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Total distance vertex irregularity strength of some corona product graphs

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Abstract

A distance vertex irregular total k-labeling of a simple undirected graph $G = G(V, E)$, is a function $f : V(G) \cup E(G) \longrightarrow \{1, 2, \dots, k\}$ such that for every pair vertices $u, v \in V(G)$ and $u \neq v$, the weights of u and v are distinct. The weight of vertex $v \in V(G)$ is defined to be the sum of the label of vertices in neighborhood of v and the label of all incident edges to v. The total distance vertex irregularity strength of G (denoted by $tdis(G)$) is the minimum of k for which G has a distance vertex irregular total k-labeling. In this paper, we present several results of the total distance vertex irregularity strength of some corona product graphs.

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1. Introduction

All graphs considered in this paper are finite, undirected simple graphs. Let $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph $G = G(V, E)$. A labeling of graph G is defined as a mapping from the set of elements in graph to a set of non negative integers [9]. Gallian [5]

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summarizes the complete survey about graph labelings. Miller et al. [6], for example, introduced a distance vertex magic labeling or known as 1-vertex-magic labeling. Here, the weight of every vertex $v \in V(G)$ is defined as the sum of all labels of vertices of distance 1 from v, which is the sum of all labels of vertices in the neighborhood of v. In [4], Chartrand et al. introduced a concept of irregular labeling, by assign positive integer labels to the edges of G, a simple connected graph of order at least 3 in such a way that G becomes irregular, i.e., the weights (incident edges label sums) at each vertex of G are distinct. The problem is to find the minimum value of the largest label over all such irregular assignments. This value is known as the irregularity strength of G and denoted by $s(G)$. Inspired by the concept of irregularity strength and total labeling, Bača et al. [2] define a new type of irregular labeling called total irregular labeling. For graph $G(V, E)$, a labeling $f : V(G) \cup E(G) \longrightarrow \{1, 2, \dots, k\}$ is a total vertex irregular labeling when the weight of pair vertices $v, w \in V(G), v \neq w$ under f are distinct. The weight of vertex $v \in V(G)$ under f is defined as

$$w_f(v) = f(v) + \sum_{uv \in E(G)} f(uv).$$

The set of open neighbors of v is denoted by $N(v)$, where $N(v) = \{u \in V(G) : uv \in E(G)\}$.

From these definitions of a distance magic labeling and an irregular labeling, Slamin [8] introduces a new concept of irregular vertex labeling and defines a distance vertex irregular labeling of graph G as a mapping $f : V(G) \cup E(G) \longrightarrow \{1, 2, \dots, k\}$ such that the weight of all vertices in $V(G)$ are distinct. The distance vertex irregularity strength of G is denoted by $dis(G)$ and the weight of every vertex $v \in V(G)$ under f is defined as

$$w_f(v) = \sum_{u \in N(v)} f(u),$$

where $N(v)$ is a set of open neighbors of v [8]. Bong et al. [3] continue the research by completing the distance vertex irregularity strength of cycles and wheels while Bača et al. [1] investigate an inclusive distance vertex irregular labelings. Inspired by the distance irregular vertex labeling and the vertex irregular total labeling, Wijayanti et al. [10], introduce the distance vertex irregular total k-labeling as a new concept of total labeling based on both vertex irregular total k-labeling and distance vertex irregular labeling. Wijayanti et al. [10] suggest that a graph G has a distance vertex irregular total k-labeling if there is a function $f : V(G) \cup E(G) \longrightarrow \{1, 2, \dots, k\}$ such that for every pair vertices $u, v \in V(G)$ and $u \neq v$, the weight of u is not equal to the weight of v. The following definition is for further perusal.

Definition 1.1. *Let $G = G(V, E)$ be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A distance vertex irregular total k-labeling is a function $f : V(G) \cup E(G) \longrightarrow \{1, 2, ..., k\}$ such that the weight of all vertices of G are distinct. The weight of $v \in V(G)$ under the labeling f is defined as*

$$w_f(v) = \sum_{u \in N(v)} f(u) + \sum_{u \in N(v)} f(uv), \tag{1}$$

where $N(v)$ is a set of open neighbors of v. The total distance vertex irregularity strength of G, denoted by $tdis(G)$, is defined as the smallest value of k for which G has a distance vertex irregular total k-labeling.

The distance vertex irregular total k-labeling defined in Definition 1.1, uses the concept of the neighboring vertex of distance $\{1\}$. For every pair vertices $u, v \in V(G)$, denotes $d(u, v)$ as a distance from u to v, $diam(G) = max\{d(u, v) : \forall u, v \in V(G)\}$ is diameter of G and $D \subseteq \{1, 2, ..., diam(G)\}$. The set of D-distance neighbors vertex of v is $N_D(v) = \{u \in V(G) : d(u, v) \in D\}$. For every $u \in N_D(v)$, defined $P_u(V_u, E_u)$ as the shortest path from $u \in N_D(v)$ to v and $I(v) = \bigcup_{u \in N_D(v)} E_u(P_u)$. In Definition 1.2, Wijayanti et al. [12] generalize distance $\{1\}$ to distance D.

Definition 1.2. *Let $G = G(V, E)$ be a simple finite graph with vertex set $V(G)$, edge set $E(G)$ and diameter $diam(G)$. Let $D \subseteq \{0, 1, ..., diam(G)\}$ and k, a positive integer. Define $f : V \cup E \to \{1, \dots, k\}$, a total labeling function. The weight of every vertex $v \in V(G)$ defined as,*

$$w_f(v) = \sum_{u \in N_D(v)} f(u) + \sum_{e \in I(v)} f(e). \tag{2}$$

If for every $v \in V(G)$, $w_f(v)$ are distinct, then we named the total labeling function as D-distance vertex irregular total k-labeling. Furthermore, the total D-distance vertex irregularity strength is denoted by $tdis_D(G)$.

Afterward, the total distance vertex irregularity strengths for path P_n and cycle C_n are determined. Likewise, some necessary and sufficient conditions for the existence of a D-distance vertex irregular total k-labeling are defined. For $D \neq \{1\}$, Observation 1.1, can be used to determine the existence of a D-distance vertex irregular total $k-$labeling on graph G.

Observation 1.1. *Let $G(V, E)$ be a simple finite graph, $u, v \in V(G)$ and $u \neq v$. For $D \neq \{1\}$, if $N_D(u) = N_D(v)$ and $I(u) = I(v)$ then G does not have the D-distance vertex irregular total k-labeling.*

Proof. Let $G(V, E)$ be a graph with a D-distance vertex irregular total k-labeling. Since $N_D(u) = N_D(v)$ and $I(u) = I(v)$, by the Definition 1.2, we obtain

$$w_f(u) = \sum_{x \in N_D(u)} f(x) + \sum_{e \in I(u)} f(e) = \sum_{x \in N_D(v)} f(x) + \sum_{e \in I(v)} f(e) = w_f(v). \tag{3}$$

The weights of u and v are the same, a contradiction. □

Figure 1. presents an example of some graphs that do not have the D-distance vertex irregular total $k-$labeling, as intended in Observation 1.1.

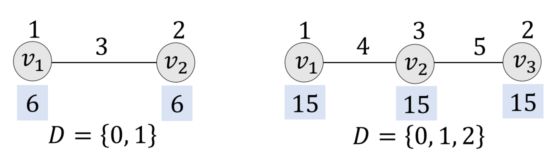

Figure 1. Some graphs without D-distance vertex irregular total k-labeling

In Figure 1, regardless of the given label value, for $D = \{0, 1\}$, the weight of all vertices in P_2 are same, and for $D = \{0, 1, 2\}$, all vertices of graph P_3 also have the same weight. Therefore, for $D = \{0, 1\}$ and $D = \{0, 1, 2\}$, P_2 and P_3 do not have D-distance vertex irregular total k-labeling, respectively.

Observation 1.2. *Let $G(V, E)$ be a graph with a D-distance vertex irregular total k-labeling. For every $u, v \in V(G)$, if $d(u, v) \in D$, $N_D(u) - v = N_D(v) - u$ and $I(u) = I(v)$ then the labels of u and v are ought to be distinct.*

Proof. Let $G(V, E)$ be a graph with a D-distance vertex irregular total k-labeling and f is any D-distance vertex irregular total k-labeling of $G(V, E)$. Since $d(u, v) \in D$, $N_D(u) - \{v\} = N_D(v) - \{u\}$ and $I(u) = I(v)$, we obtain $w_f(u) - f(v) = w_f(v) - f(u)$. If $f(v) = f(u)$ then $w_f(u) = w_f(v)$, a contradiction. □

The example of some graphs in Observation 1.2 can be seen in Figure 2.

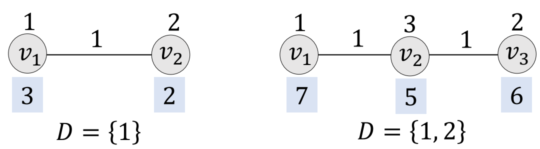

Figure 2. Some graphs with the distinct label of vertices

In Figure 2, for $D = \{1\}$, the label of $v_1, v_2 \in V(P_2)$ ought to be distinct, so that $w_{f,\{1\}}(v_1) \neq w_{f,\{1\}}(v_2)$ and for $D = \{1, 2\}$, the label of $v_1, v_2, v_3 \in V(P_3)$ ought to be distinct, so that the weight of each vertex of P_3 can be different.

Wijayanti et al. [11], in their study of the total distance vertex irregularity strength of fan and wheel graphs for $D = \{1\}$, propose two theorems and one lemma,

Theorem 1.1. *[11] Let F_n be a fan graph and $n \geq 1$. If F_n has the distance vertex irregular total k-labeling then*

$$tdis(F_n) = \begin{cases} \left\lceil \frac{n+2}{5} \right\rceil, & \text{for } n > 2, \\ 2, & \text{for } n = 1, 2, 3. \end{cases} \tag{4}$$

Theorem 1.2. *[11] Let W_n be a wheel graph and $n \geq 3$. If W_n has the distance vertex irregular total k-labeling then*

$$tdis(W_n) = \left\lceil \frac{n+4}{5} \right\rceil, \text{ for } n > 2. \tag{5}$$

Overall, the lemma for determining the lower bound of total distance vertex irregularity strength for a graph G is provided as follows,

Lemma 1.1. *Let $G(V, E)$ be a simple finite graph with maximum degree Δ and minimum degree δ. The lower bound of $tdis(G)$ is*

$$tdis(G) \geq \left\lceil \frac{|V(G)|-1+2\delta}{2\Delta} \right\rceil. \tag{6}$$

Proof. Let $G = (V, E)$ be a simple finite graph with maximum degree Δ and minimum degree δ. The smallest weight of vertices in $V(G)$ is 2δ (when we put label 1 to all neighbors of a vertex with degree δ and all incident edges of its vertex). Since the weight of every vertex ought to be distinct and G has $|V(G)|$ vertices, the minimum value of largest weight is $2\delta + |V(G)| - 1$ (when the values of all vertices weight in G construct an arithmetic progression). This weight is obtained from the sum of at most 2Δ integers. Thus, the largest label contributing to this weight must be at least $\left\lceil \frac{|V(G)|-1+2\delta}{2\Delta} \right\rceil$. □

Hence, we used the theorems above to prove some theorems in the next section.

2. Main Result

Let H and J be two graphs ordered m and n, consecutively. The corona product of graph H with graph J denoted by $H \odot J$ is a graph resulting from making m copies of J, namely J_i: $i = 1, \ldots, m$, and connecting every vertex in $V(J_i)$ with the i-th vertex of H. In this section, we define the distance vertex irregular total k-labeling and determine the total distance vertex irregularity strength of some corona product graphs namely $\bar{K}_m \odot \bar{K}_n$, $C_n \odot K_1$ and $P_n \odot K_1$.

2.1. *Total distance vertex irregularity strength of* $\bar{K}_m \odot \bar{K}_n$

Let $\bar{K}_m$ be the complement of the complete graph K_m. $|V(\bar{K}_m \odot \bar{K}_n)| = m(n+1)$ and $|E(\bar{K}_m \odot \bar{K}_n)| = mn$, with mn vertices of degree 1 and m vertices of degree n. Figure 3 shows a distance vertex irregular total labeling of $\bar{K}_3 \odot \bar{K}_2$.

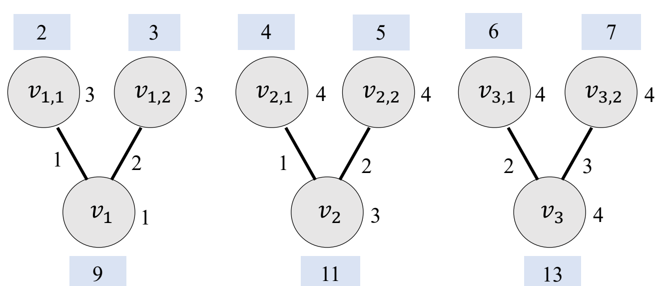

Figure 3. A distance vertex irregular total labeling of $\bar{K}_3 \odot \bar{K}_2$

Theorem 2.1 discusses the exact value of total distance vertex irregularity strength of $\bar{K}_m \odot \bar{K}_n$.

Theorem 2.1. *Let m, n be two positive integer. If $\bar{K}_m \odot \bar{K}_n$ has a distance vertex irregular total k-labeling, then,*

$$tdis(\bar{K}_m \odot \bar{K}_n) = \begin{cases} n, & \text{for } m = 1, \ n \geq 2, \\ m+1, & \text{for } m \geq 1, \ n = 1, \\ \left\lceil \frac{mn+1}{2} \right\rceil, & \text{for } m \geq 2, \ n \geq 2. \end{cases} \quad (7)$$

Proof. Graph $\bar{K}_m \odot \bar{K}_n$ has $m(n+1)$ vertices and mn edges, m vertices of degree $n = \Delta$ and mn vertices of degree $1 = \delta$. Without compromising generality, name the vertices of degree n as v_i and the vertices of degree 1 as v_{ij}, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We obtain, $V(\bar{K}_m \odot \bar{K}_n) = \{v_1, \ldots, v_m, v_{1,1}, \ldots, v_{1,n}, \ldots, v_{m,1}, \ldots, v_{m,n}\}$, and $E(\bar{K}_m \odot \bar{K}_n) = \{v_1 v_{1,1}, \ldots, v_1 v_{1,n}, \ldots, v_m v_{m,1}, \ldots, v_m v_{m,n}\}$. The smallest weight is 2 (if we give label 1 to neighbor vertex and the incident edge of a vertex with degree 1). The minimum k of which $\bar{K}_m \odot \bar{K}_n$ has a distance vertex irregular total k-labeling can be obtained if the weight of the vertices $\bar{K}_m \odot \bar{K}_n$ construct an arithmetic progression $2, 2+1, \ldots, 2+mn-1, 2+mn, \ldots, 2+mn+m-1$.

For $m = 1, n \geq 2$, v_1 is a neighbor of every vertex of degree 1. Hence, for the weight of each vertex $v_{1,j}$ to be different, the label of $v_1 v_j$, $j = 1, \ldots, n$ must be distinct, and since the lower bound of maximum weight of $v_{1,j}$ is $n+1$, this resulted in the maximum label of $v_1 v_j$ ought to be equal to or greater than n. Moreover, since the weights of v_1 must be greater than or equal to $n+2$, results $tdis(\bar{K}_m \odot \bar{K}_n) \geq max\left\{\left\lceil \frac{n+2}{2n} \right\rceil, n\right\} = n$. For $m \geq 1, n = 1$, $max\{w_f(v_i), w_f(v_{i,1})\} = max\{2, 4, \ldots, 2m, 3, 5, \ldots, 2m+1\} = 2m+1$. This resulting in, $tdis(\bar{K}_m \odot \bar{K}_n) \geq \left\lceil \frac{2m+1}{2} \right\rceil = m+1$. For $m \geq 2, n \geq 2$, the maximum weights of v_i and $v_{i,j}$ are $mn+1$ and $mn+m+1$, respectively, resulting in $tdis(\bar{K}_m \odot \bar{K}_n) \geq max\left\{\left\lceil \frac{mn+1}{2} \right\rceil, \left\lceil \frac{mn+m+1}{2n} \right\rceil\right\} = \left\lceil \frac{mn+1}{2} \right\rceil$.

Furthermore, we prove the equation by showing the existence of a distance vertex irregular total $k-$labeling of $\bar{K}_m \odot \bar{K}_n$, where $k = n$ for $m = 1$, $n \geq 2$, $k = m+1$ for $m \geq 1$, $n = 1$ and $k = \left\lceil \frac{mn+1}{2} \right\rceil$ for $m \geq 2$, $n \geq 2$. We proceed by considering three cases.

Case 1. For $m = 1$, $n \geq 2$, define f, the total labeling function of $\bar{K}_m \odot \bar{K}_1$ as follows.

$$\begin{aligned} f(v_1) &= 1, \\ f(v_{1,1}) &= 1, \\ f(v_1 v_j) &= j, \text{ for } j = 1, \ldots, n. \end{aligned}$$

The labeling function f generates a weight function w_f, which is,

$$\begin{aligned} w_f(v_1) &= \tfrac{n(n+3)}{2}, \\ w_f(v_{1,j}) &= j+1, \text{ for } j = 1, \ldots, n. \end{aligned}$$

Hence, $\{w_f(v_{1,j}), \text{for } j = 1, \ldots, n\} = \{w_f(v_{1,1}), w_f(v_{1,2}), \ldots, w_f(v_{1,n})\} = \{2, 3, 4, 5, \ldots, n+1\}$, constructing an arithmetic progression, with the maximum vertex weight $w_f(v_{1,n}) = n+1$. The maximum labels of vertices and edges of $\bar{K}_m \odot \bar{K}_n$ is $k = n$.

Case 2. For $m \geq 2, n = 1$, define f, the total labeling function of $\bar{K}_m \odot \bar{K}_1$ as follows,

$$\begin{aligned} f(v_i) &= i, && \text{for } i = 1, \ldots, m, \\ f(v_{i,1}) &= i+1, && \text{for } i = 1, \ldots, m, \\ f(v_i v_{i,1}) &= i, && \text{for } i = 1, \ldots, m. \end{aligned}$$

The labeling function f generates a weight function w_f, which is,

$$\begin{aligned} w_f(v_{i,1}) &= 2i, && \text{for } i = 1, \ldots, m, \\ w_f(v_i) &= 2i+1, && \text{for } i = 1, \ldots, m. \end{aligned}$$

Thus, for $i = 1, \ldots, m$ and $j = 1$ we find that the set of the weight of the vertices of graph $\bar{K}_m \odot \bar{K}_n$ is $\{w_f(v_{1,1}), w_f(v_1), w_f(v_{2,1}), w_f(v_2), \ldots, w_f(v_{m,1}), w_f(v_m)\} = \{2, 3, 4, 5, \ldots, 2m, 2m+1\}$, constructing an arithmetic progression, with the maximum vertex weight $w_f(v_m) = 2m+1$. Since all the vertices in $V(\bar{K}_m \odot \bar{K}_n)$, $m \geq 2, n = 1$, have degree 1, the maximum label of the vertices and edges is $k = \lceil \frac{2m+1}{2} \rceil = m + 1$.

Case 3. For $m \geq 2, n \geq 2$, we define f as follows.

$$f(v_i) = \begin{cases} 1, & \text{for } i = 1, \\ \lceil \frac{in+1}{2} \rceil, & \text{for } i = 2, \ldots, m. \end{cases} \tag{8}$$

$$f(v_i v_j) = \begin{cases} j, & \text{for } i = 1 \ , \ j = 1, \ldots, n, \\ (i-1)n + j - \lceil \frac{in-1}{2} \rceil, & \text{for } i = 2, \ldots, m, \ j = 1, \ldots, n. \end{cases} \tag{9}$$

$$f(v_{i,j}) = \begin{cases} \lceil \frac{mn+1}{2} \rceil - 1, & \text{for } i = 1, \ j = 1, 2, \ldots, n, \\ \lceil \frac{mn+1}{2} \rceil, & \text{for } i = 2, \ldots m, \ j = 1, 2, \ldots, n. \end{cases} \tag{10}$$

We obtain w_f, the function of the weight of the vertices of $\bar{K}_m \odot \bar{K}_n$ generated by f, as follows.

$$\begin{aligned} w_f(v_{i,j}) &= (i-1)n + 1 + j, \text{ for } i = 1, \ldots, m, \ j = 1 \ldots, n, \\ w_f(v_1) &= \tfrac{n(n+1)}{2} + n\left(\lceil \tfrac{mn+1}{2} \rceil - 1\right), \\ w_f(v_i) &= \tfrac{n(n-1)}{2} + n\left(\lceil \tfrac{in}{2} \rceil - n + 1 + \lceil \tfrac{mn+1}{2} \rceil\right), \text{ for } i = 2, \ldots m. \end{aligned}$$

Thus, the sets of $\{w_f(v_{i,j}) : \ i = 1, \ldots, m, \ j = 1, \ldots, n\}$ and $\{w_f(v_i) : \ i = 1, \ldots, m\}$ are

$$\begin{gathered} \{w_f(v_{1,1}), \ldots, w_f(v_{1,n}), w_f(v_{2,1}), \ldots, w_f(v_{2,n}), \ldots, w_f(v_{m,1}), \ldots w_f(v_{m,n})\} \\ = \{2, \ldots, n+1, n+2, \ldots, 2n+1, \ldots, (m-1)n+2, \ldots, mn+1\}, \end{gathered}$$

and

$$\begin{gathered} \{w_f(v_1), w_f(v_2), w_f(v_3), \ldots, w_f(v_{m-1}), w_f(v_m)\} = \Big\{ \tfrac{n(n+1)}{2} + n\left(\lceil \tfrac{mn+1}{2} \rceil - 1\right), \\ \tfrac{n(n-1)}{2} + n\left(\lceil \tfrac{2n}{2} \rceil - n + 1 + \lceil \tfrac{mn+1}{2} \rceil\right), \tfrac{n(n-1)}{2} + n\left(\lceil \tfrac{3n}{2} \rceil - n + 1 + \lceil \tfrac{mn+1}{2} \rceil\right), \ldots, \\ \tfrac{n(n-1)}{2} + n\left(\left\lceil \tfrac{(m-1)n}{2} \right\rceil - n + 1 + \lceil \tfrac{mn+1}{2} \rceil\right), \tfrac{n(n-1)}{2} + n\left(\lceil \tfrac{mn}{2} \rceil - n + 1 + \lceil \tfrac{mn+1}{2} \rceil\right) \Big\}. \end{gathered}$$

The sequence of $w_f(v_{i,j})$ forms an arithmetic progression, with the maximum weight of the vertex is $w_f(v_{m,n}) = mn + 1$ and the sequence of $w_f(v_i)$ forms an arithmetic progression with the maximum weight of the vertex is $w_f(v_m) = \frac{n(n-1)}{2} + n\left(\lceil \frac{mn}{2} \rceil - n + 1 + \lceil \frac{mn+1}{2} \rceil\right)$. The maximum label of v_i and $v_{i,j}$ is $k = \lceil \frac{mn+1}{2} \rceil$. It is clear that, $\frac{n(n+1)}{2} + n\left(\lceil \frac{mn+1}{2} \rceil - 1\right)$ always bigger than $mn + 1$, for all integer value of m and n. Every weight of the vertex of $\bar{K}_m \odot \bar{K}_n$ is distinct and $tdis(\bar{K}_m \odot \bar{K}_n) = \lceil \frac{mn+1}{2} \rceil$. □

2.2. Total distance vertex irregularity strength of $C_n \odot K_1$

The graph $C_n \odot K_1$ is known as sun graph. The total distance vertex irregularity strength of $C_n \odot K_1$, is determined in the following theorems.

Theorem 2.2. *Let n be a positive integer, $n \geq 3$. If $C_n \odot K_1$ has the distance vertex irregular total k-labeling then*

$$tdis(C_n \odot K_1) \geq \left\lceil \frac{n+1}{2} \right\rceil. \quad (11)$$

Proof. Let n be a positive integer, $n \geq 3$ and $C_n \odot K_1$ be a corona product graph. Define f as a distance vertex irregular total k-labeling on $C_n \odot K_1$. Since $C_n \odot K_1$ a corona product graph, it has n vertices of degree one and n vertices of degree three. For $i = 1, 2, \ldots, n$, we denote the vertex of degree one as v_i and the vertex of degree three as u_i, which is $V(C_n \odot K_1) = \{u_i : \ i = 1, 2, \ldots, n\} \cup \{v_i : \ i = 1, 2, \ldots, n\}$ and $E(C_n \odot K_1) = \{u_i v_i : \ i = 1, \ldots, n\} \cup \{u_i u_{i+1} : \ i = 1, \ldots, n-1\} \cup \{u_n u_1\}$. The smallest weight is obtained from a vertex of degree one, if we label $'1'$ to its neighborhood vertex and incident edge. Meanwhile, the highest weight is obtained from a vertex of degree three. The set of the optimal weight of the vertices in $V(C_n \odot K_1)$ under f is $\{w_f(v_i) : \ i = 1, \ldots, n\} \cup \{w_f(u_i) : \ i = 1, \ldots, n\} = \{2, 3, \ldots, n+1\} \cup \{n+2, \ldots, 2n+1\}$. We obtain

$$tdis(C_n \odot K_1) = k \geq max \left\{ \left\lceil \frac{n+1}{2} \right\rceil, \left\lceil \frac{2n+1}{6} \right\rceil \right\} = \left\lceil \frac{n+1}{2} \right\rceil. \quad (12)$$

The weight of all vertices of $C_n \odot K_1$, under f are distinct and $tdis(C_n \odot K_1) \geq \left\lceil \frac{n+1}{2} \right\rceil$. □

The next theorem presents the exact value of the total distance vertex irregularity strength of $C_n \odot K_1$.

Theorem 2.3. *Let n be a positive integer, $n \geq 3$ and $C_n \odot K_1$ be a corona product graph. If $C_n \odot K_1$ has a distance vertex irregular total k-labeling, then*

$$tdis(C_n \odot K_1) = \left\lceil \frac{n+1}{2} \right\rceil. \quad (13)$$

Proof. Let n be a positive integer, $n \geq 3$ and $C_n \odot K_1$ be a corona product graph, where $V(C_n \odot K_1) = \{u_i : \ i = 1, 2, \ldots, n\} \cup \{v_i : \ i = 1, 2, \ldots, n\}$ and $E(C_n \odot K_1) = \{u_i v_i : \ i = 1, \ldots, n\} \cup \{u_i u_{i+1} : \ i = 1, \ldots, n-1\} \cup \{u_n u_1\}$. The number of vertices and edges of $C_n \odot K_1$ are $|V(C_n \odot K_1)| = 2n$ and $|E(C_n \odot K_1)| = 2n$. The minimum degree of $C_n \odot K_1$ is $\delta = 1$ and maximum degree of $C_n \odot K_1$ is $\Delta = 3$. Using Theorem 2.2, we have the lower bound of total distance vertex irregularity strength of sun graphs is $tdis(C_n \odot K_1) \geq \left\lceil \frac{n+1}{2} \right\rceil$.
Hence, proving the equation of $tdis(C_n \odot K_1)$ is sufficient, provided that a distance vertex irregular total k-labeling with $k = \left\lceil \frac{n+1}{2} \right\rceil$ exists. To do so, consider the following three cases.

Case 1. For $n = 3$.

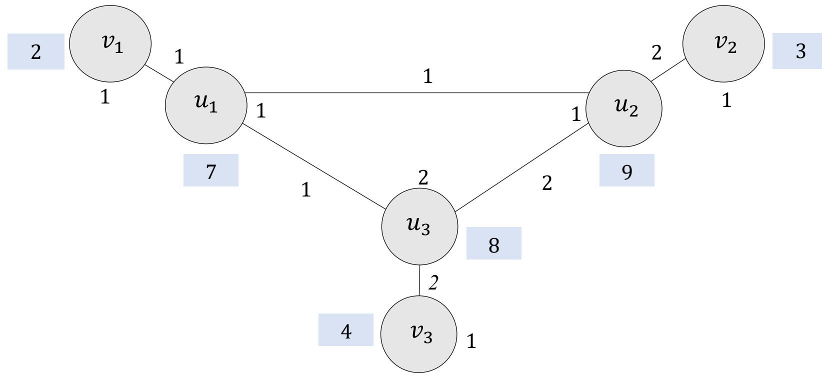

Figure 4. A distance vertex irregular total k-labeling of $C_n \odot K_1$ for $n = 3$

Hence, the weight of every vertex of $C_n \odot K_1$ is distinct. For $n = 3$, $tdis(C_n \odot K_1) = \lceil \frac{n+1}{2} \rceil$.

Case 2. For $n \equiv 1, 2 \pmod 4$, $n \geq 4$. Define f, a total labeling function of $C_n \odot K_1$ as follows.

$$f(u_i) \quad = \quad \begin{cases} i, & \text{for } \; i = 1, \ldots, \lceil \frac{n}{2} \rceil, \\ n - i + 2, & \text{for } \; i = \lceil \frac{n}{2} \rceil + 1, \ldots, n. \end{cases} \tag{14}$$

$$f(u_i v_i) \quad = \quad \begin{cases} 1, & \text{for } \; i = 1, \\ i - 1, & \text{for } \; i = 2, \ldots, \lceil \frac{n}{2} \rceil + 1, \\ n - i + 2, & \text{for } \; i = \lceil \frac{n}{2} \rceil + 2, \ldots, n. \end{cases} \tag{15}$$

$$f(u_i u_j) = \lceil \tfrac{n-3}{4} \rceil - 1, \;\; \text{for } \; i = 1, \ldots, n-1, j = i+1, \text{and } i = n, j = 1. \tag{16}$$

Subcase 2.1. If $n \equiv 1 \pmod 4$, $n \geq 4$, the label of v_i is

$$f(v_i) \quad = \quad \begin{cases} \lceil \frac{n+1}{2} \rceil - 1, & \text{for } \; i = 1, \\ \lceil \frac{n+1}{2} \rceil - i + 2, & \text{for } \; i = 2, \ldots, \lceil \frac{n}{2} \rceil - 1, \\ 3, & \text{for } \; i = \lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 1, \\ i - \lceil \frac{n}{2} \rceil + 1, & \text{for } \; i = \lceil \frac{n}{2} \rceil + 2, \ldots, n. \end{cases} \tag{17}$$

Subcase 2.2. If $n \equiv 2 \pmod 4$, $n \geq 4$, the label of v_i is

$$f(v_i) \quad = \quad \begin{cases} \lceil \frac{n+1}{2} \rceil - 1, & \text{for } \; i = 1, \\ \lceil \frac{n+1}{2} \rceil - i + 2, & \text{for } \; i = 2, \ldots, \lceil \frac{n}{2} \rceil, \\ 4, & \text{for } \; i = \lceil \frac{n}{2} \rceil + 1, \\ i - \lceil \frac{n}{2} \rceil + 1, & \text{for } \; i = \lceil \frac{n}{2} \rceil + 2, \ldots, n. \end{cases} \tag{18}$$

It is shown that the labels of every vertex and edge are not greater than $\lceil \frac{n+1}{2} \rceil$.

Case 3. For $n \equiv 0, 3 \pmod 4$, $n \geq 4$. Define f, a total labeling function of $C_n \odot K_1$ as follows,

$$f(u_i) = \begin{cases} i, & \text{for } i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil, \\ n - i + 2, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n. \end{cases} \quad (19)$$

$$f(u_i v_i) = \begin{cases} 1, & \text{for } i = 1, \\ i - 1, & \text{for } i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil + 1, \\ n - i + 2, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 2, \dots, n. \end{cases} \quad (20)$$

$$f(u_i u_j) = \left\lceil \tfrac{n-3}{4} \right\rceil, \text{ for } i = 1, \dots, n-1, \ j = i + 1, \text{ and } i = n, \ j = 1. \quad (21)$$

Subcase 3.1. If $n \equiv 0 \pmod 4$, $n \geq 4$, the label of v_i is

$$f(v_i) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil - 2, & \text{for } i = 1, \\ \left\lceil \frac{n+1}{2} \right\rceil - i + 1, & \text{for } i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil, \\ 3, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1, \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 2, \dots, n. \end{cases} \quad (22)$$

Subcase 3.2. If $n \equiv 3 \pmod 4$, $n \geq 4$, the label of v_i is

$$f(v_i) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil - 2, & \text{for } i = 1, \\ \left\lceil \frac{n+1}{2} \right\rceil - i + 1, & \text{for } i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil - 1, \\ 2, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil + 1, \\ i - \left\lceil \frac{n}{2} \right\rceil + 1, & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 2 \dots, n. \end{cases} \quad (23)$$

It shows that the labels of every vertex and edge are not greater than $\left\lceil \frac{n+1}{2} \right\rceil$.

As a result of this, the set of the weight of the vertices of $C_n \odot K_1$ is $\{2, 3, 4, \dots, n+2, n+3, \dots, 2n+1\}$ where the labels of every vertex and edge are not greater than $\left\lceil \frac{n+1}{2} \right\rceil$. □

2.3. Total distance vertex irregularity strength of $P_n \odot K_1$

In this section, we determine the total distance vertex irregularity strength of $P_n \odot K_1$, which is known as comb graphs. The lower bound of $tdis(P_n \odot K_1)$ is given as follows,

Theorem 2.4. *Let n be a positive integer. If $P_n \odot K_1$ has the distance vertex irregular total k-labeling then*

$$tdis(P_n \odot K_1) \geq \left\lceil \tfrac{n+1}{2} \right\rceil. \quad (24)$$

Proof. Let n be a positive integer, define f as a distance vertex irregular total k-labeling on $P_n \odot K_1$. Since $P_n \odot K_1$ is a corona product graph, it has n vertices of degree one, two vertices of degree two and $n-2$ vertices of degree three. For $i = 1, 2, \dots, n$, we denote the vertex of degree one as v_i and the vertex of degree two and three as u_i. Therefore $V(P_n \odot K_1) = \{u_i : \ i = 1, 2, \ldots, n\} \cup \{v_i : i = 1, 2, \ldots, n\}$ and $E(P_n \odot K_1) = \{u_i v_i : \ i = 1, \ldots, n\} \cup \{u_i u_{i+1} : \ i = 1, \ldots, n-1\}$. The smallest weight is obtained if we put 1 as a label to neighborhood vertex and incident edge of vertex with degree one. Thus the maximum weight ought to be on vertex with degree three. The set of the optimal weight of all vertices in $V(P_n \odot K_1)$ under f is $\{w_f(v_i) : \ i = 1, \ldots, n\} \cup \{w_f(u_i) : i = 1, \ldots, n\} = \{2, 3, \ldots, n+1\} \cup \{n+2, \ldots, 2n+1\}$. We obtain,

$$tdis(P_n \odot K_1) = k \geq max\left\{\left\lceil \tfrac{n+1}{2} \right\rceil, \left\lceil \tfrac{2n+1}{6} \right\rceil\right\} = \left\lceil \tfrac{n+1}{2} \right\rceil. \tag{25}$$

As a result, the weight of all vertices of $P_n \odot K_1$, under f are distinct and $tdis(P_n \odot K_1) \geq \left\lceil \frac{n+1}{2} \right\rceil$. □

The exact value of total distance vertex irregularity strength of $P_n \odot K_1$ is presented in the following theorem.

Theorem 2.5. *Let n be a positive integer and $P_n \odot K_1$ be a corona product graph. If $P_n \odot K_1$ has the distance vertex irregular total k-labeling, then*

$$tdis(P_n \odot K_1) \ = \ \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & \text{for } n \geq 2, \\ 2, & \text{for } n = 1. \end{cases} \tag{26}$$

Proof. Let n be a positive integer and $P_n \odot K_1$ be a corona product graph. $V(P_n \odot K_1) = \{u_i : i = 1, 2, \ldots, n\} \cup \{v_i : \ i = 1, 2, \ldots, n\}$ and $E(KP_n) = \{u_i v_i : \ i = 1, \ldots, n\} \cup \{u_i u_{i+1} : i = 1, \ldots, n-1\}$. The number of vertices and edges of $P_n \odot K_1$ are $|V(P_n \odot K_1)| = 2n$ and $|E(P_n \odot K_1)| = 2n-1$. The minimum degree of $P_n \odot K_1$ is $\delta = 1$ and maximum degree of $P_n \odot K_1$ is $\Delta = 3$. Using Theorem 2.4, the lower bound of $tdis(P_n \odot K_1)$ is $\left\lceil \frac{n+1}{2} \right\rceil$. Therefore, proving the equation of $tdis(P_n \odot K_1)$, when a distance vertex irregular total k-labeling with $k = \left\lceil \frac{n+1}{2} \right\rceil$ is evidence. It is done by considering the following two cases.

Case 1. For $n \leq 7$ and $n \neq 6$. For $n = 1, 2, 3$, we label the vertices and edges of $P_n \odot K_1$ as in Figure 5.

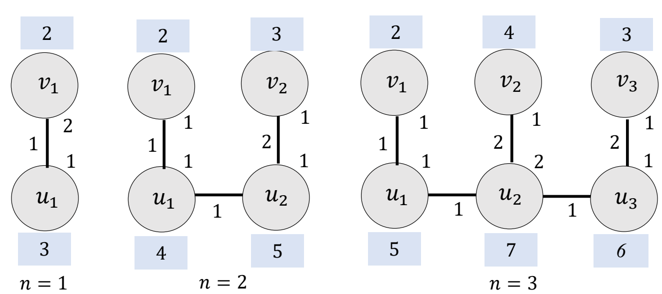

Figure 5. A distance vertex irregular total k-labeling of $P_n \odot K_1$ for $n = 1, 2, 3$

For $n = 4, 5$, the label of vertices and edges of $P_n \odot K_1$ is shown in Figure 6.

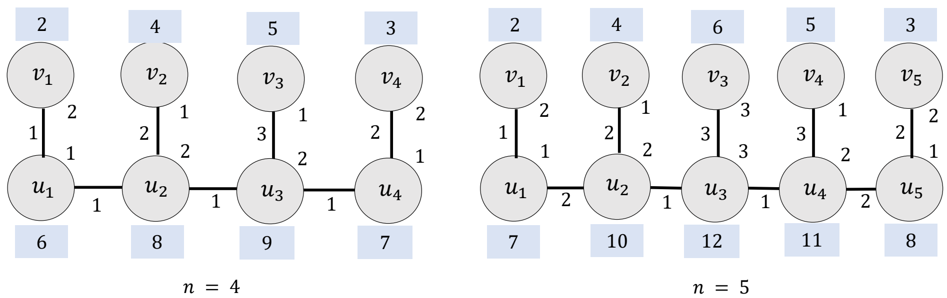

Figure 6. A distance vertex irregular total k-labeling of $P_n \odot K_1$ for $n = 4, 5$

For $n = 7$, the label of vertices and edges of $P_n \odot K_1$ is shown in Figure 7.

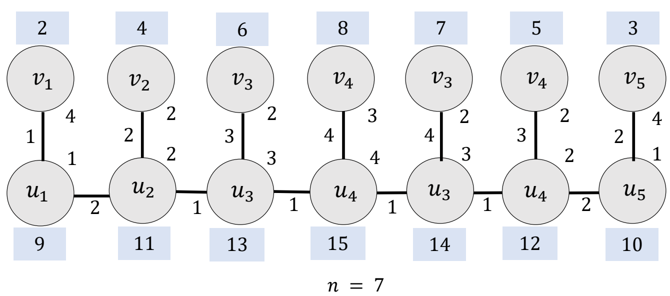

Figure 7. A distance vertex irregular total k-labeling of $P_n \odot K_1$ for $n = 7$

The weight of every vertex of $P_n \odot K_1$ is distinct. For $n = 1$, we obtain $tdis(P_n \odot K_1) = 2$ and for $n = 2, 3, 4, 5, 7$, $tdis(P_n \odot K_1) = \lceil \frac{n+1}{2} \rceil$.

Case 2. For $n \geq 6$, $n \neq 7$.

$$f(u_i) = \begin{cases} i, & \text{for } i = 1, \ldots, \lceil \frac{n}{2} \rceil, \\ n - i + 1, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \ldots, n. \end{cases} \tag{27}$$

$$f(u_i v_i) = \begin{cases} i, & \text{for } i = 1, \ldots, \lceil \frac{n}{2} \rceil, \\ n - i + 2, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \ldots, n. \end{cases} \tag{28}$$

Subcase 2.1. If $n \equiv 0, 2, 4 \pmod 6$, $n \geq 6$ and $n \neq 7$ the labels of v_i, $i = 1, \ldots, n$ and $u_i u_{i+1}$, $i = 1, \ldots, n-1$ are

$$f(v_i) = \begin{cases} \lceil \frac{n+1}{2} \rceil, & \text{for } i = 1, n, \\ 2, & \text{for } i = 2, n-1, \\ i - 1, & \text{for } i = 3, \ldots, \lceil \frac{n}{2} \rceil, \\ n - i, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \ldots, n-2. \end{cases} \tag{29}$$

$$f(u_iu_{i+1}) = \begin{cases} \lceil \frac{n}{2} \rceil - i - 1, & \text{for } i = 1, \\ \lceil \frac{n}{2} \rceil - i, & \text{for } i = 2, \ldots, \lceil \frac{n}{2} \rceil - 1, \\ 1, & \text{for } i = \lceil \frac{n}{2} \rceil, \\ i - \lceil \frac{n}{2} \rceil, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \ldots, n-2, \\ i - \lceil \frac{n}{2} \rceil - 1, & \text{for } i = n-1. \end{cases} \quad (30)$$

Subcase 2.2. If $n \equiv 1, 3, 5 \pmod 6$, $n > 7$, the labels of v_i, $i = 1, \ldots, n$ and u_iu_{i+1}, $i = 1, \ldots, n-1$ are

$$f(v_i) = \begin{cases} \lceil \frac{n+1}{2} \rceil, & \text{for } i = 1, n, \\ i, & \text{for } i = 2, \ldots, \lceil \frac{n}{2} \rceil - 2, \\ i - 1, & \text{for } i = \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil, \\ n - i, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \\ n - i + 1, & \text{for } i = \lceil \frac{n}{2} \rceil + 2, \ldots, n-1. \end{cases} \quad (31)$$

$$f(u_iu_{i+1}) = \begin{cases} \lceil \frac{n}{2} \rceil - i - 1, & \text{for } i = 1, \ldots, \lceil \frac{n}{2} \rceil - 2, \\ 1, & \text{for } i = \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil, \\ i - \lceil \frac{n}{2} \rceil, & \text{for } i = \lceil \frac{n}{2} \rceil + 1, \ldots, n-1. \end{cases} \quad (32)$$

It is shown that the labels of every vertex and edge are not greater than $\lceil \frac{n+1}{2} \rceil$. □

3. Conclusion

In the previous sections, we determine the lower bound of the total distance vertex irregularity strength and establish the exact value of the total distance vertex irregularity strength for $\bar{K}_m \odot \bar{K}_n$, $C_n \odot K_1$ and $P_n \odot K_1$ graphs. For $m \geq 2, n \geq 2$, the total distance vertex irregularity strength of $\bar{K}_m \odot \bar{K}_n$ is $\lceil \frac{mn+1}{2} \rceil$. For $m \geq 1, n = 1$, $tdis(\bar{K}_m \odot \bar{K}_1) = m + 1$ and for $m = 1, n \geq 2, tdis(\bar{K}_1 \odot \bar{K}_m) = n$. For sun graphs with $n \geq 3$, $tdis(C_n \odot K_1) = \lceil \frac{n+1}{2} \rceil$. For comb graphs with $n \geq 2$, $tdis(P_n \odot K_1) = \lceil \frac{n+1}{2} \rceil$ and for $n = 1$, $tdis(P_n \odot K_1) = 2$. Hence, we conclude this paper with an open problem as follows.

Problem. Investigate the total distance vertex irregularity strength for corona product of arbitrary graph with zero graphs.

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