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On families of 2-nearly Platonic graphs

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Abstract

A 2-nearly Platonic graph of type $(k|d)$ is a k-regular planar graph with f faces, $f-2$ of which are of size d and the remaining two are of sizes d_1, d_2, both different from d. Such a graph is called balanced if $d_1 = d_2$. We show that all connected 2-nearly Platonic graphs are necessarily balanced. This proves a recent conjecture by Keith, Froncek, and Kreher.

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1. Introduction

Throughout this paper, all graphs we consider are finite, simple, connected, planar, undirected and non-trivial graph.

A graph is said to be planar, or embeddable in the plane, if it can be drawn in the plane such that each common point of two edges is a vertex. This drawing of a planar graph G is called a planar embedding of G and can itself be regarded as a graph isomorphic to G. Sometimes, we

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call a planar embedding of a graph a *plane graph*. By this definition, it is clear that we need some matters of the topology of the plane. Immediately, after deleting the points of a plane graph from the plane, we have some maximal open sets (or regions) of points in the plane called *faces* of the plane graph. There exist exactly one unbounded region that we call it the *outerface* of the plane graph and other faces (if they exist) are called as *internal faces*. The boundary of a face is the set of points of vertices and edges touching the face. In the graph-theoretic language, the boundary of a face is a closed walk. The number of edges located on the boundary of a face is called the *degree* of the face. A face is said to be incident with the vertices and edges in its boundary, and two faces are adjacent if their boundaries have an edge in common.

A graph G is k*-regular* when the degrees of all vertices are equal to k. A regular graph is one that is k-regular for some k. Let $G = (V, E, F)$ be a graph with the vertex set V, edge set E, and face set F. The well-known *Euler's formula* states that if G is a connected planar graph, then

$$|V| - |E| + |F| = 2.$$

The *size* of a face in a plane graph G is the total length of the closed walk(s) bounding the face in G. A cut-edge belongs to the boundary of only one face, and it contributes twice to its size.

A graph G is k*-connected* if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.

Platonic solids are a well-known family of five three-dimensional polyhedra. There is no reliable information about their first mention, and different opinions have been taken [1, 17]. However, they are attractive for mathematicians and others, in terms of some symmetries that they have. In the last two centuries, many of authors have paid attention to the polyhedra and they have extended it to convex and concave polytopes, see, e.g., [8].

But what matters from the combinatorial point of view is that a convex polyhedron can be embedded on a sphere, and then we can map it on a plane so that the images of lines on the sphere do not cut each other in the plane. In this way, we have corresponded a polyhedron on the sphere with a planar graph in the plane. Steinitz's theorem (see, e.g., [8]) states that a graph G with at least four vertices is the network of vertices and edges of a convex polyhedron if and only if G is planar and 3-connected.

A k-regular planar graph with f faces is a t*-nearly Platonic graph* of type $(k|d)$ if $f > 2t$, $f - t$ of its faces are of size d and the remaining t faces are of sizes other than d. The faces of size d are often called *common faces*, and the remaining ones *exceptional* or *disparate* faces. When $t \geq 2$ and all disparate faces are of the same size, then the graph is called a *balanced* t*-nearly Platonic graph*.

In 1967, Grünbaum considered 3-regular and connected planar graphs and he got some results. For example, for a 3-regular connected planar graph and $b \in \{2, 3, 4, 5\}$, it is proved that if the size of all faces but t faces is divisible by b, then $t \geq 2$ and if $t = 2$ then two exceptional faces have not a common vertex [8]. In 1968, in his Ph.D thesis, Malkevitch proved the same results for 4 and 5-regular 3-connected planar graphs [18]. Several papers are devoted to the study of this topic, but all of them have considered planar graphs such that the sizes of all faces but some exceptional faces are a multiple of b and $b \in \{2, 3, 4, 5\}$ (see [2, 10, 11, 13, 14]).

Keith, Froncek, and Kreher [15, 16] and Froncek, Khorsandi, Musawi, and Qiu [6] proved recently that there are no 1-nearly Platonic graphs.

Deza, Dutour Sikirič, and Shtogrin [4] classified for each admissible pair $(k|d)$ all possible sizes of the exceptional faces of balanced 3-nearly Platonic graphs and sketched a proof of the completeness of the list. Froncek and Qiu [5] provided a detailed combinatorial proof of existence of infinite families of such graphs for each of the listed exceptional sizes.

There are 14 well-known families of balanced connected 2-nearly Platonic graphs (see, e.g., [4] or [15]). Deza, Dutour Sikirič, and Shtogrin [4] provide a list and offer a sketch of a proof of the completeness of the list. Keith, Froncek, and Kreher conjectured [15] that every connected 2-nearly Platonic graph must be balanced.

We show that the only admissible types of 2-nearly Platonic graphs are $(3|3)$, $(3|4)$, $(3|5)$, $(4|3)$, and $(5|3)$, and that all connected 2-nearly Platonic graphs are balanced. We also prove in detail that the list of 14 families of connected 2-nearly Platonic graphs presented by Deza, Dutour Sikirič, and Shtogrin [4] is complete.

In a recent preprint [12], Jendroľ lists 15 families of 2-nearly Platonic graphs. According to his paper, two of them are *topologically non-isomorphic* although they are isomorphic in the usual sense, when one only considers mutual adjacency of vertices. That is, two graphs $G_1(V_1, E_1, F_1)$ and $G_2(V_2, E_2, F_2)$ are *isomorphic* if there exists a bijection $\varphi : V_1 \to V_2$ such that $\varphi(x)\varphi(y) \in E_2$ if and only if $xy \in E_1$.

Note that every disconnected 2-nearly Platonic graph is the disjoint union of a connected 2-nearly Platonic graph and a number of Platonic graphs of the same type. So, throughout this paper, all 2-nearly Platonic graphs we consider are connected.

2. Terminology and notation

We say that the two exceptional faces are *touching each other* or simply *touching*, if they share at least one vertex. Similarly, an exceptional face will be called *self-touching* if a vertex appears on the boundary of the face more than once.

Definition 2.1. *Let G be a graph and $S \subseteq V(G)$. Then $\langle S\rangle$, the subgraph induced by S, denotes the graph on S whose edges are precisely the edges of G with both ends in S. Also, $G - S$ is obtained from G by deleting all the vertices in S and their incident edges. If $S = \{x\}$ is a singleton, then we write $G - x$ rather than $G - \{x\}$.*

Definition 2.2. *A $(k; k_1, k_2|d)$-block B of order n is a 2-connected planar graph with $n-2$ vertices of degree k, two vertices x and y with $\deg(x) = k_1, \deg(y) = k_2$ where $2 \le k_2 \le k_1 < k$, all faces but one of degree d, and the remaining face of degree $h \neq d$, where vertices x, y belong to the face of degree h.*

Definition 2.3. *A $(k; k_1|d)$-endblock of order n is a 2-connected planar graph with $n - 1$ vertices of degree k, one vertex x with $\deg(x) = k_1$ where $2 \le k_1 < k$, all faces but one of degree d, and the remaining face of degree $h \neq d$, where the vertex x belongs to the face of degree h.*

When we speak about blocks, we always assume that the exceptional face is the outerface. Let the boundary path of the exceptional face of $(k; k_1, k_2|d)$-block B be of size h. We denote the boundary of the exceptional face $x = x_0, x_1, \dots, x_a = y, x_{a+1}, \dots, x_{a+b-1}$ such that $h = a + b$ and always assume that $a \le b$. When we need to specify a and b in our arguments, we denote such a block as $(k; k_1, k_2|d, \langle a, b\rangle)$-block.

Similarly, if we need to specify the size h of the exceptional face in an endblock, we will denote it as $(k; k_1|d, \langle h\rangle)$-endblock.

We observe that when the exceptional faces of a 2-nearly Platonic graphs touch in exactly one vertex, say z, then by splitting z into two vertices x and y we obtain a $(k; k_1, k_2|d, \langle a, b\rangle)$-block, where $\deg(x) \ge 2, \deg(y) \ge 2$ and a and b are the sizes of the two original exceptional faces, $k_2 = 2$ and $2 \le k_1 \le 3$. Consequently, such a graph would have to be of type $(4|3)$ or $(5|3)$.

3. Related results

We will use the following results in our proofs.

Theorem 3.1. [19, Exercise 6.1.6] *A plane graph is* 2*-connected if and only if for every face, the bounding walk is a cycle.*

Theorem 3.2 ([16])**.** *There are no* 2*-connected* 1*-nearly Platonic graphs.*

The following result, without any proof, is stated in [3]. For a detailed proof, see [6, Theorem 6.1].

Theorem 3.3. *There are no $(k; k_1|d)$-endblocks of any admissible type.*

The non-existence of 1-nearly Platonic graphs with connectivity 1 follows directly from the above Theorem.

Theorem 3.4 ([6])**.** *There are no* 1*-nearly Platonic graphs with connectivity* 1*.*

Theorem 3.5. *Every connected* 2*-nearly Platonic graph is* 2*-connected.*

Proof. If a graph G is connected but not 2-connected, it must contain at least two endblocks. Let the endblocks be B_1 and B_2 with outer faces F_1 and F_2, respectively, and the boundary be a part of the outer face F_1' of G. The length of the boundary of F_1 is at least 3, and the same applies for the boundary of the rest of F_1'. Therefore, F_1' has a boundary of length at least 6 and must be one of the two exceptional faces. Since two endblocks B_1 and B_2 have at most a vertex in common, they cannot share in a face and so, at least one of B_1 and B_2 does not contain the other exceptional face. But then this endblock is a $(k; k_1|d)$-endblock as defined above. This is a contradiction, since it follows from Theorem 3.3 that no $(k; k_1|d)$-endblock can exist in a k-regular planar graph. Hence, a connected 2-nearly Platonic graph must be at least 2-connected. □

Proposition 3.1. *Suppose that $G = (V, E, F)$ is a* 2*-nearly Platonic graph of type (k, d). Then $3 \le d \le 5$ for $k = 3$ and $d = 3$ for $k = 4, 5$.*

Proof. Let $|V| = n$, $|E| = m$, $|F| = f$ and d_1 and d_2 be the sizes of two exceptional faces. The graph is k-regular and so we have $kn = 2m$. On the other hand, $2m = (f-2)d + d_1 + d_2$ and by Euler's formula $f - 2 = m - n = \frac{1}{2}(k-2)n$. Therefore, $kn = \frac{1}{2}(k-2)nd + d_1 + d_2$ or $d = \frac{2k}{k-2} - \frac{2(d_1+d_2)}{(k-2)n}$ which implies that $d < \frac{2k}{k-2}$. Now, if $k = 3$, then $3 \le d \le 5$ and if $k \in \{4, 5\}$, then $d = 3$. □

Remark 3.1. *Since blocks and endblocks in this paper are induced subgraphs of 2-nearly platonic graphs, Proposition 3.1 applies to them as well.*

4. Touching exceptional faces

Most proofs in this section are done by contradiction. We will assume a specific $(k;k_1,k_2|d)$-block B exists and attempt to build it step by step, adding vertices in our already built subgraph of B, until we find out that a vertex cannot be properly placed, or a face of size d cannot be created.

We introduce two notions that we will repeatedly use. In a $(k;k_1,k_2|d)$-block, a vertex is *saturated*, if it is of its desired degree, that is, k,k_1, or k_2. A path of length $d-1$ is *weakly saturated*, if all its internal vertices are saturated.

4.1. No self-touching exceptional face

First we observe that if there are two touching exceptional faces, each of them must be touching the other but not itself.

Lemma 4.1. *There is no self-touching exceptional face in any connected* 2*-nearly Platonic graph.*

Proof. Suppose that in a connected 2-nearly Platonic graph G there is a self-touching exceptional face. Then G has a cut-vertex, which contradicts Theorem 3.5. □

4.2. Excluding non-admissible values of a

Now we reduce the family of blocks that we need to investigate just to the cases where $a \leq d$. Recall that we denote the boundary of the exceptional face of a $(k;k_1,k_2|d,\langle a,b\rangle)$-block B by $x=x_0,x_1,\ldots,x_a=y,x_{a+1},\ldots,x_{a+b-1}$. We will use this notation in the proofs of the following lemmas.

Lemma 4.2. *If there exists a $(k;k-1,k_2|d,\langle a,b\rangle)$-block B with $a>d$, then there exists a $(k;k-1,k_2|d,\langle a',b'\rangle)$-block B' with $a'<a$.*

Proof. We observe that the edge x_0x_{d-1} is not present in B. If $k=3$, vertex x_0 is only adjacent to x_1 and x_{a+b-1}. If $k=4$, then $d=3$. Suppose for the sake of contradiction that the edge x_0x_2 belongs to B. Then the remaining neighbors of x_1, say y_1 and y_2, must be inside the cycle $C: x_0,x_1,x_2$. This means that one of the paths y_i,x_1,x_0 is weakly saturated and should be completed into a triangle. However, this is impossible, since x_0 is already of degree three.

Essentially the same argument can be made when $k=5$ and the fourth neighbor of x_0 is placed outside of C. Hence, we are left with the case when $k=5$, vertex x_0 has a neighbor y_0 inside C. Because x_0 is saturated, both paths x_1,x_0,y_0 and x_2,x_0,y_0 are weakly saturated and must be completed into triangles. This implies that y_0 is adjacent to both x_1 and x_2. Also we observe that now the path x_{a+b-1},x_0,x_2 is weakly saturated and we must have the edge $x_{a+b-1}x_2$. This means that x_2 is saturated. Now, both of triangles x_1,x_2,y_0 and x_0,x_1,y_0 are faces of B. Therefore, x_1 is of degree 3, a contradiction.

We are now ready to prove our claim. Take the block B, remove edge $x_{d-1}x_d$ and replace it by edge x_0x_{d-1}. Notice that the edge x_0x_{d-1} can be added, since it is not present in B as proved above.

Vertex x_0 is now of degree k while x_d is of degree $k-1$.

If $k = 4$ or 5, we first consider the case when the triangular face containing the edge $x_{d-1}x_d$ had the third vertex on the boundary of the outerface, call it x_j, where $d+1 < j \leq a+b-1$. Then the subgraph bounded by the cycle $x_0, x_{d-1}, x_j, x_{j+1}, x_{a+b-1}$ is an endblock with the exceptional vertex x_j of degree 2 or 3, and we contradict Theorem 3.3.

Therefore, the new face $x_0, x_1, \ldots, x_{d-1}$ and the new outerface are cycles. The lengths of the boundary segments are now $a-d$ and $b+d$, respectively. Other faces are the same as the faces of B. Thus, in this new graph all facial boundaries are cycles.

Now, by Theorem 3.1, this new graph is 2-connected. Therefore, we have found the desired $(k; k-1, k_2|d, \langle a-d, b+d\rangle)$-block B'. □

Hence, from now on we can only consider $(k; k-1, k_2|d)$-blocks with $1 \leq a \leq d$. First we exclude the existence of $(k; k-1, k_2|d)$-blocks with $a = d$.

Lemma 4.3. *There is no $(k; k-1, k_2|d)$-block with $a = d$ for any admissible k.*

Proof. Suppose such a block exists. First assume $k_2 = 2$. We remove the vertex $x_d = y$ and add the edge $x_0x_{d-1} = xx_{d-1}$.

Notice that the edge x_0x_{d-1} can be added, since it is not present in B as proved in the proof of Lemma 4.2. This way we obtain a new internal face of size d. Vertex x_0 is now of degree k, vertex x_{d+1} is now of degree $k-1$ and all other vertices are of degree k.

Similarly as in the proof of Lemma 4.2, all facial boundaries are cycles and so by Theorem 3.1, this new graph is 2-connected. But then we have a $(k; k-1|d)$-endblock, which does not exist by Theorem 3.3.

If $k_2 \geq 3$, then $4 \leq k \leq 5$ and we must have $d = a = 3$. Remove the edge $x_2x_3 = x_2y$ and replace it by the edge $x_0x_2 = xx_2$. This creates a new internal triangular face. Vertex x_0 is now of degree k, vertex $x_3 = y$ is of degree $k_2 - 1 \geq 2$ and all other vertices are of degree k.

Again as in the proof of Lemma 4.2, we first consider the case when the triangular face containing the edge $x_{d-1}x_d$ had the third vertex on the boundary of the outerface, say x_j, where $3 < j \leq a+b-1$. Then the subgraph bounded by the cycle $x_0, x_2, x_j, x_{j+1}, x_{a+b-1}$ is an endblock with the exceptional vertex x_j of degree 2 or 3, which contradicts Theorem 3.3.

Therefore, all facial boundaries are cycles and by Theorem 3.1 this new graph is 2-connected.

Now, the edge x_2x_3 must have belonged to a triangle x_2, x_3, z and we have a $(k; k_2-1|d)$-endblock with boundary $x_0, x_2, z, x_3, \ldots, x_{a+b-1}$ and x_3 of degree $k_2 - 1 \geq 2$, which does not exist by Theorem 3.3. □

The case of $a = d-1$ can be easily excluded for $k_2 < k-1$.

Lemma 4.4. *There is no $(k; k-1, k_2|d)$-block with $a = d-1$ and $k_2 < k-1$ for any admissible k.*

Proof. If such a block exists, then by adding edge xy we create a new internal face of size d. Notice that the edge $xy = x_0x_{d-1}$ can be added, since it is not present in B as proved in the proof of Lemma 4.2. Vertex y is still of degree less than k while all other vertices are of degree k. This new graphs would be a $(k; k_2|d)$-endblock, which does not exist by Theorem 3.3. □

Now we exclude the existence of $(k; k-1, k-1|d)$-blocks with $a = 1$.

Lemma 4.5. *There is no* $(k; k-1, k-1|d)$*-block with* $a = 1$ *for any admissible* k*.*

Proof. Suppose such a block B exists. If $d = 3$, then add a new vertex z and edges xz and yz. This creates a $(k, 2|3)$-endblock, which does not exist by Theorem 3.3.

For $d = 4$, create a copy B' of B with vertices x' and y' corresponding to x and y, respectively. Then add edges xx' and yy' to create a new internal face of size four. All vertices in this new graph are of degree $k = 3$, and the outerface is of size at least six. The resulting graph would now be 2-connected and 1-nearly Platonic, but such a graph does not exist by Theorem 3.2.

For $d = 5$, again create B' as above, and an extra vertex z. Add edges $xx', yz, y'z$ to obtain a new internal face x, x', y', z, y of size five, vertex z of degree two, and all other vertices of degree $k = 3$. The new graph now would be a $(k; 2|5)$-endblock, which does not exist by Theorem 3.3. □

For $k = 5$, we can even exclude large values of a even for $k_1 = k - 2 = 3$.

Lemma 4.6. *If there is a* $(5; 3, k_2|3, \langle a, b\rangle)$*-block* B *with* $a > 3$*, then there is a* $(5; 3, k_2|3, \langle a-3, b+3\rangle)$*-block. Consequently, there exists a* $(5; 3, k_2|3, \langle a', b'\rangle)$*-block* B' *with* $1 \le a' \le 3$*.*

Proof. As always, $b \ge a$ by our assumptions above.

First, call z the common neighbor of x_2 and x_3. Take the edge x_2x_3 and replace it by edge x_0x_2. We have a new triangular face x_0, x_1, x_2 and $\deg(x_0) = \deg(x_3) = 4$ and $\deg(x_a) = k_2$. Now take the edge zx_3 and replace it by edge x_0z. We have a new triangular face x_0, x_2, z and $\deg(x_0) = 5, \deg(x_3) = 3$ and $\deg(x_a) = k_2$. If the original triangular face x_2, x_3, z had $z = x_j$ for some $j > a$, then the cycle $x_0, z = x_j, x_{j+1}, \ldots, x_{a+b-1}$ is now bounding a $(5; 3|d')$-endblock for some $d' \le 4$, which is impossible by Theorem 3.3.

Hence, z is an inner vertex of the block B. But in this case we have obtained another block $(5; 3, k_2|d, \langle a-3, b+3\rangle)$. If $a - 3 \le 3$, we are done. Otherwise, we repeat the reduction until we arrive at a block $(5; 3, k_2|d, \langle a', b'\rangle)$ with $a' = a - 3t \le 3$ as desired. □

4.3. Uniqueness of $(3; 2, 2|d)$*-blocks*

We can now show uniqueness of the $(3; 2, 2|d)$-blocks for all $d = 3, 4, 5$.

Lemma 4.7. *The* $(3; 2, 2|d)$*-blocks are unique for each* $d = 3, 4, 5$*.*

Proof. **Case 1.** $d = 3$

Since the case of $a \ge 3$ is impossible by Lemmas 4.2 and 4.3 and of $a = 1$ by Lemma 4.5, we only need to investigate the case $a = 2$.

When $a = 2$, then we can add the edge $xy = x_0x_2$ to obtain a 3-regular 2-connected graph with all faces size $d = 3$, except possibly the outer one. If the new outerface is of size greater than three, then we have obtained a 2-connected 1-nearly Platonic graph. By Theorem 3.2, there is no such graph, hence the outerface must be a triangle x_0, x_2, x_3 and the new graph is the tetrahedron. Thus, the original $(3; 2, 2|3)$-block was the tetrahedron without one edge shown in Figure 1.

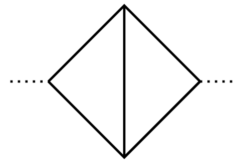

Figure 1: Unique $(3; 2, 2|3)$-block

Case 2. $d = 4$

Similarly as above, by Lemmas 4.2, 4.3 and 4.5, we only need to consider $2 \leq a \leq 3$.

If $a = 2$, by adding a new vertex z and edges x_0z and zx_2, both x_0 and x_2 now have degree 3, and we obtain a $(3; 2|4)$-endblock, which does not exist by Theorem 3.3.

When $a = 3$, then by adding the edge $xy = x_0x_3$ we obtain a 3-regular 2-connected graph with all faces except possibly the outer one of size $d = 4$. By Theorem 3.2, there is no 2-connected 1-nearly Platonic graph, hence the outerface must be a 4-cycle x_0, x_3, x_4, x_5 and the new graph is the cube. Thus, the original $(3; 2, 2|4)$-block was the cube without one edge shown in Figure 2.

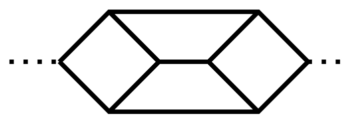

Figure 2: Unique $(3; 2, 2|4)$-block

Case 3. $d = 5$

We must consider only $2 \leq a \leq 4$ as proved in Lemmas 4.2, 4.3 and 4.5.

For $a = 2$, we take two copies of B and add two new vertices z_1, z_2 and edges z_1z_2, xz_1, yz_2, $x'z_1$ and $y'z_2$. This creates two new faces of size five, bounded by cycles $x = x_0, x_1, x_2 = y, z_2, z_1$ and $x' = x'_0, x'_1, x'_2 = y', z_2, z_1$. The outerface of this new amalgamated graph is of size at least eight, which is impossible, since the graph would be 2-connected and 1-nearly Platonic, which is impossible by Theorem 3.2.

For $a = 3$, adding a new vertex z and edges $xz = x_0z$ and $zy = zx_3$ we would obtain a $(3; 2|5)$-endblock, which does not exist by Theorem 3.3.

Finally, when $a = 4$, by adding the edge $xy = x_0x_4$ we get a new face of size five bounded by x_0, x_1, x_2, x_3, x_4 and all vertices are now of degree three. By Theorem 3.2, there are no 2-connected 1-nearly Platonic graphs. Since our new graph is 2-connected, the new outerface must be a pentagon as well. Thus, the new graph is a dodecahedron and the original graph was a dodecahedron without one edge shown in Figure 3. □

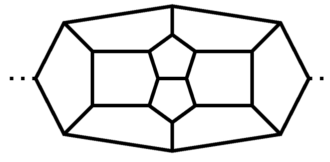

Figure 3: Unique $(3; 2, 2|5)$-block

4.4. Uniqueness of $(4; k_1, k_2|d)$*-blocks*

Now we discuss existence of $(4; k_1, k_2|d)$-blocks. The only admissible value of d is $d = 3$, and the only possibilities are $(4; 3, 3|3)$-blocks, $(4; 3, 2|3)$-blocks, and $(4; 2, 2|3)$-blocks.

Lemma 4.8. *The* $(4; 3, 3|3)$*-block is unique.*

Proof. Again by Lemmas 4.2, 4.3 and 4.5, we only need to consider the case $a = 2$.

We again add to B the edge $xy = x_0x_2$ similarly as in the case of $k = 3$ and obtain a 4-regular 2-connected graph with all internal faces of size $d = 3$. By Theorem 3.2, there is no 2-connected 1-nearly Platonic graph, hence the outerface must be a triangle x_0, x_2, x_3 and the new graph is the octahedron. Thus, the original $(4; 3, 3|3)$-block was the octahedron without one edge shown in Figure 4. □

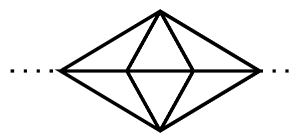

Figure 4: Unique $(4; 3, 3|3)$-block

The following result is a direct corollary of Lemmas 4.3, 4.4, and 4.5.

Lemma 4.9. *There is no* $(4; 3, 2|3)$*-block.*

Proof. First assume such a block B exists for $a = 1$. Then we take two copies of B, say B and B' and amalgamate vertices y and y', obtaining another vertex of degree three and add edge xx'. But then we have constructed a 2-connected 1-nearly Platonic graph of type $(4|3)$, which does not exist by Theorem 3.2.

We cannot have $a = 2$ by Lemma 4.4, or $a = 3$ by Lemma 4.3. Hence, the proof is complete. □

The only remaining case for 4-regular blocks is more complex.

Lemma 4.10. *The* $(4; 2, 2|3)$*-block is unique.*

Proof. Let such a block be called B. If $a = 1$, we create three copies B^0, B^1, B^2 of B with the vertices of degree two denoted x^i and y^i in each copy B^i. Assume that B has t internal triangular faces and observe that $b > 1$. Then we amalgamate x^i with y^{i+1} for all $i = 0, 1, 2$, where the superscripts are calculated modulo 3. This way we obtain a 2-connected 4-regular graph with $3t + 1$ inner triangular faces and the outerface of size $3b \geq 6$. Because no such graph exists by Theorem 3.2, this case is impossible.

When $a = 2$, then we add the edge $xy = x_0x_2$ and obtain a $(4; 3, 3|3)$-block with the vertices of degree three joined by an edge, which cannot exist by Lemma 4.5. Hence, $a \neq 2$.

For $a = 3$ and $b = 3$, the boundary is the 6-cycle $x_0, x_1, \ldots, x_5$ with $\deg(x_0) = \deg(x_3) = 2$. Therefore, inside the 4-cycle x_1, x_2, x_4, x_5 with edges $x_1x_2, x_2x_4, x_4x_5, x_5x_1$ there must be a vertex

v, adjacent to all vertices of the cycle. This gives the unique $(4;2,2|3)$-block shown in Figure 5. Notice that amalgamating x_0 with x_3 produces an octahedron.

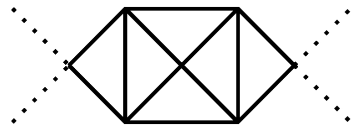

Figure 5: Unique $(4;2,2|3)$-block

Now we need to show that when $a=3$, we cannot have $b>3$. Suppose we can. But then by amalgamating x_0 with x_3 as above into a vertex x' we obtain one new inner triangular face x', x_1, x_2 and an outerface $x_0, x', x_3, \ldots, x_{3+b-1}, x_0$ of size $b+1$. Because $b>3$, the outerface is of size at least four, and we have a 2-connected 1-nearly Platonic graph of type $(4|3)$. No such graph exists by Theorem 3.2, which implies $b=3$, contradicting our assumption that $b>3$.

Finally, let $a>3$. Let $a=3c+r$ for some $c\geq 1$ and $0\leq r\leq 2$. Remove edges $x_2x_3, x_5x_6, \ldots, x_{3c-1}x_{3c}$ and replace them by edges $x_0x_2, x_3x_5, \ldots, x_{3c-3}, x_{3c-1}$.

Let i be the smallest subscript such that the edge $x_{3i-1}x_{3i}$ belonged to a triangle x_{3i-1}, x_{3i}, x_j for some $j>a$ and the previous edges (if any) $x_{3s-1}x_{3s}$ belonged to triangles x_{3s-1}, x_{3s}, z_{3s}, where z_{3s} is not a boundary vertex. Then the graph bounded by the cycle $x_0, x_2, z_3, x_3, x_5, \ldots, x_{3i-1}, x_j$, $x_{j+1}, \ldots, x_{a+b-1}$ has x_0 of degree 3 and x_j of degree 2 or 3. However, if $\deg(x_j)=2$, no such graph can exist by Lemmas 4.2 and 4.3.

If $\deg(x_j)=3$, the graph B^* bounded by $x_0, x_2, z_3, x_3, x_5, \ldots, x_{3i-1}, x_j,\ x_{j+1}, \ldots, x_{a+b-1}$ would be a $(4;3,3|3)$-block. However, such a block is unique with $a=2$, while here we have $a^*\geq 3$, because of the path $x_0, x_2, z_3, x_3, x_5, \ldots, x_{3i-1}, x_j$, where $i\geq 1$. Therefore, this possibility can be ruled out as well.

Thus, no edge $x_{3s-1}x_{3s}$ belongs to a triangle x_{3i-1}, x_{3i}, x_j.

Now if $a=3c$, after performing the edge operations above, we are left with x_a having degree one and its only remaining neighbor is x_{a+1}. Removing x_a we obtain again a $(4;3,3|3)$-block as in the previous paragraph, and the same contradiction.

When $a=3c+1$, we end up with $\deg(x_0)=\deg(x_{a-1})=3$ and $\deg(x_a)=2$. We create two copies of the block, say B and B', amalgamate x_a with x'_a and add a new edge $x_{a-1}x'_{a-1}$. This way we obtain a $(4;3,3|3,\langle 2a-1,2b\rangle)$-block. Since $2b>2a-1\geq 7$, no such block can exist by Lemma 4.8.

Finally, for $a=3c+2$ we transform the graph so that $\deg(x_0)=\deg(x_{a-2})=3$ and $\deg(x_a)=2$. By adding the edge $x_{a-2}x_a$ we create a new inner triangle x_{a-2}, x_{a-1}, x_a and $\deg(x_{a-2})=4$ and $\deg(x_a)=3$. The resulting graph is a $(4;3,3|3,\langle a-1,b\rangle)$-block.. But $b>a-1\geq 4$, and no such block can again exist by Lemma 4.8. □

4.5. Uniqueness of $(5;k_1,k_2|d)$-blocks

Again, we must have $d=3$. Hence, we are left with blocks of type $(5;k_1,k_2|3)$ for $4\geq k_1\geq k_2\geq 2$.

Lemma 4.11. *The $(5;4,4|3)$-block is unique.*

Proof. It follows from Lemma 4.5 that we only have to consider $a = 2$.

By adding the edge $x_0x_2 = xy$, we obtain a 5-regular graph with all internal faces of size three. By Theorem 3.2, the outerface now must be also a triangle, as otherwise we would have a 2-connected 1-nearly Platonic graph with the outerface of size more than three. Therefore, the new graph is the icosahedron and the original one was the icosahedron without an edge shown in Figure 6. □

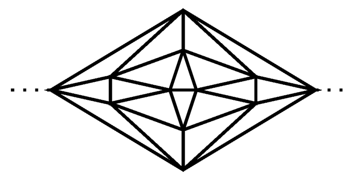

Figure 6: Unique $(5; 4, 4|3)$-block

Lemma 4.12. *A* $(5; 4, 3|3)$*-block does not exist.*

Proof. We have $1 \leq a \leq 3$ by Lemma 4.2. But we cannot have $a = 3$ by Lemma 4.3, or $a = 2$ by Lemma 4.4. Hence, $a = 1$.

We again create two copies of B with t inner triangular faces and assume $\deg(x) = \deg(x') = 4$ and $\deg(y) = \deg(y') = 3$. Now we add a new vertex z and edges zx, zx', zy, zy', yy'. This way we obtain a 2-connected graph with $2t + 3$ inner triangular faces in which $\deg(z) = 4$ and all other vertices are of degree five. This graph would be a $(5; 4|3)$-endblock, which does not exist by Theorem 3.3. □

Lemma 4.13. *A* $(5; 4, 2|3)$*-block does not exist.*

Proof. We have $a = 1$ by Lemmas 4.2, 4.3 and 4.4.

Suppose $\deg(y) = 2$. We remove y and obtain a $(5; 4, 3|3)$-block. But by Proposition 4.12, it is impossible. □

Now we investigate $(5; 3, k_2|3)$-blocks.

Lemma 4.14. *A* $(5; 3, 3|3)$*-block does not exist.*

Proof. By Lemma 4.6 we have $1 \leq a \leq 3$. If $a = 1$, we create two copies of the block B with t inner triangular faces (and $b > 1$) and amalgamate the edges xy and $x'y'$. This creates a 2-connected 5-regular graph with $2t$ inner triangular faces and the outerface of size $2b \geq 4$. Such graph would be 1-nearly Platonic and cannot exist by Theorem 3.2.

When $a = 2$, then by adding the edge $xy = x_0x_2$ we would obtain a $(5; 4, 4|3)$-block whose non-existence was proved in Lemma 4.11.

For $a = 3$, we replace the edge $x_2x_3 = x_2y$ by edge x_0x_2, creating a new triangular face. Now $\deg(x) = 4, \deg(y) = 2$ and the boundary path from x to y is $x = x_0, x_2, v, y$ for some v. If $v = x_j$ for some $j > 3$, then the block bounded by $x_0, x_2, x_j, x_{j+1}, \ldots, x_{a+b-1}$ is a $(5; 4, 3|3)$-block or $(5;4,2|3)$-block where $v = x_j$ is of degree three or two, respectively. Such blocks do not exist by Lemmas 4.12 and 4.13.

If v is an inner vertex of B, then we obtain a $(5;4,2|3)$-block with $a = 3$, which cannot exist by Lemma 4.3. All cases have been covered and the proof is complete. □

Lemma 4.15. *The* $(5;3,2|3)$*-block is unique.*

Proof. By Lemma 4.6 we have $1 \leq a \leq 3$. If $a = 1$, we create three copies B^0, B^1, B^2 of the block B with $\deg(x^i) = 3$ and $\deg(y^i) = 2$ in each copy B^i. Assume that B has t internal triangular faces and observe that $b > 1$. Then we amalgamate x^i with y^{i+1} for all $i = 0, 1, 2$, where the superscripts are calculated modulo 3. This way we obtain a 2-connected 5-regular graph with $3t + 1$ inner triangular faces and the outerface of size $3b \geq 6$. Because no such graph exists by Theorem 3.2, this case is impossible.

When $a = 2$, then we add the edge $xy = x_0x_2$ and obtain a $(5;4,3|3)$-block, which cannot exist by Lemma 4.12.

Now let $a = 3$ and the outerface be $x_0, x_1, \dots, x_{a+b-1}$, where $\deg(x_0) = 3, \deg(x_3) = 2$, and $\deg(x_i) = 5$ otherwise.

Amalgamate x_0 and x_3 into a new vertex x' of degree five so that the new triangular face is x', x_1, x_2 and it is an inner face. The outerface is now $x', x_4, x_5, \dots, x_{a+b-1}$, and has size $a+b-3$.

If $b > 3$, we have $a+b-3 \geq 4$. But then the resulting graph is a 1-nearly Platonic graph of type $(5|3)$ with exceptional face of size at least four, which is non-existent by Theorem 3.2. Therefore, $b = 3$. But then the new amalgamated graph has outer boundary of size three, namely x', x_4, x_5. Clearly, we have obtained the icosahedron, and the original block shown in Figure 7 is unique.

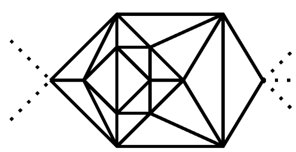

Figure 7: Unique $(5;3,2|3)$-block

Since there are no other values of a to investigate, the proof is now complete. □

4.6. Classification: touching exceptional faces

Lemma 4.16. *In a* 2*-nearly Platonic graphs, two exceptional faces cannot touch at exactly one vertex.*

Proof. Let G be a 2-nearly Platonic graphs with two exceptional faces F_1 and F_2 touching at exactly one vertex. Suppose that the boundary of F_1 is the cycle $x_1, x_2, \dots, x_n$ and the boundary of F_2 is the cycle $y_1, y_2, y_3, \dots, y_m$, where $x_1 = y_1$. Since $\{x_2, x_n, y_2, y_m\} \subseteq N(x_1)$, we have $k \in \{4, 5\}$ and $d = 3$.

If $k = 4$, then by splitting x_1 into two vertices x' and x'' such that $N(x') = \{x_n, y_m\}$ and $N(x'') = \{x_2, y_2\}$, we obtain a $(4;2,2|3\langle|F_1|,|F_2|\rangle)$-block. Now, by Lemma 4.10 this block is unique. This implies that $|F_1| = |F_2| = 3$ and G is an octahedron, a contradiction.

If $k = 5$, then without loss of generality (WLOG) suppose that the fifth neighbor of x_1 is z and adjacent to x_2 and y_2. By splitting x_1 into two vertices x' and x'' such that $N(x') = \{x_n, y_m\}$ and $N(x'') = \{x_2, y_2, z\}$, we obtain a $(5; 3, 2|3\langle|F_1|, |F_2|\rangle)$-block. Now, by Lemma 4.15 this block is unique. This implies that $|F_1| = |F_2| = 3$ and G is an icosahedron, a contradiction. □

Lemma 4.17. *Every* 2*-nearly Platonic graph of type* $(k|d)$ *with touching exceptional faces is constructed by the* $(k; k_1, k_2|d)$*-blocks and* K_2*.*

Proof. Let G be a 2-nearly Platonic graphs with two exceptional faces F_1 and F_2 touching at vertices $z_1, z_2, \ldots, z_\ell$. Note that by Lemma 4.16, $\ell \geq 2$. Suppose that the boundary of F_1 is the cycle $x_1, x_2, \ldots, x_n$ and the boundary of F_2 is the cycle $y_1, y_2, \ldots, y_m$ and suppose they are located clockwise from x_1 and y_1. Also, WLOG assume that the $z_1 = x_1 = y_1$, $z_2 = x_i = y_j$ and it is located clockwise from x_1. If $i = 2$, then x_1 is adjacent to x_2 and since $x_1 = y_1$ and $x_2 = y_j$, it follows that y_1 is adjacent to y_j. Thus $j = 2$ and so the edge x_1x_2 is common in F_1 and F_2. It is a block K_2. Similarly, if $j = 2$, then $i = 2$. If $i > 2$, then $j > 2$ and $\{x_2, \ldots, x_{i-1}\} \cap \{y_2, \ldots, y_{j-1}\} = \emptyset$. We consider the subgraph H induced by vertices belong on and inside of the cycle $x_1, x_2, \ldots, x_i, y_{j-1}, y_{j-2}, \ldots, y_2$. Note that all faces in H are cycles and by Theorem 3.1, H is 2-connected. All vertices in H are of degree k except x_1 and x_i which are of degree less than k. Also, all interior faces are of size d. Now, H is a $(k; k_1, k_2|d)$-block. This completes the proof. □

Based on our lemmas, we can now state the main result of this section.

Theorem 4.1. *There are exactly seven infinite families of* 2*-nearly Platonic graphs with touching exceptional faces; one of each of types* $(3|3), (3|4), (3|5)$*, two of type* $(4|3)$*, and two of type* $(5|3)$*, shown in Figures 8–14.*

Moreover, all these graphs have the two exceptional faces of the same size.

Proof. By Lemma 4.17, every 2-nearly Platonic graph of type $(k|d)$ with touching exceptional faces is constructed by the $(k; k_1, k_2|d)$-blocks and K_2.

For constructing the graphs of type $(3|d)$, we must use K_2 and $(3; 2, 2|d)$-blocks which are all unique by Lemma 4.7.

For type $(3|3)$ the only possible block is the $(3; 2, 2|3, \langle 2, 2\rangle)$-block isomorphic to the tetrahedron with one removed edge, and the graph must be a chain alternating the $(3; 2, 2|3, \langle 2, 2\rangle)$-blocks and graphs K_2.

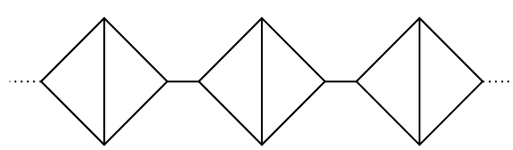

Figure 8: Chain of blocks of type $(3; 2, 2|3)$ and K_2, tetrahedron edge cycle

Next, for type $(3|4)$ the only possible block is the $(3; 2, 2|4, \langle 3, 3\rangle)$-block isomorphic to the cube with one removed edge, and the graph is a chain alternating the $(3; 2, 2|4, \langle 3, 3\rangle)$-blocks and graphs K_2.

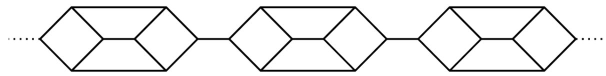

Figure 9: Chain of blocks of type $(3;2,2|4)$ and K_2, cube edge cycle

Once again, for type $(3|5)$ the only possible block is the $(3;2,2|5,\langle 4,4\rangle)$-block isomorphic to the dodecahedron with one removed edge, and the graph is a chain alternating the $(3;2,2|5,\langle 4,4\rangle)$-blocks and graphs K_2.

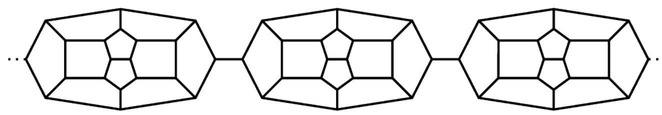

Figure 10: Chain of blocks of type $(3;2,2|5)$ and K_2, dodecahedron edge cycle

For type $(4|3)$, the blocks could possibly be only of type $(4;3,3|3)$, $(4;3,2|3)$, or $(4;2,2|3)$. A $(4;3,2|3)$-block does not exist by Lemma 4.9; the other two are unique by Lemmas 4.8 and 4.10. Hence, the graph is either a chain consisting of the $(4;3,3|5,\langle 2,2\rangle)$-blocks (that is, octahedrons without an edge) and graphs K_2, or a chain of $(4;2,2|5,\langle 3,3\rangle)$-blocks, arising from octahedron by splitting one vertex.

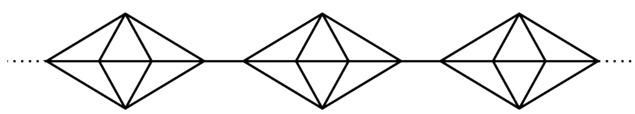

Figure 11: Chain of blocks of type $(4;3,3|3)$ and K_2, octahedron edge cycle

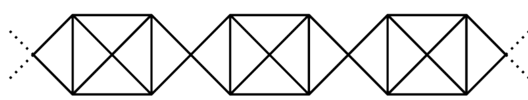

Figure 12: Chain of blocks of type $(4;2,2|3)$, octahedron vertex cycle

For type $(5|3)$, blocks of type $(5;4,3|3)$ and $(5;4,2|3)$ do not exist by Lemmas 4.12 and 4.13, respectively. The block of type $(5;4,4|3)$ is unique by Lemma 4.11; it is the $(5;4,4|3,\langle 2,2\rangle)$-block, isomorphic to the icosahedron with one removed edge. The resulting graph then is a chain of the $(5;4,4|3,\langle 2,2\rangle)$-blocks alternating with graphs K_2.

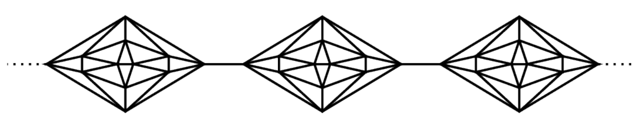

Figure 13: Chain of blocks of type $(5;4,4|3)$ and K_2, icosahedron edge cycle

For blocks of type $(5; k_1, k_2|3)$, where $k_1 \leq 3$, we cannot use the block K_2. Otherwise, there exist some vertices of degree less than 5, a contradiction.

If two blocks have one vertex z_1 in common, then z_1 belongs to a block B_1 of type $(5; 3, k_2|3)$. Block of type $(5; 3, 3|3)$ does not exist by Lemma 4.14. Thus B_1 is of type $(5; 3, 2|3)$ which is unique by Lemma 4.15; it is the $(5; 3, 2|3, \langle 3, 3\rangle)$-block, obtained from the icosahedron by splitting one vertex into two vertices z_1 and z_2 of degree three and two, respectively. Similarly, z_2 belongs to a block B_2 of type $(5; 3, 2|3, \langle 2, 2\rangle)$. By repeating this process, we obtain a chain of the $(5; 3, 2|3, \langle 2, 2\rangle)$-blocks.

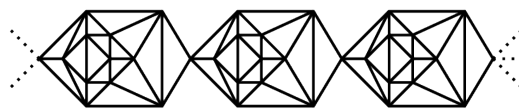

Figure 14: Chain of blocks of type $(5; 3, 2|3)$, icosahedron vertex cycle

□

5. Non-touching exceptional faces

5.1. New notions

Let F_1, F_2 be the disjoint outer and inner disparate faces, respectively. We denote their respective boundaries by $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_m$ in clockwise order. We define the distance between F_1 and F_2 as

$$\operatorname{dist}(F_1, F_2) = \min\{\operatorname{dist}(x_i, y_j)|x_i \in F_1, y_j \in F_2\}.$$

In a subgraph of a 2-nearly Platonic graph of type $(k|d)$, a vertex is *saturated*, if it is of degree k. It should be obvious that in a 2-nearly Platonic graph with non-touching exceptional faces, each vertex must belong to at least two faces of size d. Similarly, in a subgraph of a 2-nearly Platonic graph of type $(k|d)$, a path of length $d-1$ is *weakly saturated*, if all its internal vertices are of degree k.

We start with some easy observations regarding the graphs of types $(3|3)$, $(3|4)$, and $(4|3)$.

5.2. Graphs of types $(3|3)$*,* $(3|4)$*, and* $(4|3)$

Observation 5.1. *There is no* 2*-nearly Platonic graph of type* $(3|3)$ *with non-touching exceptional faces.*

Proof. By contradiction. Let $\operatorname{dist}(F_1, F_2) = 1$. Then there is an edge $x_i y_i$ for some i. Because the faces are non-touching, we have $x_a \neq y_b$ for any a, b. Vertex x_i is saturated, having neighbors x_{i-1}, x_{i+1}, y_i, and must belong to a triangular face x_i, x_{i+1}, y_i. But then y_i is of degree at least four, a contradiction.

If $\operatorname{dist}(F_1, F_2) = \operatorname{dist}(x_i, y_i) \geq 2$, then we have a path $x_i, v_1, v_2, \ldots, y_i$ (where possibly $v_2 = y_i$). Again, x_i is saturated, hence must belong to triangular faces x_i, x_{i-1}, v_1 and x_i, x_{i+1}, v_1, and v_1 must be of degree at least four, a contradiction again. □

Observation 5.2. *The only* 2*-nearly Platonic graph of type* $(3|4)$ *with non-touching exceptional faces is a prism.*

Proof. If $\mathrm{dist}(F_1, F_2) = 1$, the graph will be a prism. Let the shortest path be x_1y_1, weakly saturating the path x_2, x_1, y_1, y_2, and since the common face is of degree four, x_2y_2 is forced. Using the same argument repeatedly, edge x_iy_i is forced for every i. Hence, the graph must be a prism.

If $\mathrm{dist}(F_1, F_2) = \mathrm{dist}(x_1, y_1) \geq 2$, let $x_1, v_1, v_2, \ldots, y_1$ be the shortest path, where v_2 can be equal to y_1. Then v_1 will have one more neighbor, say w_1, WLOG in the clockwise direction. This saturates v_1 and thus v_2 and x_n are adjacent. But now we have a shorter path $x_n, v_2, \ldots, y_1$, a contradiction. □

Observation 5.3. *The only* 2*-nearly Platonic graph of type* $(4|3)$ *with non-touching exceptional faces is an antiprism.*

Proof. If $\mathrm{dist}(F_1, F_2) = 1$, the graph will be an anti-prism. Let x_1y_1 be a shortest path. Vertex x_1 has neighbors x_2, x_n, y_1 and some v_1, which can be placed WLOG so that the edge x_1v_1 is placed between edges x_1x_n and x_1y_1. Then x_1 is saturated, and we must have edge x_2y_1. Now y_1 is saturated, which forces edge x_2y_2. After repeating the argument n times, we obtain an anti-prism.

Now suppose that $\mathrm{dist}(F_1, F_2) = \mathrm{dist}(x_1, y_1) \geq 2$, and $x_1, v_1, v_2, \ldots, y_1$ is the shortest path, where v_2 can again be equal to y_1.

Let w_1 be the fourth neighbor of x_1 and WLOG suppose it is located counter-clockwise from x_1. Now x_1 is saturated and v_1 and x_2 are adjacent. For the same reason, saturation of x_1, we must have the edge w_1v_1. Notice that v_1 is now saturated, which forces also the edge x_2v_2. This would mean that $\mathrm{dist}(x_2y_1) < \mathrm{dist}(x_,y_1) = \mathrm{dist}(F_1, F_2)$, which is a contradiction, and the proof is complete. □

5.3. Graphs of type $(3|5)$

For the $(3|5)$ case, we need several lemmas to determine the distance between the two non-touching exceptional faces.

Lemma 5.1. *Suppose* G *is a* 2*-nearly Platonic graph of type* $(3|5)$ *with non-touching exceptional faces* F_1, F_2*.*

Let $l = \mathrm{dist}(F_1, F_2)$ *and suppose* $l \geq 3$*. Let* $x_1, v_1, v_2, \ldots, y_1$ *be a path of length* l*. Denote by* w_i *the third neighbor of* v_i *for* $i = 1, 2, \ldots, l-1$*, and assume* w_1 *is located clockwise from* v_1*. Then all vertices* w_{2j+1} *are located clockwise from the path* $x_1, v_1, v_2, \ldots, y_1$*, while all vertices* w_{2j} *are located counter-clockwise from* $x_1, v_1, v_2, \ldots, y_1$*.*

Proof. First observe that w_2 must be placed counter-clockwise from v_2. For if not, then the path x_n, x_1, v_1, v_2, v_3 is weakly saturated and forces edge x_nv_3. Then $\mathrm{dist}(x_n, y_1) < \mathrm{dist}(x_1, y_1) = \mathrm{dist}(F_1, F_2)$, which is a contradiction.

Now let i be the smallest subscript such that w_i and w_{i+1} are both placed in the same direction, say counter-clockwise from v_i and v_{i+1}, respectively. Then the path $w_{i-1}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$ is weakly saturated, which forces edge $w_{i-1}v_{i+1}$. However, this creates a path $x_1, v_1, \ldots, v_{i-1}, w_{i-1}$, v_{i+2}, v_{i+3}, y_1 of length $l-1 < \mathrm{dist}(F_1, F_2)$, which is impossible. This contradiction completes the proof. □

Lemma 5.2. *Suppose G is a 2-nearly Platonic graph of type $(3|5)$ with non-touching exceptional faces F_1, F_2 and $\operatorname{dist}(F_1, F_2) = l$. Then the only possible values of l are 1 and 3.*

Proof. Let the shortest path be as in the previous proof, and third neighbors w_i of v_i be placed clockwise for odd subscripts, and counter-clockwise for even subscripts. We first want to show that the shortest path cannot have length more than three.

Suppose it does. Then w_2 is placed counter-clockwise, and x_n, x_1, v_1, v_2, w_2 is a weakly saturated path, forcing edge $x_n w_2$. Similarly, w_4 (which can be equal to y_m) is placed counter-clockwise, and w_2, v_2, v_3, v_4, w_4 is a weakly saturated path, forcing edge $w_2 w_4$. This creates a path $x_n, w_2, w_4, v_4, v_5 \dots, y_1$ of length at most $l-1$, a contradiction.

Now suppose $l = 2$, and denote the shortest path between F_1 and F_2 by x_1, v_1, y_1. Again suppose that the third neighbor w_1 of v_1 is placed clockwise from v_1. Then since the path x_n, x_1, v_1, y_1, y_m is weakly saturated, we muse have $x_n y_m$ as an edge. Then $\operatorname{dist}(F_1, F_2) = \operatorname{dist}(x_n, y_m) = 1$, which is impossible. □

So we just proved that the distance can only be one or three. In fact, the structures of the graphs for both cases are determined for both cases, which will be shown in the next lemma. Specifically, for the distance one case, we will shown that by some operations, every such graph could become a 2-nearly Platonic graphs with touching faces while the sizes of the exceptional faces do not change. And for the distance three case, we could reduce it to the smallest such graph and determine its structure.

Lemma 5.3. *There is exactly one infinite class of 2-nearly Platonic graphs of type $(3|5)$ with non-touching exceptional faces F_1, F_2 and $\operatorname{dist}(F_1, F_2) = 1$.*

Proof. Let H be a graph satisfying the assumptions of our lemma, the sizes of F_1 and F_2 are equal to $m(\neq 5)$ and $n(\neq 5)$, respectively and assume there is an edge $x_1 y_1$. We can now split the edge into two edges $x_1' y_1'$ and $x_1'' y_1''$. We create five copies of this graph $H_1, H_2, \dots, H_5$, and amalgamate the edge $x_1'' y_1''$ in the i-th copy with $x_1' y_1'$ in the $(i+1)$-st copy, with i taken modulo 5. It should be clear that the new graph is still 2-nearly Platonic with two exceptional faces, one of size $5n$, and the other of size $5m$.

Then we relabel the vertices of the exceptional faces. Denote one of the edges $x_1' y_1'$ by $w_1 z_1$, and let our two exceptional faces be $w_1 w_2 \dots w_{5n}$ and $z_1 z_2 \dots z_{5m}$. We add edges $w_1 w_5$, $w_6 w_{10}, \dots, w_{5n-4} w_{5n}$ and remove edges $w_n w_1, w_5 w_6, \dots, w_{5n-5} w_{5n-4}$. Notice that none of the edges $w_{5s-4} w_{5s}$ for $s = 1, 2, \dots, n$ existed in the graph before we added it. If w_{5s-4} and w_{5s} belong to the same copy H_j, then they correspond to some boundary vertices x_t and x_{t+4} and so $n > 5$. If $x_t x_{t+4}$ is an edge in H, then the induced subgraph by all vertices on and inside of the cycle $x_t, x_{t+1}, \dots, x_{t+4}$ is a $(3; 2, 2|5)$-block with $a = 1$, which does not exist by Lemma 4.5.

Also, w_{5s-4} and w_{5s} cannot belong to two consecutive copies H_j and H_{j+1}, since we were not adding any new edges when amalgamating H_j and H_{j+1} along the edges $x_1' y_1'$ and $x_1'' y_1''$.

This way we obtain a 2-nearly Platonic graph with touching faces since the two new faces share the vertex z_1. Note that this operation does not change the size of the exceptional faces.

As we proved before, the conjecture is true for the touching exceptional faces case, thus we can conclude that $5m = 5n$, or $m = n$. Since we have classified the structure of the 2-nearly Platonic graphs for the touching exceptional faces, we can determine the structure of all the non-touching case of type $(3|5)$ by reversing the operations. The fundamental block of this type is shown in Figure 15. □

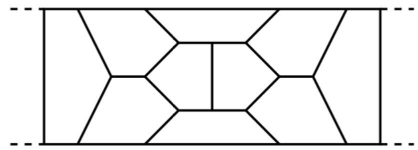

Figure 15: Fundamental block with non-touching faces of type $(3|5)$ with $l = 1$

Lemma 5.4. *There is exactly one infinite class of* 2*-nearly Platonic graphs of type* $(3|5)$ *with non-touching exceptional faces* F_1, F_2 *and* $\mathrm{dist}(F_1, F_2) = 3$*.*

Proof. Recall that the exceptional faces F_1 and F_2 are bounded by cycles $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$, respectively with $n \leq m$. We have $\mathrm{dist}(F_1, F_2) = 3$ and denote a shortest path by x_1, v_1, w_1, y_1 and the third neighbors of x_2 by v_2. Since the path v_2, x_2, x_1, v_1, w_1 is a weakly saturated path, thus w_1v_2 is an edge. Now, we denote the third neighbor of y_2 by w_2 and since the path w_2, y_2, y_1, w_1, v_2 is a weakly saturated path, thus v_2w_2 is an edge. By repeating this process, we construct the path $v_1, w_1, v_2, w_2, \ldots, v_n, w_n$. For each $i = 1, 2, \ldots, n$, v_i and w_i are the third adjacents of x_i and y_i, respectively.

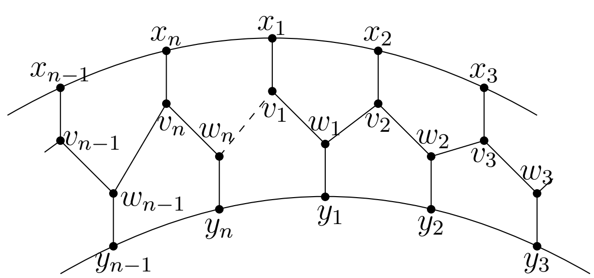

Figure 16: Fundamental block with non-touching faces of type $(3|5)$ with $l = 3$

We have a weakly saturated path v_1, x_1, x_n, v_n, w_n and so w_nv_1 is an edge. Finaly, the path y_n, w_n, v_1, w_1, y_1, is a weakly saturated path and two vertices y_n and y_1 have to be adjacent. This concludes that $m = n$ and the constructed graph is a balanced 2-nearly Platonic graph as desired. The fundamental block of this type is shown in Figure 17. □

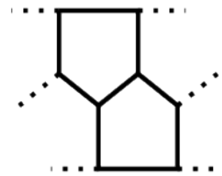

Figure 17: Fundamental block with non-touching faces of type $(3|5)$ with $l = 3$

Hence in either case, we know the structure of the graph and the two exceptional faces have the same degree for each case.

5.4. Graphs of type $(5|3)$

For the $(5|3)$ case, we also first discuss the distance between the two exceptional faces.

We start by showing that the number of vertices on F_1 and F_2 is half of the order of the graph. Denote the order of the graph by $|V|$, and we know there are m and n vertices on F_1 and F_2, respectively. Since the graph is 5-regular, there are $5|V|/2$ edges. Also, we can count the number of edges using the number of faces. By Euler's formula, the number of faces, denoted by $|F|$, is $|E| - |V| + 2$, which is $5|V|/2 - |V| + 2$, or $3|V|/2 + 2$. Since all faces except two are triangles, and the other two faces are of degree m and n, respectively, we have $3(|F| - 2) + m + n = 2|E|$. Because $|F| = 3|V|/2+2$ and $|E| = 5|V|/2$, we have $9|V|/2+m+n = 5|V|$, or $2(m+n) = |V|$, as desired.

We summarize these findings as follows.

Observation 5.4. *Let G with vertex set V be a 2-nearly Platonic graph with non-touching exceptional faces of sizes m and n, respectively. Then $|V| = 2(m + n)$.*

Now we use the fact that $|V| = 2(m + n)$ to show the distance between F_1 and F_2 cannot be greater than two.

Lemma 5.5. *Let G be a 2-nearly Platonic graph of type $(5|3)$ with non-touching exceptional faces F_1 and F_2 and $\operatorname{dist}(F_1, F_2) = l$. Then $1 \leq l \leq 2$. In particular, if $l = 2$, then G is balanced and $N(F_1) = N(F_2)$.*

Proof. Suppose the distance is three or more, and define the neighborhoods of F_1, F_2 as

$$N(F_j) = \{u \mid ux \in E(G) \text{ for some } x \in F_j \text{ and } u \notin F_j\}, \text{where } j = 1, 2.$$

Thus, four sets $F_1, F_2, N(F_1)$ and $N(F_2)$ are pairwise distinct and so

$$|F_1| + |F_2| + |N(F_1)| + |N(F_2)| \leq |V|. \quad (1)$$

We want to show that $|N(F_j)| = 2|F_j|$. Let the n-cycle $x_1, x_2, \ldots, x_n$ be the boundary of F_1 and $u_i^0, u_i^1, u_i^2 \in N(F_1)$ be the three neighbors of x_i, placed in that order. Since the common faces are all triangles, u_i^2 and u_{i+1}^0 must be the same vertex. Also, u_i^0 and u_i^1 are forced to be adjacent, as well as u_i^1 and u_i^2. So in $N(F_1)$, we would have n distinct vertices that have exactly two neighbors in F_1 each, and n distinct vertices with exactly one neighbor in F_1 each. Thus, together there are $2n$ vertices.

By applying the same argument to $N(F_2)$, we have $|N(F_2)| = 2m$. Because we have $|F_1| = n, |F_2| = m$ and by Observation 5.4, $|V| = 2(m+n)$, it follows that by the inequality (1), $n + m + 2n + 2m \leq 2(n+m)$, a contradiction.

If $l = 2$, then the three sets F_1, F_2 and $N(F_1) \cup N(F_2)$ are pairwise distinct and so we have $|F_1| + |F_2| + |N(F_1) \cup |N(F_2)| \leq |V|$. Thus, $|N(F_1) \cup |N(F_2)| \leq m + n$. Since $N(F_1) \subseteq N(F_1) \cup N(F_2)$ and

$$2n = |N(F_1)| \leq |N(F_1) \cup N(F_2)| \leq n + m, \quad (2)$$

we have $n \leq m$. By symmetry, looking at $N(F_2)$, we obtain $m \leq n$, which implies $m = n$. Now, by the inequalities (2), $N(F_1) = N(F_1) \cup N(F_2) = N(F_2)$. This completes the proof. □

In the previous lemma, we not only proved the statement, but we observed that if the distance is two, the conjecture holds. Moreover we can determine the structure in this case. By the proof of the lemma, all vertices other than the boundary of F_1 and F_2 are at distance one from both F_1 and F_2. Let the vertices having one neighbor on F_1 be v_i for $i = 1, 2, \ldots, n$ and $x_i v_i$ be the edges. Let the common neighbor of x_i and x_{i+1} be w_i. Then there is the inner cycle $C_{2n} = v_1, w_1, v_2, w_2, \ldots, v_n, w_n$ closing the triangles.

By symmetry, all vertices except the boundary of F_1 and F_2 are at distance one from F_2. Thus they are all in the set $\{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_n\}$. Clearly, for $i = 1, 2, \ldots, n$ each w_i is already of degree four and must be a neighbor of exactly one vertex on F_2, say y_j. This forces v_i to be the remaining neighbor of both y_{i-1} and y_i. This uniquely determines the structure of the graph. We summarize our findings in the following lemma.

Lemma 5.6. *The class of* 2*-nearly Platonic graphs of type* $(5|3)$ *with non-touching exceptional faces* F_1 *and* F_2 *and* $\operatorname{dist}(F_1, F_2) = 2$ *is unique.*

Proof. The proof was given above, and the stucture can be seen in Figure 18 below. □

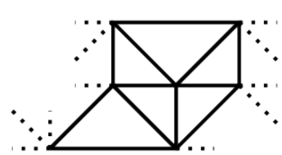

Figure 18: Fundamental block with non-touching faces of type $(5|3)$ with $l = 2$

There is only one case left now, namely when $\operatorname{dist}(F_1, F_2) = 1$. The method we will use is similar to what we did in distance one case for type $(3|5)$. That is, we will use the results for the touching case to prove the non-touching case.

Lemma 5.7. *There are exactly two classes of* 2*-nearly Platonic graphs of type* $(5|3)$ *with non-touching exceptional faces* F_1 *and* F_2 *and* $\operatorname{dist}(F_1, F_2) = 1$.

Proof. Because $\mathrm{dist}(F_1, F_2) = 1$, we must have an edge joining the two faces, say x_1y_1. Up to symmetry, there are three possible structures for the neighbors of x_1 and y_1. The first case is that the two neighbors of x_1 inside the boundary are located clockwise from x_1y_1 while two neighbors of y_1 are counter-clockwise from x_1y_1. The second case is that only x_1 has its two internal neighbors on the same side of x_1y_1, say clockwise for it, and y_1 has neighbors on both sides of x_1y_1. The last case is that the two neighbors of x_1 inside the boundary are on different sides of x_1y_1, and so are the two neighbors of y_1. The reason why the four neighbors cannot be on the same side of x_1y_1, say clockwise from x_1y_1, is that if so, then the path x_n, x_1, y_1 is weakly saturated and we would need the edge x_ny_1 to complete the triangular face. However, this would make y_1 of degree six, which is impossible. The three possible cases are shown in Figure 19 below.

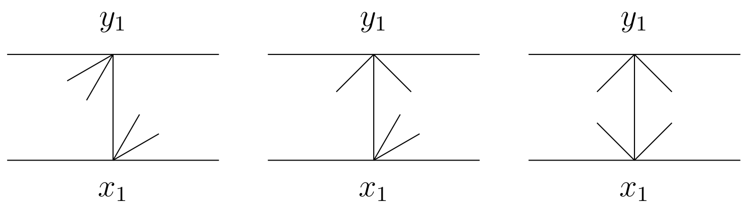

Figure 19: From left to right: case 1, 2, and 3

In fact, if we have the structure as described in the second case, we obtain the same structure as in case one. We have the two neighbors of x_1 located clockwise from x_1y_1, which implies that the path $x_nx_1y_1$ is weakly saturated, and x_ny_1 must be an edge. Now, x_n, x_1, y_1 form a triangle and because both x_1 and y_1 are already saturated, x_n cannot have a neighbor inside the triangle. Thus both remaining internal neighbors of x_n are located counter-clockwise from x_ny_1, and the two remaining neighbors of y_1 (one of which is x_1) are both on the clockwise side of x_ny_1, which is what we have in case one up to symmetry.

We reduced the problem to two cases, and will discuss them now one by one. For the first case where the two neighbors of x_1 inside the boundary are located clockwise from x_1y_1 while two neighbors of y_1 are counter-clockwise from x_1y_1, we can split the edge x_1y_1 to obtain a strip. Then we make three copies of the strip and attach them together, for the vertices that are incident with the splitting edge are symmetric. This way we obtain a larger 2-nearly Platonic graph with exceptional faces of degrees $3n$ and $3m$. We label the vertices again so that x_1y_1 is a path from F_1 to F_2 and x_1 has two neighbors clockwise from x_1y_1. So x_1y_2 will be an edge connecting F_1 to F_2 as well. Then we remove edges $x_{3n}x_1, x_3x_4, \ldots, x_{3n-3}x_{3n}$ and add edges $x_1x_3, x_4x_5, \ldots, x_{3n-2}x_n$. The graph remains 5-regular and all but the two exceptional faces are triangles. Also, the long faces will have the same size as the graph before the operation. However, now the two exceptional faces share the vertex y_1, so by the previous result, the two faces must have the same size, i.e. $3m = 3n$, thus $m = n$, as desired.

For the third case, where the two neighbors of x_1 inside the boundary are on different sides of x_1y_1, and the same holds for y_1, we can also split the edge x_1y_1, make three copies and glue them together. Then instead of adding and removing edges on only one of the exceptional faces as we did in the first case, we will add and remove edges on both inner and outerface. Again after the operation the two new exceptional faces share a vertex, which is one of the common neighbors of x_1 and y_1. Since the operation does not change the face size, we could conclude that $3m = 3n$, and so $m = n$.

Since the class of 2-nearly Platonic graphs with touching faces obtained by these operations is unique as described in Lemma 4.15, it should be obvious that starting with graphs in Figure 19 and reversing the steps, we obtain graphs in Figures 20 and 21, respectively. □

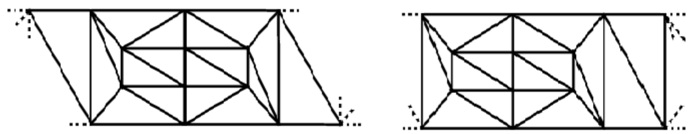

Figure 20: Case 1 and Case 2 are isomorphic.

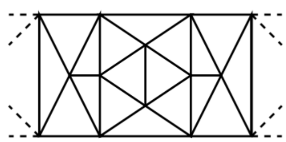

Figure 21: Case 3

5.5. Classification: non-touching exceptional faces

Theorem 5.1. *There are exactly seven infinite families of* 2*-nearly Platonic graphs with non-touching exceptional faces; the prism of type* $(3|4)$ *in Figure 22, two graphs of type* $(3|5)$ *in Figures 23 and 24, antiprism of type* $(4|3)$ *in Figure 25, and three graphs of type* $(5|3)$ *in Figures 26, 27 and 28. Moreover, all these graphs have the two exceptional faces of the same size.*

Proof. The non-existence of 2-nearly Platonic graphs of type $(3|3)$ follows from Observation 5.1. The result for type $(3|4)$ follows from Observation 5.2.

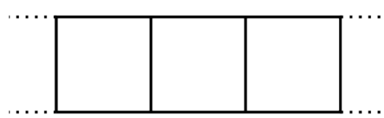

Figure 22: Prism, type $(3|4)$

For type $(3|5)$ the result follows from Lemmas 5.3 and 5.4.

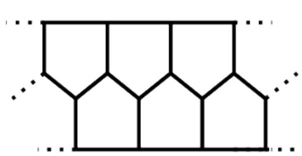

Figure 23: Barrel, type $(3|5)$

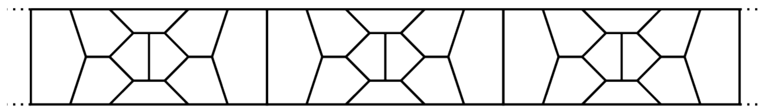

Figure 24: Dodecahedron thick cycle, type $(3|5)$

The result for type $(4|3)$ follows from Observation 5.3.

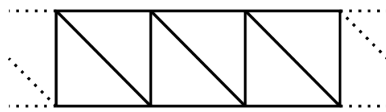

Figure 25: Antiprism, type $(4|3)$

Finally, for type $(5|3)$ the result follows from Lemmas 5.6 and 5.7. □

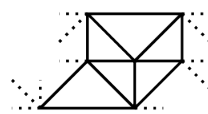

Figure 26: Icosahedron wide cycle, type $(5|3)$

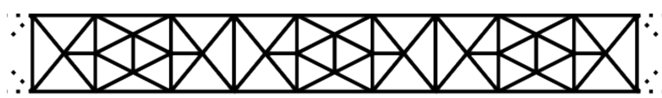

Figure 27: Icosahedron first thick cycle, type $(5|3)$

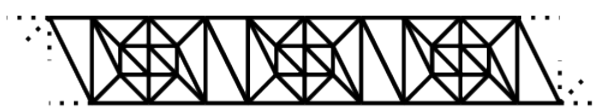

Figure 28: Icosahedron second thick cycle, type $(5|3)$

6. Conclusion

We summarize our results in the form answering in the affirmative the conjecture by Keith, Froncek, and Kreher [15]. For the respective classes of graphs, we slightly modified the terminology introduced in [15].

Theorem 6.1. *All* 2*-nearly Platonic graphs are balanced and belong to one of the following* 14 *families, listed by type* $(k|d)$*:*

- *Type* $(3|3)$*:*
  1. *tetrahedron edge cycle (Figure 8)*
- *Type* $(3|4)$*:*
  2. *cube edge cycle (Figure 9)*
  3. *prism (Figure 22)*
- *Type* $(3|5)$*:*
  4. *dodecahedron edge cycle (Figure 10)*
  5. *barrel (Figure 23)*
  6. *dodecahedron thick cycle (Figure 24)*
- *Type* $(4|3)$*:*
  7. *octahedron edge cycle (Figure 11)*
  8. *octahedron vertex cycle (Figure 12)*
  9. *antiprism (Figure 25)*
- *Type* $(5|3)$*:*
  10. *icosahedron edge cycle (Figure 13)*
  11. *icosahedron vertex cycle (Figure 14)*
  12. *icosahedron wide cycle (Figure 26)*
  13. *icosahedron first thick cycle (Figure 27)*
  14. *icosahedron second thick cycle (Figure 28)*

Remark. *This paper was originally written as two independent papers by two disjoint pairs of co-authors. The methods used in both papers were very similar, and the papers differed mainly in the structure of theorems and proofs. After we had learned about each other, we decided to join forces and merge the papers into one [7], which is in its final form presented here.*

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