_3^- \geq n-(k-1) = n-k+1$. Since $b_3^- + b_3^+ \leq n-1$ then $b_3^+ \leq (n-1)-(n-k+1) = k-2$.

Since $A_2, ..., A_{k+1}$ have the same number of vertices, then we obtain $b_i^+ \leq k-(i-1)$ and $b_i^- \geq n-(k-(i-1))$ for $2 \leq i \leq k+1$. However, for $i \in [2, k]$ $n_i \geq n$ since otherwise there exists a red C_4 in G composed by two vertices in A_i together with c_i and c_{i+1}, or two vertices in A_{i+1} together with c_{i+1} and c_{i+2}. Thus, $n \leq n_i \leq n+1$. Since $b_{k+1}^+ \leq 0$ and $b_{k+2}^- \leq n-1$, then $b_{k+1}^+ + b_{k+2}^- = n_{k+1} \leq (0)+(n-1) = n-1$. Therefore, there is a red C_4 in G composed by two vertices in A_{k+1} together with c_{k+1} and c_{k+2}.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge $e \in G$. Define $\alpha_1 \cong [b_1^+, b_2^-|b_2^+, b_3^-|...|b_k^+, b_{k+1}^-|b_{k+1}^+, b_{k+2}^-]$ where $b_1^+ = n-1, b_i^- = n-k+(i-2), b_i^+ = k-(i-1)$ with $i \in [2, k+1]$, and $b_{k+2}^- = n-1$. Next, for $j \in [2, k+1]$ define $\alpha_j \cong [d_1^+, d_2^-|d_2^+, d_3^-|...|d_k^+, d_{k+1}^-|d_{k+1}^+, d_{k+2}^-]$ where

$$d_i^- = \begin{cases} b_i^- + 1, & 2 \leq i \leq j, \\ b_i^-, & j+1 \leq i \leq k+2, \end{cases} \quad d_i^+ = \begin{cases} b_i^+ - 1, & 2 \leq i \leq j, \\ b_i^+, & j+1 \leq i \leq k+1 \text{ or } i = 1. \end{cases}$$

Let $e \in (c_i, A_i)$ or (A_i, c_{i+1}) for some $i \in [1, k+1]$, then consider the maximal red-blue coloring α_i on G such that $\alpha_i(e)$ is red. By considering the restriction of the coloring α_i for $i \in [1, k+1]$ on $G - e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

2.2. Some infinite classes of graphs in $\mathcal{R}(C_4, K_{1,n})$

In this section, we are going to construct some infinite classes of graphs which belong to $\mathcal{R}(C_4, K_{1,n})$ for any integer $n \geq 3$.

The first class is the theta-path graph $G[2n-k, a_2, ..., a_{z+1}, n+k]$ of length $z+2$ for any $z \geq 2$. The second class is the theta-path graph $G[n+(k-1), a_2, \ldots, a_{z_1+1}, 2n-k, a_{z_1+3}, \ldots, a_{z_2+z_1+2}, n+1]$ of length $z_1 + z_2 + 3$ for any $z_1, z_2 \geq 1$.

To illustrate these theta-path graphs, we give $G[2n-k, a_2, ..., a_{z+1}, n+k]$ with $n = 4$, $k = 1$, and $z = 4$ in Figure 2.

Theorem 2.7. *Let n, k and z be integers, with $n \geq 3$, $2 \leq k \leq \lfloor (n-1)/2 \rfloor$ and $z \geq 2$. Then, the theta-path graph $G[2n-k, a_2, ..., a_{z+1}, n+k]$ in $\mathcal{R}(C_4, K_{1,n})$, with $a_i = n$ for $i \in [2, z+1]$.*

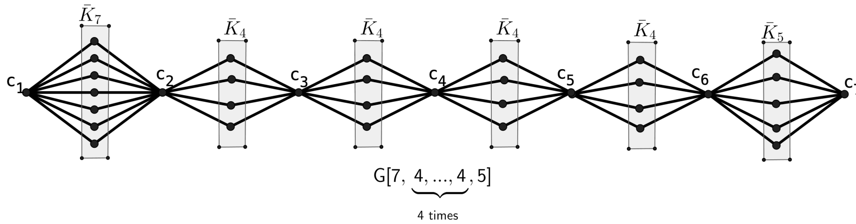

Figure 2. Graph $G[7, 4, 4, 4, 4, 5]$.

Proof. Let $G \cong G[2n-k, a_2, ..., a_{z+1}, n+k]$ for any fixed integers $n \geq 3$, $z \geq 1$ and $k \in [1, \lfloor (n-1)/2 \rfloor]$. First, we will show that $G \to (C_4, K_{1,n})$. Consider any red-blue coloring on the edges of G with containing no blue $K_{1,n}$. We will show that there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^- | b_2^+, b_3^- | ... | b_z^+, b_{z+1}^- | b_{z+1}^+, b_{z+2}^- | b_{z+2}^+, b_{z+3}^-]$. For $i \in [1, z+2]$, denote by n_i the number of vertices in A_i incident to blue edges.

Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_2^- + b_2^+ \leq n-1$, $b_{z+3}^- \leq n-1$, $b_i^- + b_i^+ \leq n-1$ for $i \in [2, z+2]$, $n_1 \leq 2n-k$, $n_i \leq n$ for $i \in [2, z+1]$, and $n_{z+2} \leq n+k$. However, $n_1 \geq 2n-k-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2, or two vertices in A_2 together with c_2 and c_3. Thus, $2n-k-1 \leq n_1 \leq 2n-k$.

Since $2n-k-1 \leq n_1 \leq 2n-k$ then $b_2^- \geq (2n-k-1)-(n-1) = n-k$. Since $b_2^- + b_2^+ \leq n-1$ then $b_2^+ \leq (n-1)-(n-k) = k-1$. However, $n_2 \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_2 together with c_2 and c_3, or two vertices in A_3 together with c_3 and c_4. Thus, $n-1 \leq n_2 \leq n$. Since $n-1 \leq n_2 \leq n$, then $b_3^- \geq (n-1)-(k-1) = n-k$. Since $b_3^- + b_3^+ \leq n-1$ then $b_3^+ \leq (n-1)-(n-k) = k-1$.

Since $A_2, ..., A_{z+1}$ have the same number of vertices, then we obtain $b_i^+ \leq k-1$ and $b_{i+1}^- \geq n-k$ for $2 \leq i \leq z+1$. However, for $j \in [2, z+1]$ $n_j, \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_j together with c_j and c_{j+1}, or two vertices in A_{j+1} together with c_{j+1} and c_{j+2}. Thus, $n-1 \leq n_j \leq n$. Since $b_{z+2}^+ \leq k-1$ and $b_{z+3}^- \leq n-1$ then $b_{z+2}^+ + b_{z+3}^- = n_{z+2} \leq (k-1)+(n-1) = n+k-2$. Therefore, there is a red C_4 in G composed by two vertices in A_{z+2} together with c_{z+2} and c_{z+3}.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge $e \in G$. Now, define the labeling α_1 as follows:

$$\alpha_1 \cong [b_1^+, b_2^- | b_2^+, b_3^- | ... | b_{z+1}^+, b_{z+2}^- | b_{z+2}^+, b_{z+3}^-],$$

where $b_1^+ = n-1, b_i^- = n-k-1, b_i^+ = k$ with $i \in [2, z+2]$, and $b_{z+3}^- = n-1$. For $j = 2, 3, \cdots, z+2$, define

$$\alpha_j \cong [d_1^+, d_2^- | d_2^+, d_3^- | ... | d_{z+1}^+, d_{z+2}^- | d_{z+2}^+, d_{z+3}^-],$$

where $d_i^- = \begin{cases} b_i^- + 1, & 2 \leq i \leq j, \\ b_i^-, & j+1 \leq i \leq z+3, \end{cases}$ $d_i^+ = \begin{cases} b_i^+ - 1, & 2 \leq i \leq j+1, \\ b_i^+, & j+2 \leq i \leq z+2 \text{ or } i = 1. \end{cases}$

Let $e \in (c_i, A_i)$ or (A_i, c_{i+1}) for some $i \in [1, z+2]$, then consider the maximal red-blue coloring α_i on G such that $\alpha_i(e)$ is red. By considering the restriction of the coloring α_i for

$i \in [1, z+2]$ on $G - e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

Theorem 2.8. *Let n, k, z_1 and z_2 be integers, with $n \geq 3$, $2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ and $z_1, z_2 \geq 1$. Then, the theta-path graph $G[n+(k-1), a_2, ..., a_{z_1+1}, 2n-k, a_{z_1+3}, ..., a_{z_2+z_1+2}, n+1]$ in $\mathcal{R}(C_4, K_{1,n})$, with $a_i = n$ for $i \in [2, z_1+1] \cup [z_1+3, z_2+z_1+2]$.*

Proof. Let $G = G[n+(k-1), a_2, ..., a_{z_1+1}, 2n-k, a_{z_1+3}, ..., a_{z_2+z_1+2}, n+1]$ for any fixed integers $n \geq 3$ and $z_1, z_2 \geq 1$. First, we will show that $G \rightarrow (C_4, K_{1,n})$. Consider any red-blue coloring on the edges of G with containing no blue $K_{1,n}$. We will show that there is a red C_4 in G. Let α be a coloring $[b_1^+, b_2^- | b_2^+, b_3^- | ... | b_{z_1+1}^+, b_{z_1+2}^- | b_{z_1+2}^+, b_{z_1+3}^- | ... | b_{m+2}^+, b_{m+3}^- | b_{m+3}^+, b_{m+4}^-]$ where $m = z_1 + z_2$. For $i \in [1, m+3]$, denote by n_i the number of vertices in A_i incident to blue edges.

Since there is no blue $K_{1,n}$ in G then $b_1^+ \leq n-1$, $b_2^- + b_2^+ \leq n-1$, $b_{m+4}^- \leq n-1$, $b_i^- + b_i^+ \leq n-1$ for $i \in [2, m+3]$, $n_1 \leq n+k-1$, $n_i \leq n$ for $i \in [2, z_1+1] \cup [z_1+3, m+2]$, $n_{z_1+2} \leq 2n-k$, and $n_{m+3} \leq n+1$. However, $n_1 \geq n+k-2$ since otherwise there exists a red C_4 in G composed by two vertices in A_1 together with c_1 and c_2, or two vertices in A_2 together with c_2 and c_3. Thus, $n+k-2 \leq n_1 \leq n+k-1$.

Since $n+k-2 \leq n_1 \leq n+k-1$, then $b_2^- \geq (n+k-2)-(n-1) = k-1$. Since $b_2^- + b_2^+ \leq n-1$ then $b_2^+ \leq (n-1)-(k-1) = n-k$. However, $n_2 \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_2 together with c_2 and c_3, or two vertices in A_3 together with c_3 and c_4. Thus, $n-1 \leq n_2 \leq n$. Since $n-1 \leq n_2 \leq n$, then $b_3^- \geq (n-1)-(k-1) = n-k$. Since $b_3^- + b_3^+ \leq n-1$ then $b_3^+ \leq (n-1)-(n-k) = k-1$.

Since $A_2, ..., A_{z_1+1}$ have the same number of vertices, then we obtain $b_i^+ \leq n-k$ and $b_{i+1}^- \geq k-1$ for $i \in [2, z_1+1]$ and $j = i$. However, for $j \in [2, z_1+1]$, $n_j \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_j together with c_j and c_{j+1}, or two vertices in A_{j+1} together with c_{j+1} and c_{j+2}. Thus, $n-1 \leq n_j \leq n$.

Since $b_{z_1+2}^- \geq k-1$, then $b_{z_1+2}^+ \leq (n-1)-(k-1) = n-k$. However, $n_{z_1+2} \geq 2n-k-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_{z_1+2} together with c_{z_1+2} and c_{z_1+3}, or two vertices in A_{z_1+3} together with c_{z_1+3} and c_{z_1+4}. Thus, $2n-k-1 \leq n_{z_1+2} \leq 2n-k$.

Since $2n-k-1 \leq n_{z_1+2} \leq 2n-k$, then $b_{z_1+3}^- \leq (2n-k-1)-(n-k) = n-1$. Since $b_{z_1+3}^- + b_{z_1+3}^+ \leq n-1$ then $b_{z_1+3}^+ \leq (n-1)-(n-1) = 0$. However, $n_{z_1+3} \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_{z_1+3} together with c_{z_1+3} and c_{z_1+4}, or two vertices in A_{z_1+4} together with c_{z_1+4} and c_{z_1+5}. Thus, $n-1 \leq n_{z_1+3} \leq n$.

Since $A_{z_1+3}, ..., A_{m+2}$ have the same number of vertices, then we obtain $b_i^+ \leq 0$ and $b_{i+1}^- \geq n-1$ for $i \in [z_1+3, m+2]$. However, for $j \in [z_1+3, m+2]$, $n_j \geq n-1$ since otherwise there exists a red C_4 in G composed by two vertices in A_j together with c_j and c_{j+1}, or two vertices in A_{j+1} together with c_{j+1} and c_{j+2}. Thus, $n-1 \leq n_j \leq n$.

Since $b_{m+3}^+ \leq k-1$ and $b_{m+4}^- \leq n-1$, then $b_{m+3}^+ + b_{m+4}^- = n_{m+3} \leq (0) + (n-1) = n-1$. Therefore, there is a red C_4 in G composed by two vertices in A_{m+3} together with c_{m+3} and c_{m+4}.

Next, we will show the minimality, that is, $G - e \nrightarrow (C_4, K_{1,n})$ for any edge $e \in G$. Now, define the labeling α_1 as follows:

$$\alpha_1 \cong [b_1^+, b_2^- | b_2^+, b_3^- | ... | b_{z_1+1}^+, b_{z_1+2}^- | b_{z_1+2}^+, b_{z_1+3}^- | ... | b_{m+2}^+, b_{m+3}^- | b_{m+3}^+, b_{m+4}^-],$$

where

$$b_i^- = \begin{cases} k-2, & 2 \le i \le z_1+2, \\ n-2, & z_1+3 \le i \le m+3, \\ n-1, & i = m+4. \end{cases} \quad b_i^+ = \begin{cases} n-1, & i=1, \\ n-k+1, & 2 \le i \le z_1+2, \\ 1, & z_1+3 \le i \le m+3. \end{cases}$$

For $j \in [2, m+3]$, define

$$\alpha_j \cong [d_1^+, d_2^- | d_2^+, d_3^- | ... | d_{z_1+1}^+, d_{z_1+2}^- | d_{z_1+2}^+, d_{z_1+3}^- | ... | d_{m+2}^+, d_{m+3}^- | d_{m+3}^+, d_{m+4}^-],$$

where

$$d_i^- = \begin{cases} b_i^- + 1, & 2 \le i \le j, \\ b_i^-, & j+1 \le i \le m+4, \end{cases} \quad d_i^+ = \begin{cases} b_i^+ - 1, & 2 \le i \le j+1, \\ b_i^+, & j+2 \le i \le m+3 \text{ or } i=1. \end{cases}$$

Let $e \in (c_i, A_i)$ or (A_i, c_{i+1}) for some $i \in [1, m+3]$, then consider the maximal red-blue coloring α_i on G such that $\alpha_i(e)$ is red. By considering the restriction of the coloring α_i for $i \in [1, m+3]$ on $G-e$. Thus, $G - e \nrightarrow (C_4, K_{1,n})$. Therefore, G is a Ramsey $(C_4, K_{1,n})$-minimal graph. □

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