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Decomposing K_{18n} and K_{18n+1} into connected unicyclic graphs with 9 edges

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Abstract

Other than C_9 there are 239 connected unicyclic graphs with exactly 9 edges. We use established graph labeling results to prove that every one of them decomposes the complete graph K_n if $n \equiv 0$ or $1 \pmod{18}$.

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Mathematics Subject Classification : 05C51, 05C78
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1. Introduction

The idea of labeling graphs in order to decompose larger graphs was first introduced by Rosa and has been used extensively since its initial development [2]. First, one must understand the basic definition of graph decomposition.

Definition 1.1. *Let K be a simple graph, and let $\mathcal{G} = \{G_0, G_1, \cdots, G_s\}$ be a family of pairwise edge-disjoint subgraphs of K. We say that $\mathcal{G}$ is a decomposition of K when every edge of K belongs to exactly one member of $\mathcal{G}$. If all $G_i \in \mathcal{G}$ are mutually isomorphic, then we say K allows a G-decomposition or a G-design.*

Furthermore, a *unicyclic graph* is a simple graph containing exactly one cycle. A *bipartite graph* is a graph whose vertex set can be partitioned into two disjoint sets so that no two vertices within the same set are adjacent. A *tripartite graph* is a graph whose vertex set can be partitioned

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into three disjoint sets such that no two vertices within the same set are adjacent. In this article, we focus our attention on unicyclic graphs with nine edges.

Now, let G be a unicyclic graph on nine edges. If a G-decomposition of K_n exists, then it is necessary that $n \equiv 0, 1, 9,$ or $10 \pmod{18}$. However, we only consider complete graphs of order $n \equiv 0$ or $1 \pmod{18}$ as our tools do not apply to complete graphs with order $n \equiv 9$ or $10 \pmod{18}$. Furthermore, we restrict our attention to connected graphs. In Section 2, we discuss the definitions and tools that we use to find our decompositions. In Section 3, we discuss results and previous research related to this article. In Section 4, we discuss our main findings. Finally, we include an appendix of our labeled graphs used in this research.

2. Definitions and Tools

The decompositions we will be using belong to the following well-known types.

Definition 2.1. *(El-Zanati, Vanden Eynden [13]) A G-decomposition of the complete graph K_n is cyclic if there exists an ordering $u_0, u_1, \ldots, u_{n-1}$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(u_j) = u_{j+1}$ for $j = 0, 1, \ldots, n-1$ inducing an automorphism on $\mathcal{G}$, where the addition is performed modulo n.*

Definition 2.2. *(Bunge [11]) A G-decomposition of the complete graph K_n is 1-rotational if there exists an ordering $(u_0, u_1, \ldots, u_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(u_j) = u_{j+1}$ for $j = 0, 1, \ldots, n-2$ and $\varphi(u_{n-1}) = u_{n-1}$ inducing an automorphism on $\mathcal{G}$, where the addition is performed modulo $n-1$.*

Rosa discovered the following labeling techniques to create cyclic graph decompositions [2].

Definition 2.3. *(Rosa [2]) Consider the vertex set of a simple graph G on n edges, say $V(G)$, and the edge set $E(G)$. A ρ-labeling is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, 2n\}$ such that the induced length function $\ell : E(G) \to \{1, 2, \ldots, n\}$ is used to label the edge set $E(G)$ such that*

$$\ell(uv) = \textit{min}\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\},$$

where $u, v \in V(G)$ and $uv \in E(G)$.

Definition 2.3 leads to the following theorem.

Theorem 2.1. (Rosa [2]) *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

The next several definitions and theorems apply to bipartite graphs.

Definition 2.4. *(Rosa [2]) A σ-labeling of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.*

Definition 2.5. *(El-Zanati, Vanden Eynden [14]) A ρ- or σ-labeling of a bipartite graph G with bipartition (A, B) is called an ordered ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.*

Theorem 2.2. (El-Zanati, Vanden Eynden [14]) *Let G be a graph with n edges which has a ρ^+-labeling. Then G decomposes K_{2nk+1} for all positive integers k.*

Definition 2.6. *(Freyberg, Tran [5]) A σ^{+-}-labeling of a bipartite graph G with n edges and bipartition (A, B) is a σ^+-labeling with the property that $f(a) - f(b) \neq n$ for all $a \in A$ and $b \in B$, and $f(x) \notin \{2n, 2n-1\}$ for any $x \in V(G)$.*

Theorem 2.3. (Freyberg, Tran [5]) *Let G be a graph with n edges and a σ^{+-}-labeling such that the edge of length n is a pendant edge e. Then there exists a G-decomposition of both K_{2nk} and K_{2nk+1} for every positive integer k.*

The next set of definitions and theorems apply to tripartite graphs.

Definition 2.7. *(Bunge et al. [12]) Let G be a tripartite graph on n edges and with vertex partition A, B, C. A ρ-tripartite labeling of G is a ρ-labeling f of G such that:*

- *$f(a) < f(v)$ for any edge $av \in E(G)$ where $a \in A$.*
- *For every edge $bc \in E(G)$ where $b \in B$, $c, \in C$, there exists a complementary edge $b'c' \in E(G)$ where $b' \in B$, $c' \in C$ such that*

$$|f(b) - f(c)| + |f(b') - f(c')| = 2n.$$

- *For all $b \in B$, $c \in C$,*

$$|f(b) - f(c)| \neq 2n.$$

This definition leads to the following useful theorem.

Theorem 2.4. (Bunge et al. [12]) *Let G be a tripartite graph on n edges which admits a ρ-tripartite labeling. Then there exists a cyclic G-decomposition of K_{2nk+1} for all $k \geq 1$.*

We have a similar set of definitions to help address the case of complete graphs of even order. However, we note here that the following definition and theorem only apply to graphs with a pendant vertex.

Definition 2.8. *(Bunge [11]) Let G be a graph on n edges. A 1-rotational ρ-labeling of G is a one-to-one function $f : V(G) \to [0, 2n-2] \cup \{\infty\}$ such that:*

- *For some pendant vertex w, $f(w) = \infty$.*
- *f is a ρ-labeling of $G - w$.*

Theorem 2.5. (Bunge [11]) *Let G be a graph with n edges. There exists a 1-rotational G-decomposition of K_{2n} if and only if G admits a 1-rotational ρ-labeling.*

Much like with ρ-labeling, Definition 2.8 and Theorem 2.5 only help us decompose the complete graph K_{18}. However, we can again tighten our restrictions on the definition and obtain a stronger theorem that applies to a wider range of cases.

Definition 2.9. *(Bunge [11]) A 1-rotational ρ-tripartite labeling of a graph G is a one-to-one function $h : V(G) \to [0, 2n-2] \cup \{\infty\}$ such that:*

- *h is a 1-rotational ρ-labeling of G with $h(w) = \infty$, where w has degree of one.*
- *If the edge $av \in E(G)\backslash uw$, where $a \in A$, then $h(a) < h(v)$.*
- *If $bc \in E(G)$ with $b \in B$, $c \in C$, then there exists an edge $b'c' \in E(G)$ with $b' \in B$, $c' \in C$ such that*

$$|h(b) - h(c)| + |h(b') - h(c')| = 2n.$$

This definition prompts the following theorem.

Theorem 2.6. (Bunge [11]) *Let G be a tripartite graph with n edges and a vertex of degree 1. If G admits a 1-rotational ρ-tripartite labeling, then there exists a 1-rotational G-decomposition of K_{2nk} for any integer $k \geq 1$.*

3. Related Results

The decomposition spectrum of graphs with up to eight edges has been widely studied, which is what led to this study of graphs with nine edges. We direct the reader to [4], [5], [6], [7], [8], [9], and [10] for more information on graphs with up to eight edges.

To our knowledge, graphs on nine edges are largely unexplored. It is discussed in [13] that trees with up to 20 edges permit a β^+-labeling, and if a graph G with n edges permits a β^+-labeling, then G decomposes K_{2nk+1} for any positive integer k. Thus, trees on nine edges decompose K_{18k+1}. It is also discussed in [10] that a graph on n edges with a σ^+-labeling such that the edge of length n is a pendant edge decomposes K_{2nk} for any positive integer k. Therefore, we can consider any tree on nine edges, remove one pendant edge, then assign the induced eight-edged tree a β^+-labeling. Reattaching the ninth edge and assigning it length 9, we obtain a σ^+-labeling, so the graph decomposes K_{18k}.

While there are a plethora of theorems that provide information on trees, we did not find previous research on forests with nine edges, and we thus believe forests on nine edges to be unexplored. Furthermore, there is another group of students from the University of Minnesota Duluth that is studying decompositions of complete graphs K_{18k} and K_{18k+1} into unicyclic, disconnected, bipartite graphs on nine edges [1].

4. Main Result

Let $\mathcal{F} = \{G : G$ is a unicyclic connected graph with 9 edges$\}$. There are 240 members of this family (see Appendix). An exceptional member of this family is C_9 which obviously does not decompose K_n if $n \equiv 0 \pmod{18}$ and is known to decompose K_n if $n \equiv 1 \pmod{18}$ [3]. Our main result shows that all of the remaining members of $\mathcal{F}$ decompose K_n whenever $n \equiv 0$ or $1 \pmod{18}$.

Theorem 4.1. *Let $G \in \mathcal{F} \setminus \{C_9\}$. There exists a G-decomposition of K_n if $n \equiv 0$ or $1 \pmod{18}$.*

Proof. If G is bipartite, the proof follows from Theorem 2.3 and the labelings given in Section 5.1. If G is tripartite, the proof follows from Theorem 2.5 or Theorem 2.6 and the labelings given in Section 5.2. □

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5. Appendix

5.1. Bipartite graphs

The following subsection shows a σ^{+-}−labeling of every bipartite unicyclic graph with 9 edges. We use the naming convention $G_i(n; t_1, t_2, \ldots, t_n)$ to denote a unicyclic graph which contains C_n and a tree of size t_j appended to the j^{th} vertex of the cycle. We let t_1 be the size of the largest such tree. The index i is used to distinguish between non-isomorphic graphs with the same $n+1$-tuple.

5.1.1. Graphs containing a 4-cycle

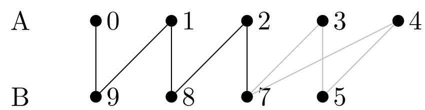

Figure 1. $G_1(4; 5, 0, 0, 0)$

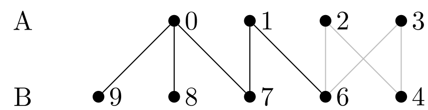

Figure 2. $G_2(4; 5, 0, 0, 0)$

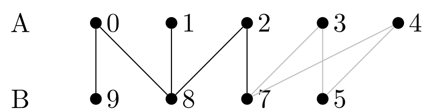

Figure 3. $G_3(4; 5, 0, 0, 0)$

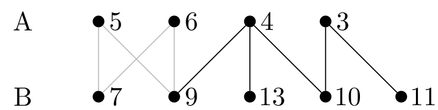

Figure 4. $G_4(4; 5, 0, 0, 0)$

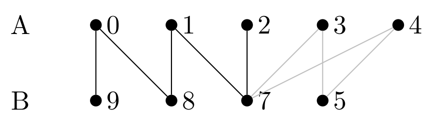

Figure 5. $G_5(4; 5, 0, 0, 0)$

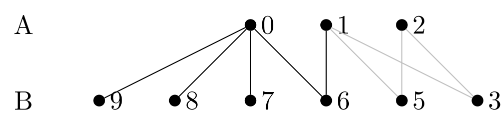

Figure 6. $G_6(4; 5, 0, 0, 0)$

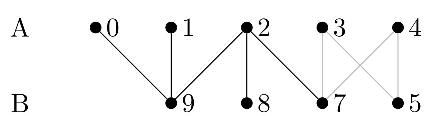

Figure 7. $G_7(4; 5, 0, 0, 0)$

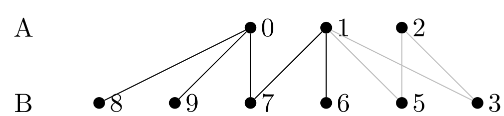

Figure 8. $G_8(4; 5, 0, 0, 0)$

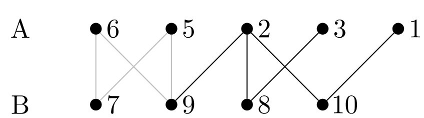

Figure 9. $G_9(4; 5, 0, 0, 0)$

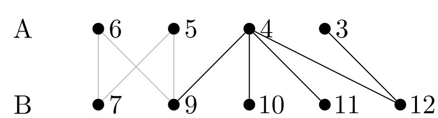

Figure 10. $G_{10}(4; 5, 0, 0, 0)$

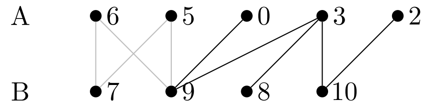

Figure 11. $G_{11}(4; 5, 0, 0, 0)$

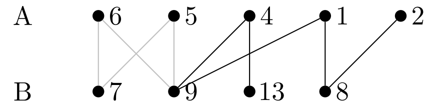

Figure 12. $G_{12}(4; 5, 0, 0, 0)$

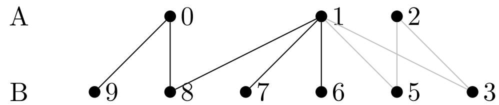

Figure 13. $G_{13}(4; 5, 0, 0, 0)$

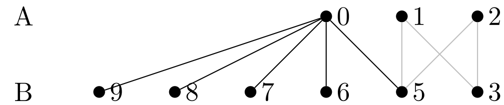

Figure 14. $G_{14}(4; 5, 0, 0, 0)$

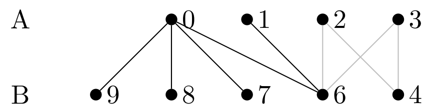

Figure 15. $G_{15}(4; 5, 0, 0, 0)$

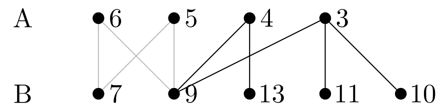

Figure 16. $G_{16}(4; 5, 0, 0, 0)$

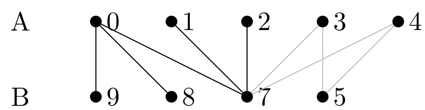

Figure 17. $G_{17}(4; 5, 0, 0, 0)$

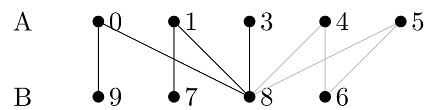

Figure 18. $G_{18}(4; 5, 0, 0, 0)$

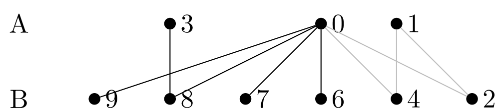

Figure 19. $G_{19}(4; 5, 0, 0, 0)$

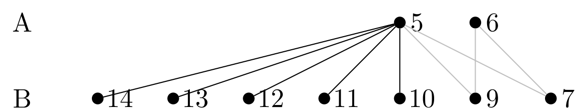

Figure 20. $G_{20}(4; 5, 0, 0, 0)$

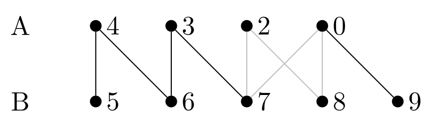

Figure 21. $G_1(4; 4, 1, 0, 0)$

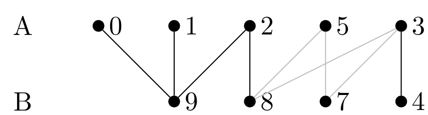

Figure 22. $G_2(4; 4, 1, 0, 0)$

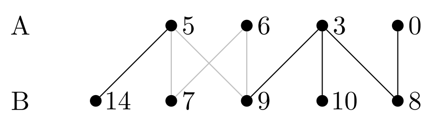

Figure 23. $G_3(4; 4, 1, 0, 0)$

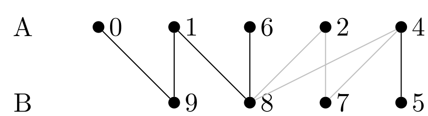

Figure 24. $G_4(4; 4, 1, 0, 0)$

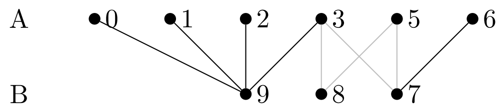

Figure 25. $G_5(4; 4, 1, 0, 0)$

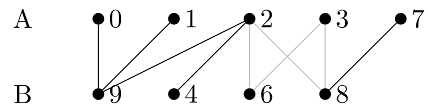

Figure 26. $G_6(4; 4, 1, 0, 0)$

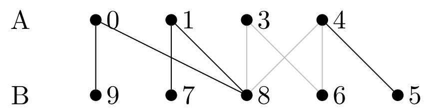

Figure 27. $G_7(4; 4, 1, 0, 0)$

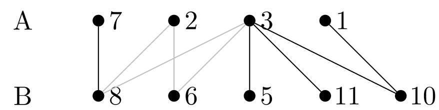

Figure 28. $G_8(4; 4, 1, 0, 0)$

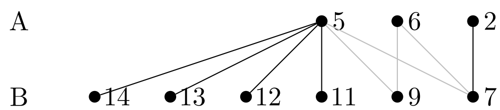

Figure 29. $G_9(4; 4, 1, 0, 0)$

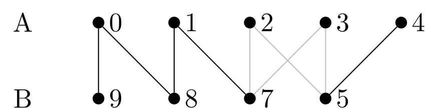

Figure 30. $G_1(4; 4, 0, 1, 0)$

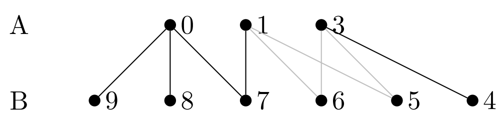

Figure 31. $G_2(4; 4, 0, 1, 0)$

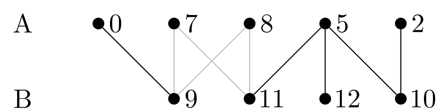

Figure 32. $G_3(4; 4, 0, 1, 0)$

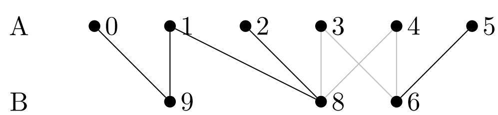

Figure 33. $G_4(4; 4, 0, 1, 0)$

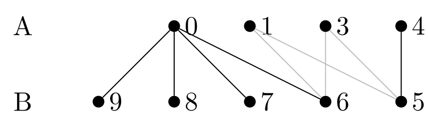

Figure 34. $G_5(4; 4, 0, 1, 0)$

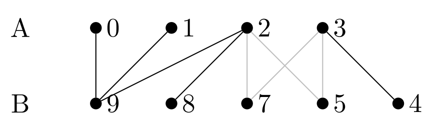

Figure 35. $G_6(4; 4, 0, 1, 0)$

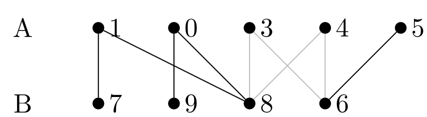

Figure 36. $G_7(4; 4, 0, 1, 0)$

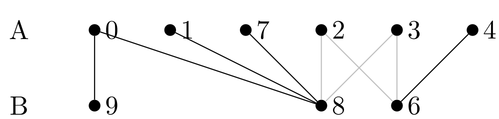

Figure 37. $G_8(4; 4, 0, 1, 0)$

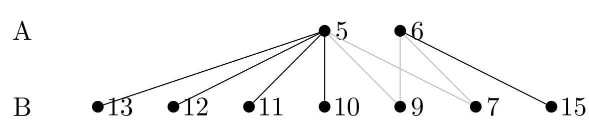

Figure 38. $G_9(4; 4, 0, 1, 0)$

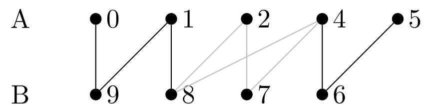

Figure 39. $G_1(4;3,2,0,0)$

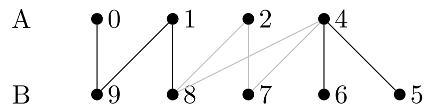

Figure 40. $G_2(4;3,2,0,0)$

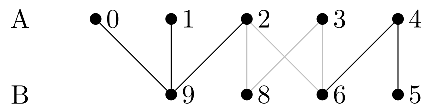

Figure 41. $G_3(4;3,2,0,0)$

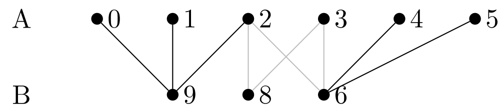

Figure 42. $G_4(4;3,2,0,0)$

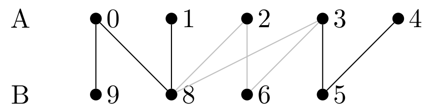

Figure 43. $G_5(4;3,2,0,0)$

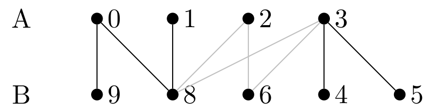

Figure 44. $G_6(4;3,2,0,0)$

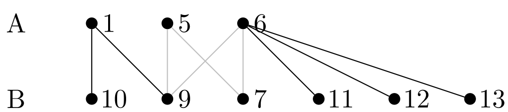

Figure 45. $G_7(4;3,2,0,0)$

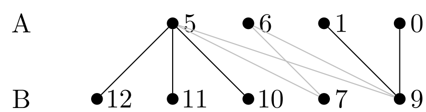

Figure 46. $G_8(4;3,2,0,0)$

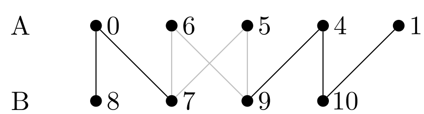

Figure 47. $G_1(4;3,0,2,0)$

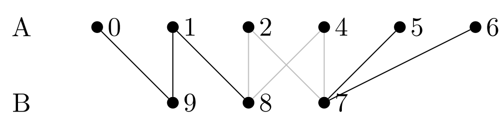

Figure 48. $G_2(4;3,0,2,0)$

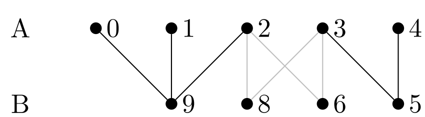

Figure 49. $G_3(4;3,0,2,0)$

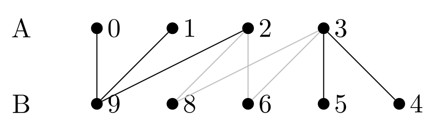

Figure 50. $G_4(4;3,0,2,0)$

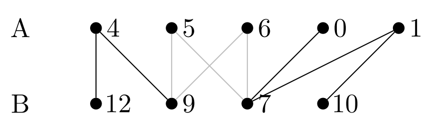

Figure 51. $G_5(4;3,0,2,0)$

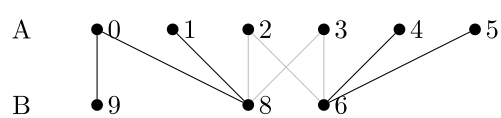

Figure 52. $G_6(4;3,0,2,0)$

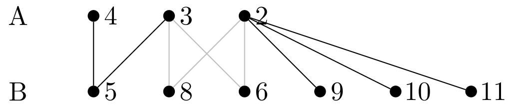

Figure 53. $G_7(4; 3, 0, 2, 0)$

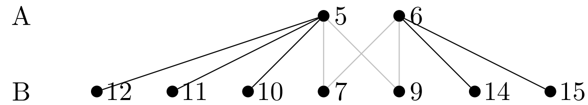

Figure 54. $G_8(4; 3, 0, 2, 0)$

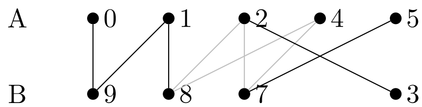

Figure 55. $G_1(4; 3, 1, 1, 0)$

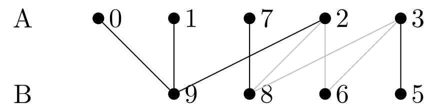

Figure 56. $G_2(4; 3, 1, 1, 0)$

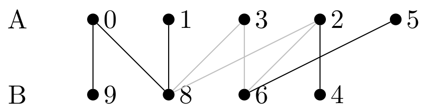

Figure 57. $G_3(4; 3, 1, 1, 0)$

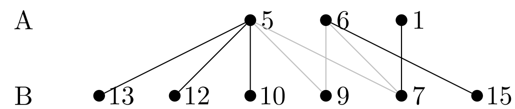

Figure 58. $G_4(4; 3, 1, 1, 0)$

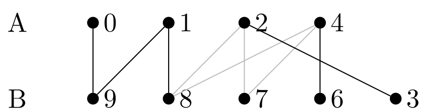

Figure 59. $G_1(4; 3, 1, 0, 1)$

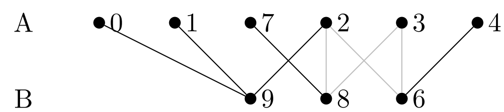

Figure 60. $G_2(4; 3, 1, 0, 1)$

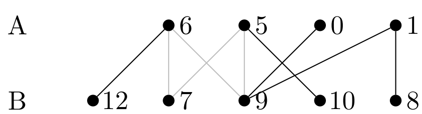

Figure 61. $G_3(4; 3, 1, 0, 1)$

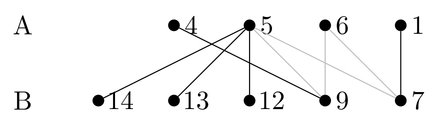

Figure 62. $G_4(4; 3, 1, 0, 1)$

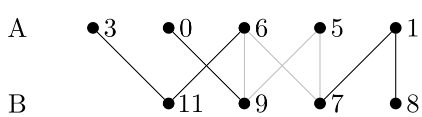

Figure 63. $G_1(4; 2, 2, 1, 0)$

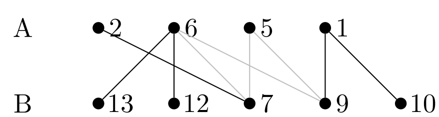

Figure 64. $G_2(4; 2, 2, 1, 0)$

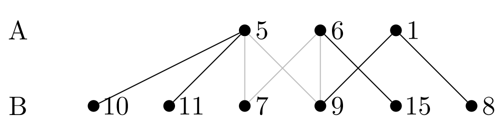

Figure 65. $G_3(4; 2, 2, 1, 0)$

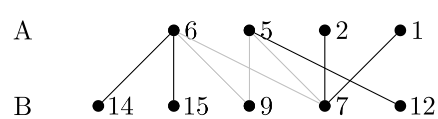

Figure 66. $G_4(4; 2, 2, 1, 0)$

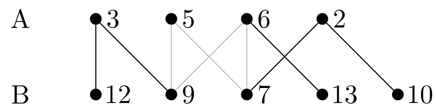

Figure 67. $G_1(4; 2, 1, 2, 0)$

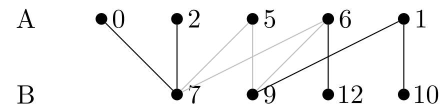

Figure 68. $G_2(4; 2, 1, 2, 0)$

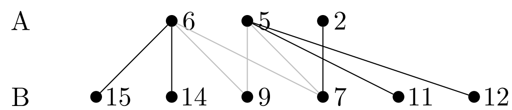

Figure 69. $G_3(4; 2, 1, 2, 0)$

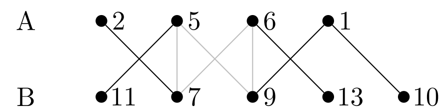

Figure 70. $G_1(4; 2, 1, 1, 1)$

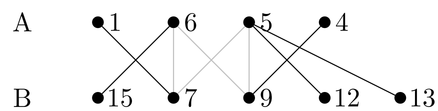

Figure 71. $G_2(4; 2, 1, 1, 1)$

5.1.2. Graphs containing a 6-cycle

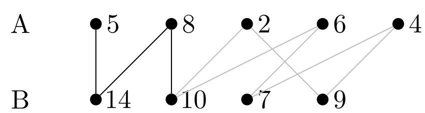

Figure 72. $G_1(6; 3, 0, 0, 0, 0, 0)$

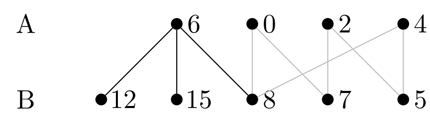

Figure 73. $G_2(6; 3, 0, 0, 0, 0, 0)$

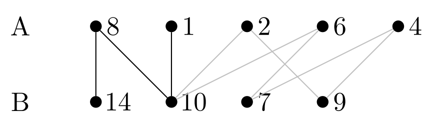

Figure 74. $G_3(6; 3, 0, 0, 0, 0, 0)$

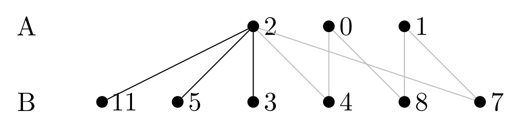

Figure 75. $G_4(6; 3, 0, 0, 0, 0, 0)$

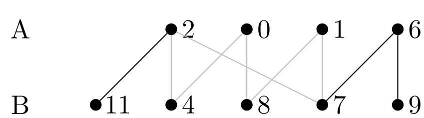

Figure 76. $G_1(6; 2, 1, 0, 0, 0, 0)$

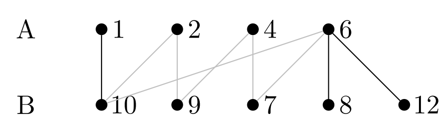

Figure 77. $G_2(6; 2, 1, 0, 0, 0, 0)$

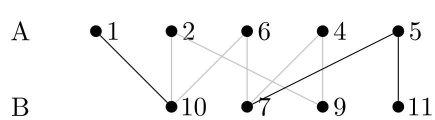

Figure 78. $G_1(6; 2, 0, 1, 0, 0, 0)$

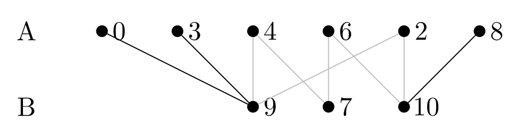

Figure 79. $G_2(6; 2, 0, 1, 0, 0, 0)$

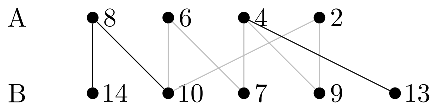

Figure 80. $G_1(6;2,0,0,1,0,0)$

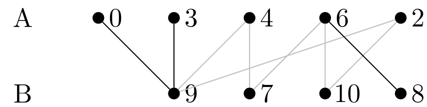

Figure 81. $G_2(6;2,0,0,1,0,0)$

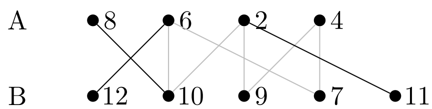

Figure 82. $G_1(6;1,1,1,0,0,0)$

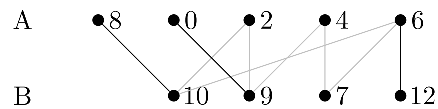

Figure 83. $G_1(6;1,1,0,1,0,0)$

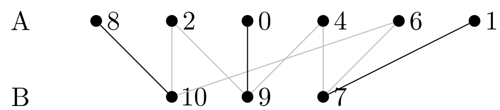

Figure 84. $G_1(6;1,0,1,0,1,0)$

5.1.3. Graphs containing an 8-cycle

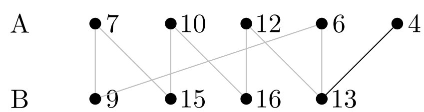

Figure 85. $G_1(8;1,0,0,0,0,0,0,0)$

5.2. Tripartite graphs

The following subsection contains a ρ-tripartite labeling (left hand side of each figure) and a 1-rotational ρ-tripartite labeling (right hand side of each figure) of every tripartite unicyclic graph with 9 edges.

5.2.1. Graphs containing a 3-cycle

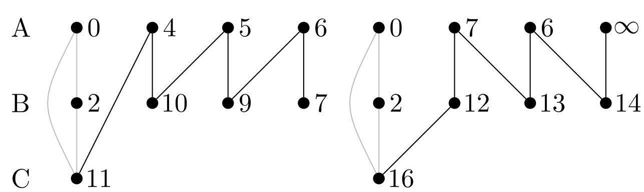

Figure 86. $G_1(3;6,0,0)$

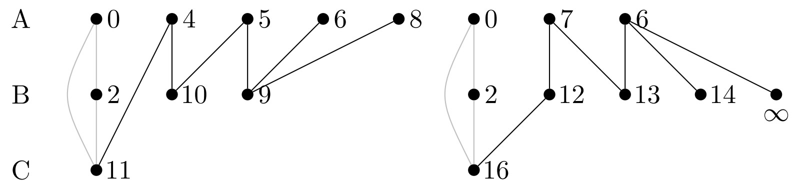

Figure 87. $G_2(3;6,0,0)$

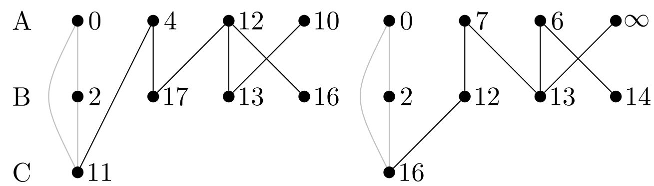

Figure 88. $G_3(3;6,0,0)$

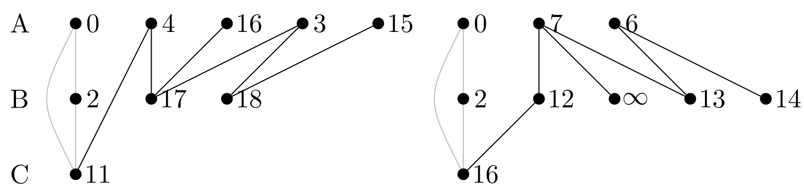

Figure 89. $G_4(3;6,0,0)$

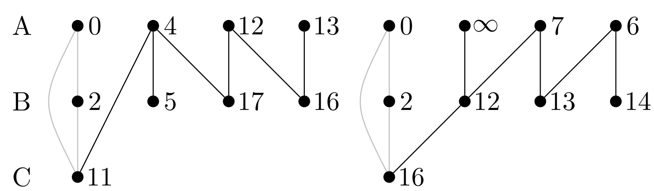

Figure 90. $G_5(3;6,0,0)$

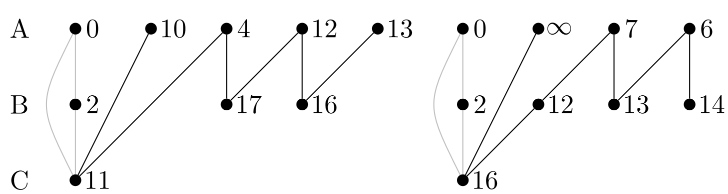

Figure 91. $G_6(3;6,0,0)$

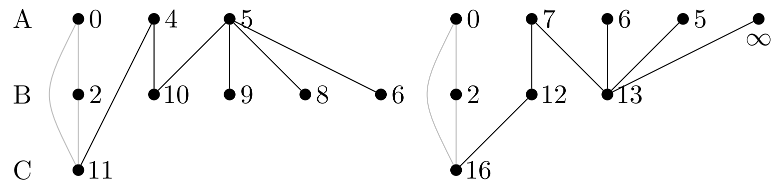

Figure 92. $G_7(3; 6, 0, 0)$

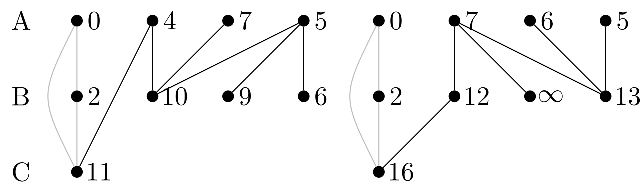

Figure 93. $G_8(3; 6, 0, 0)$

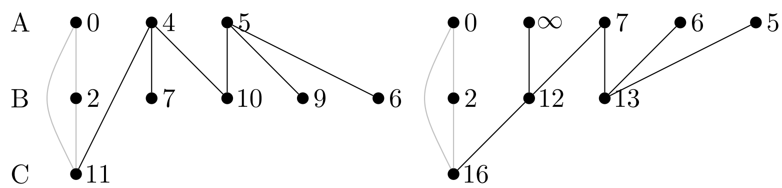

Figure 94. $G_9(3; 6, 0, 0)$

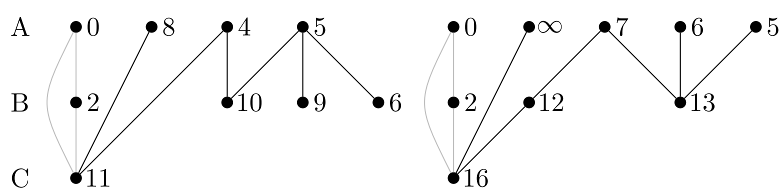

Figure 95. $G_{10}(3; 6, 0, 0)$

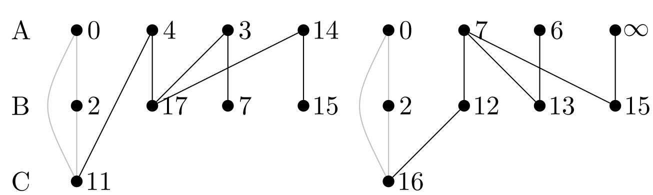

Figure 96. $G_{11}(3; 6, 0, 0)$

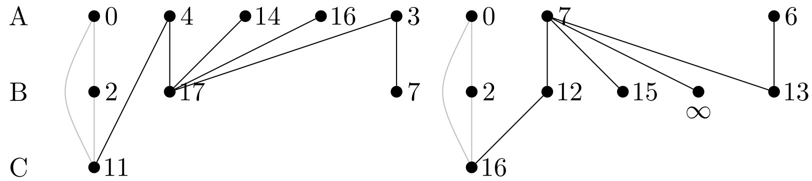

Figure 97. $G_{12}(3; 6, 0, 0)$

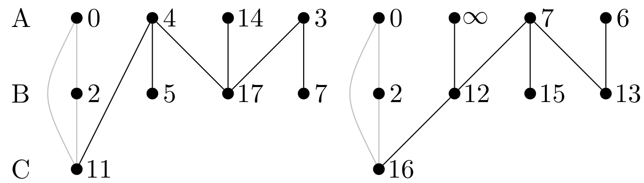

Figure 98. $G_{13}(3; 6, 0, 0)$

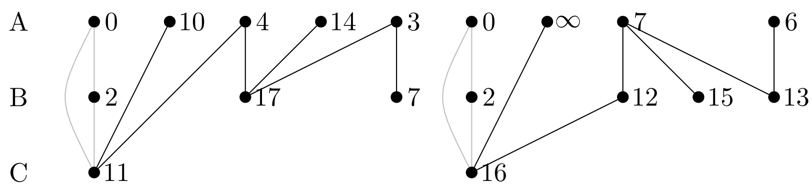

Figure 99. $G_{14}(3; 6, 0, 0)$

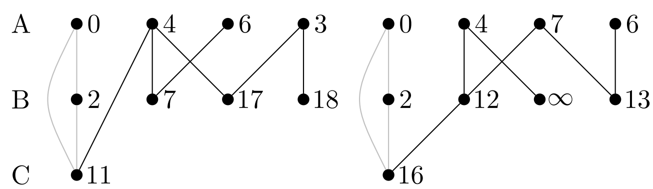

Figure 100. $G_{15}(3; 6, 0, 0)$

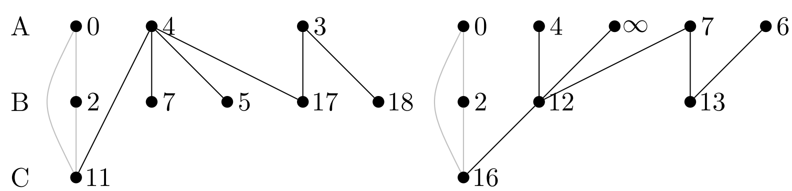

Figure 101. $G_{16}(3; 6, 0, 0)$

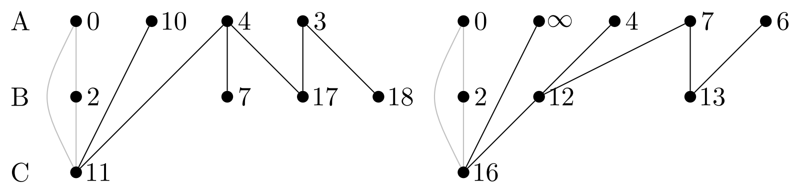

Figure 102. $G_{17}(3; 6, 0, 0)$

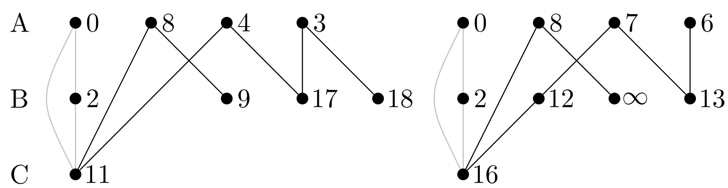

Figure 103. $G_{18}(3; 6, 0, 0)$

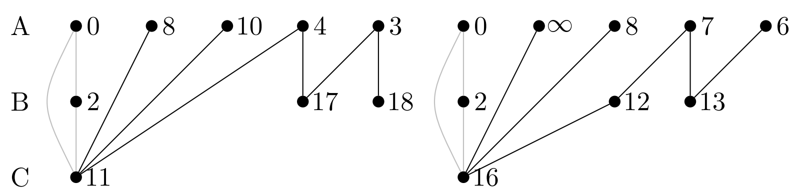

Figure 104. $G_{19}(3; 6, 0, 0)$

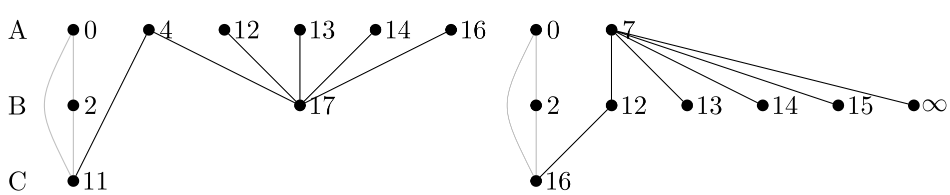

Figure 105. $G_{20}(3; 6, 0, 0)$

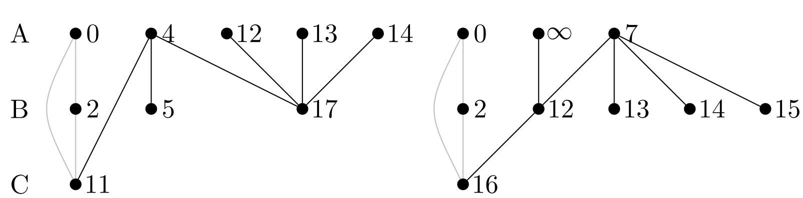

Figure 106. $G_{21}(3; 6, 0, 0)$

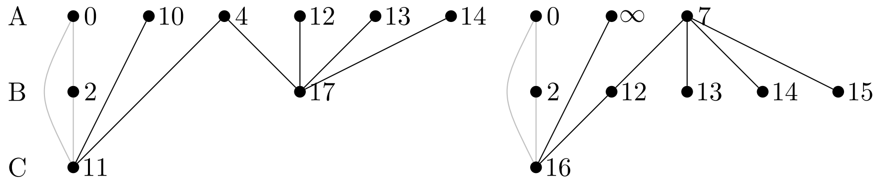

Figure 107. $G_{22}(3; 6, 0, 0)$

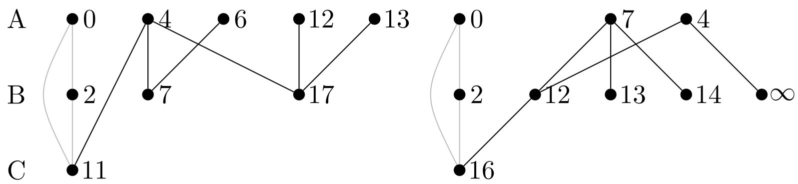

Figure 108. $G_{23}(3; 6, 0, 0)$

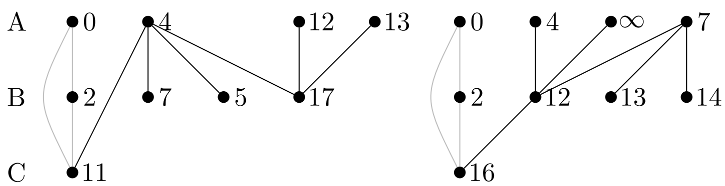

Figure 109. $G_{24}(3; 6, 0, 0)$

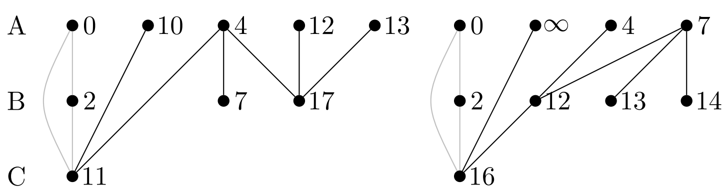

Figure 110. $G_{25}(3; 6, 0, 0)$

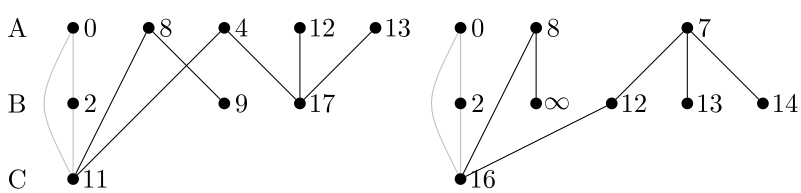

Figure 111. $G_{26}(3; 6, 0, 0)$

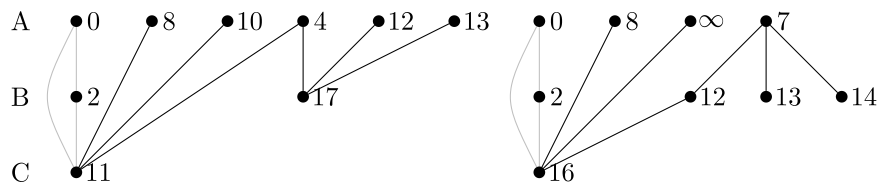

Figure 112. $G_{27}(3; 6, 0, 0)$

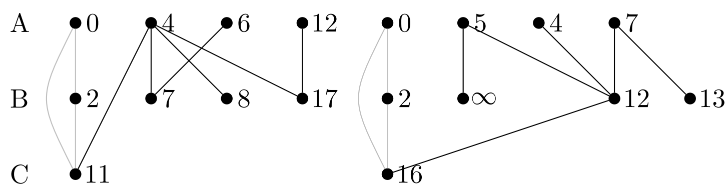

Figure 113. $G_{28}(3; 6, 0, 0)$

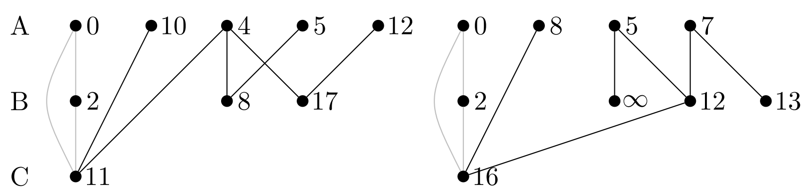

Figure 114. $G_{29}(3; 6, 0, 0)$

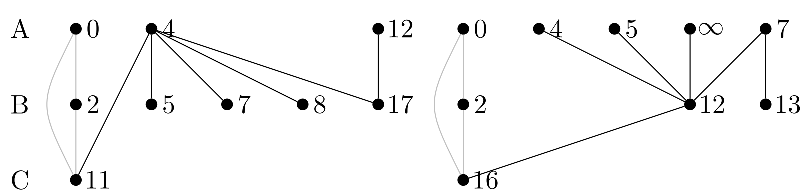

Figure 115. $G_{30}(3; 6, 0, 0)$

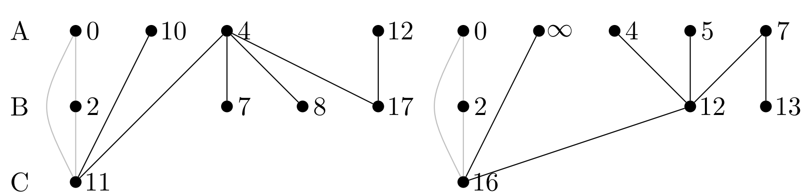

Figure 116. $G_{31}(3; 6, 0, 0)$

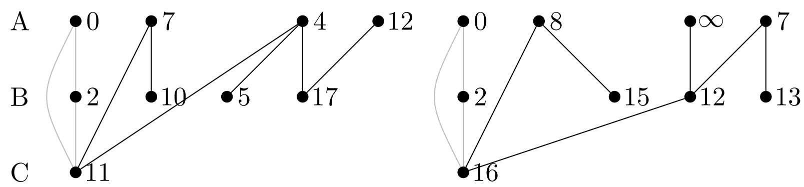

Figure 117. $G_{32}(3;6,0,0)$

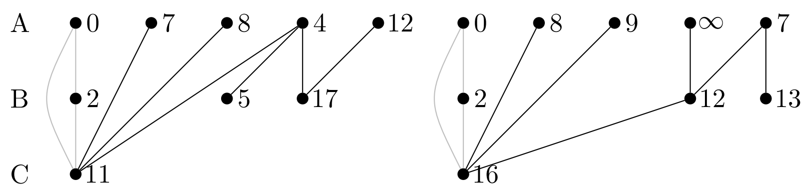

Figure 118. $G_{33}(3;6,0,0)$

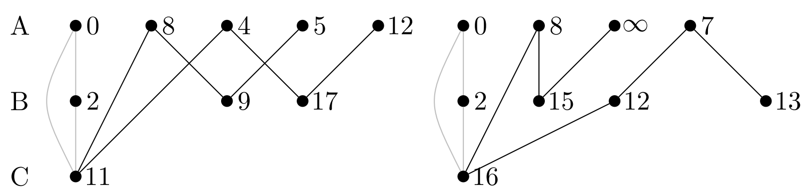

Figure 119. $G_{34}(3;6,0,0)$

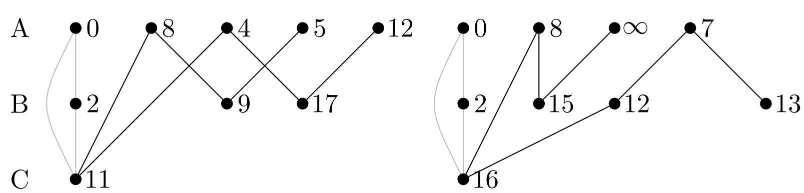

Figure 120. $G_{35}(3;6,0,0)$

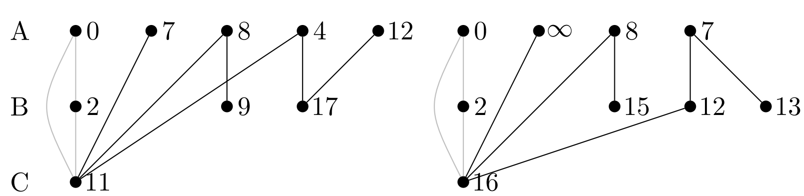

Figure 121. $G_{36}(3;6,0,0)$

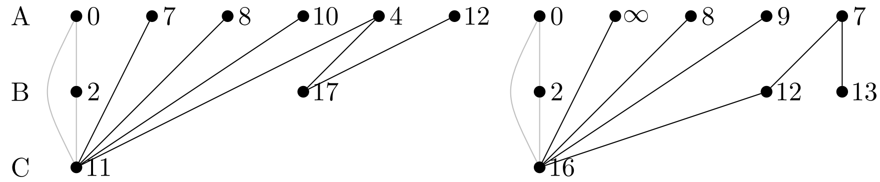

Figure 122. $G_{37}(3; 6, 0, 0)$

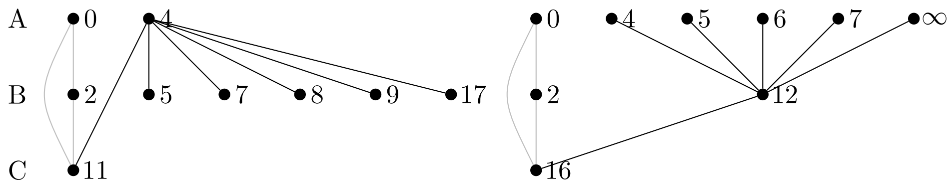

Figure 123. $G_{38}(3; 6, 0, 0)$

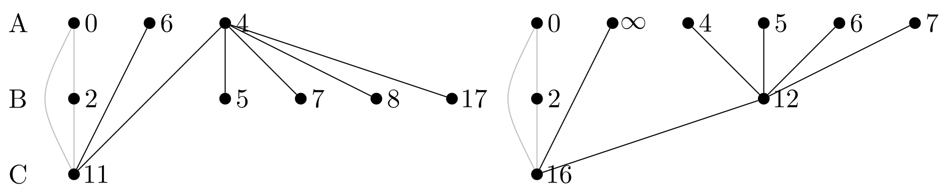

Figure 124. $G_{39}(3; 6, 0, 0)$

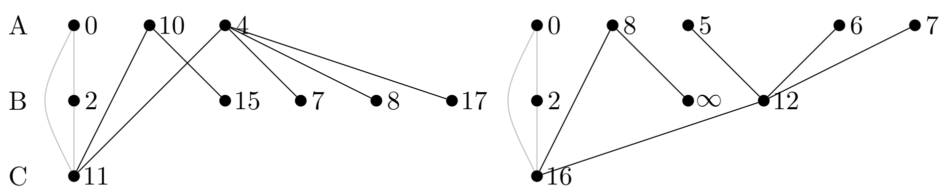

Figure 125. $G_{40}(3; 6, 0, 0)$

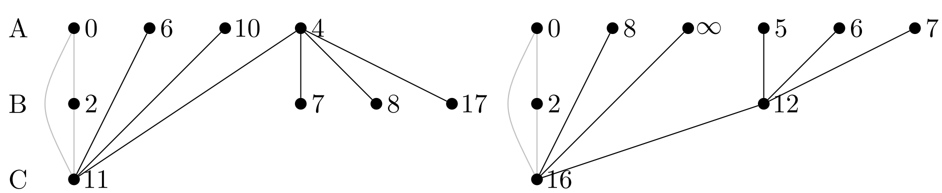

Figure 126. $G_{41}(3; 6, 0, 0)$

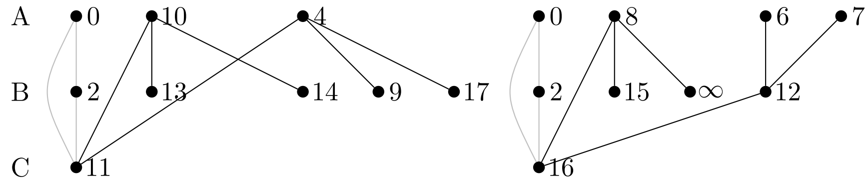

Figure 127. $G_{42}(3; 6, 0, 0)$

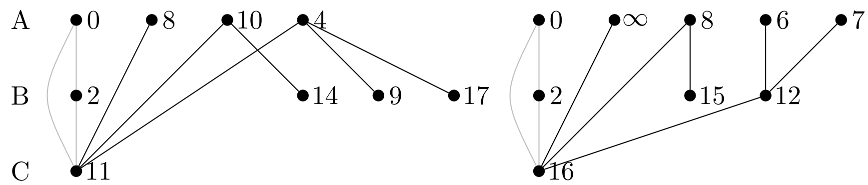

Figure 128. $G_{43}(3; 6, 0, 0)$

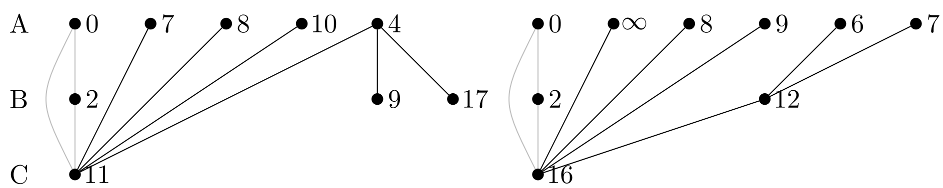

Figure 129. $G_{44}(3; 6, 0, 0)$

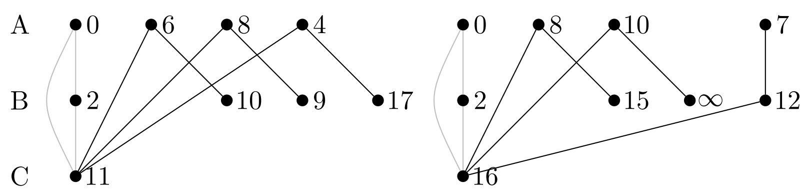

Figure 130. $G_{45}(3; 6, 0, 0)$

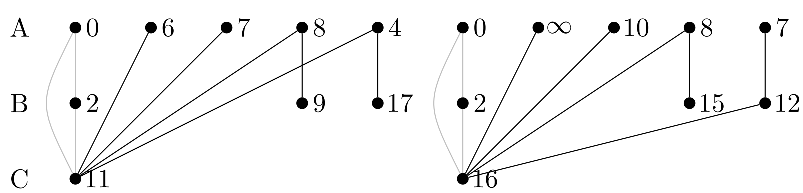

Figure 131. $G_{46}(3; 6, 0, 0)$

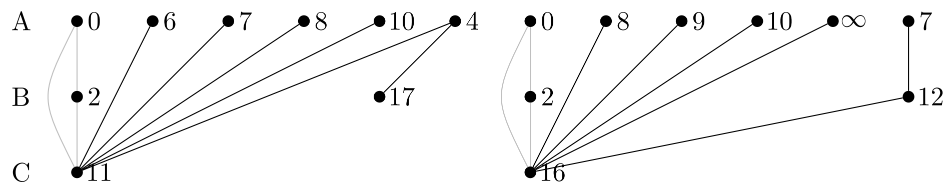

Figure 132. $G_{47}(3;6,0,0)$

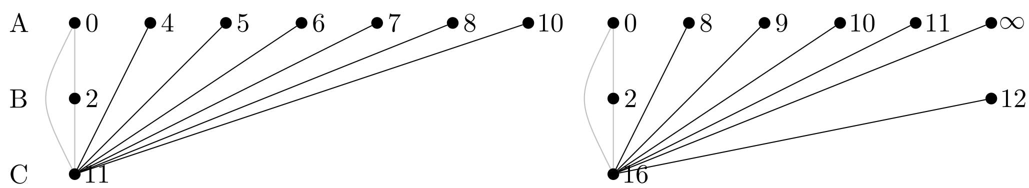

Figure 133. $G_{48}(3;6,0,0)$

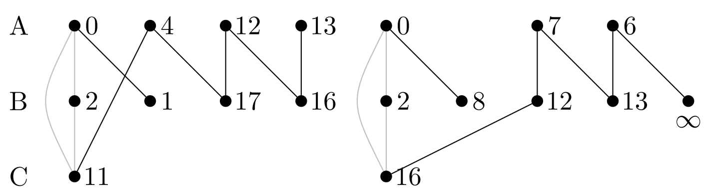

Figure 134. $G_1(3;5,1,0)$

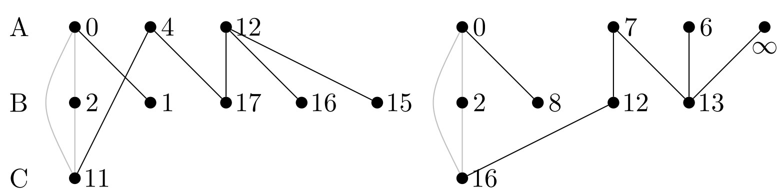

Figure 135. $G_2(3;5,1,0)$

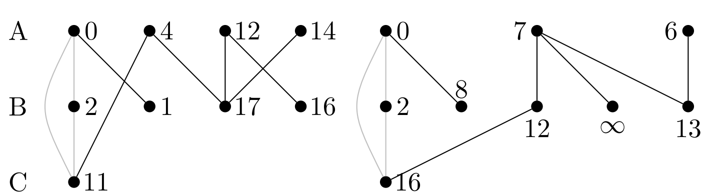

Figure 136. $G_3(3;5,1,0)$

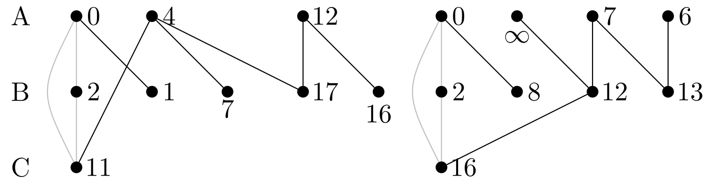

Figure 137. $G_4(3; 5, 1, 0)$

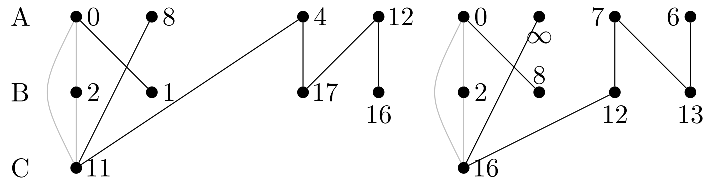

Figure 138. $G_5(3; 5, 1, 0)$

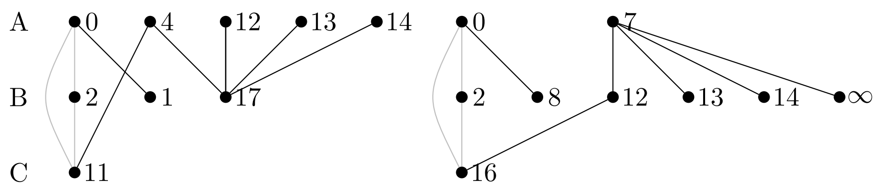

Figure 139. $G_6(3; 5, 1, 0)$

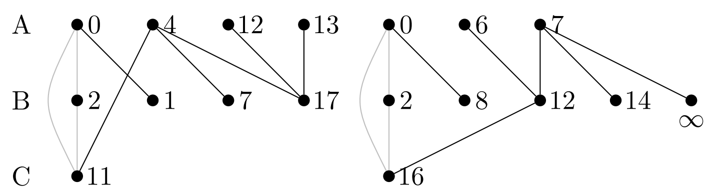

Figure 140. $G_7(3; 5, 1, 0)$

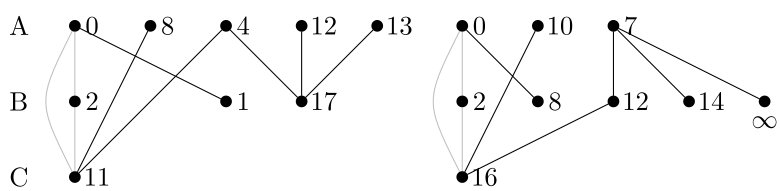

Figure 141. $G_8(3; 5, 1, 0)$

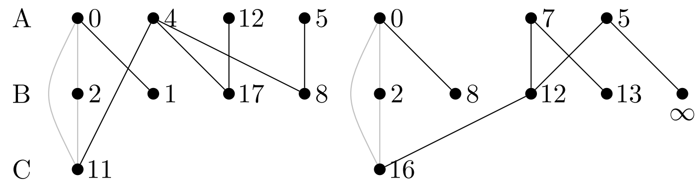

Figure 142. $G_9(3; 5, 1, 0)$

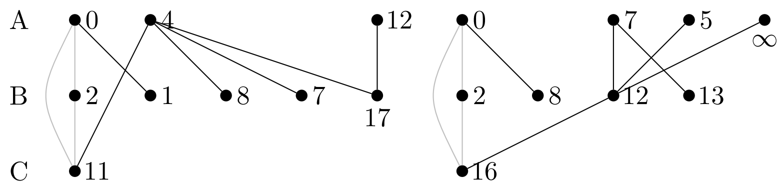

Figure 143. $G_{10}(3; 5, 1, 0)$

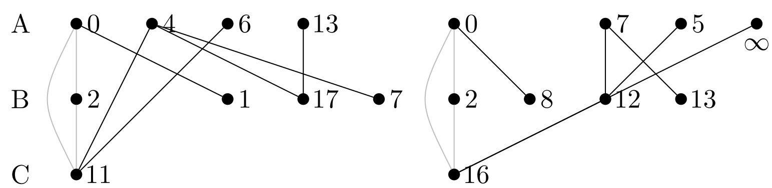

Figure 144. $G_{11}(3; 5, 1, 0)$

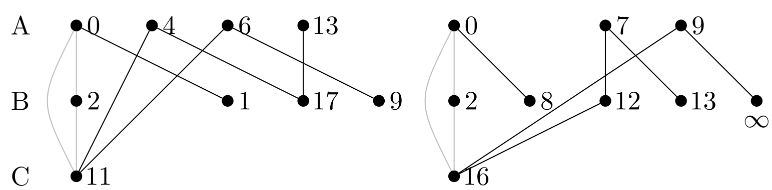

Figure 145. $G_{12}(3; 5, 1, 0)$

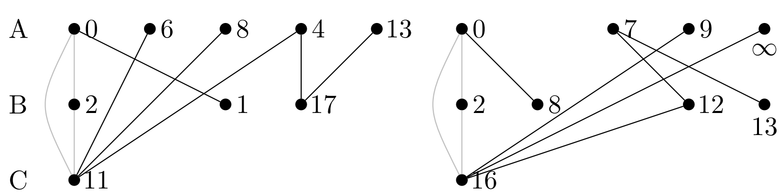

Figure 146. $G_{13}(3; 5, 1, 0)$

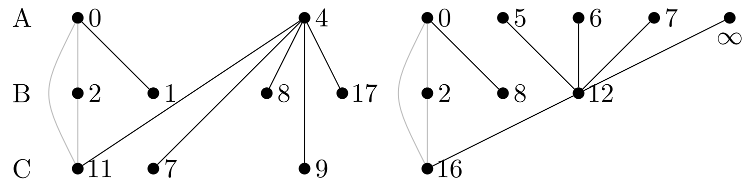

Figure 147. $G_{14}(3; 5, 1, 0)$

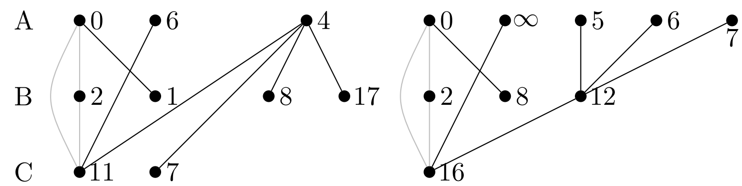

Figure 148. $G_{15}(3; 5, 1, 0)$

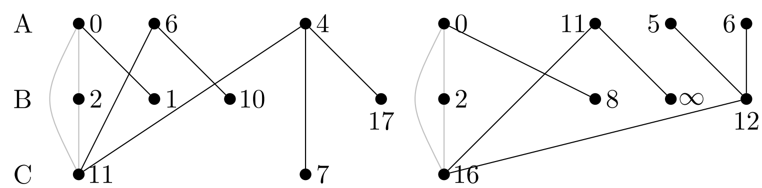

Figure 149. $G_{16}(3; 5, 1, 0)$

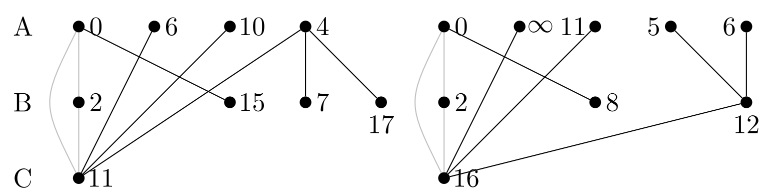

Figure 150. $G_{17}(3; 5, 1, 0)$

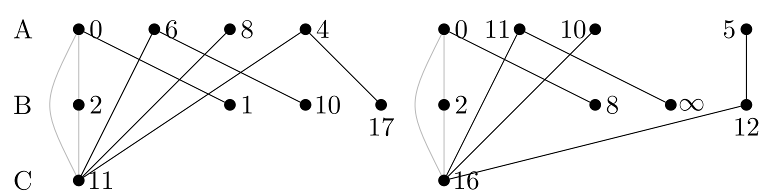

Figure 151. $G_{18}(3; 5, 1, 0)$

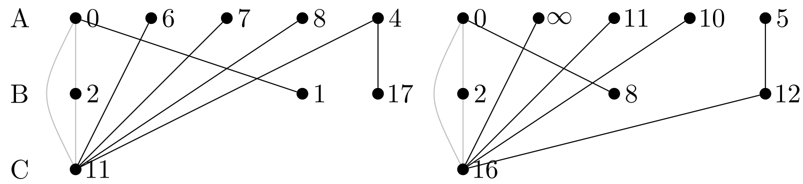

Figure 152. $G_{19}(3; 5, 1, 0)$

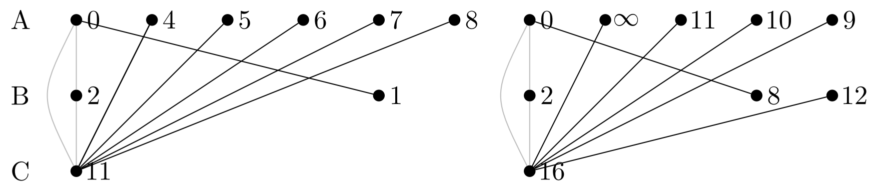

Figure 153. $G_{20}(3; 5, 1, 0)$

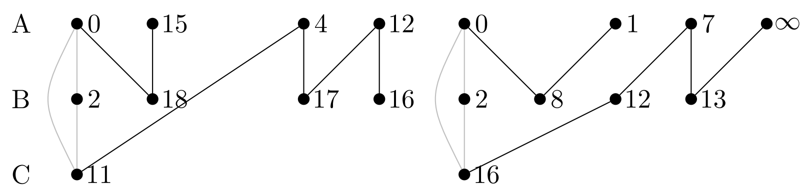

Figure 154. $G_1(3; 4, 2, 0)$

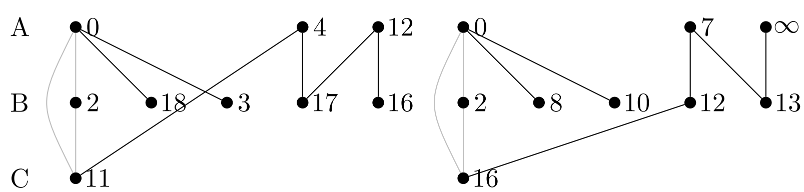

Figure 155. $G_2(3; 4, 2, 0)$

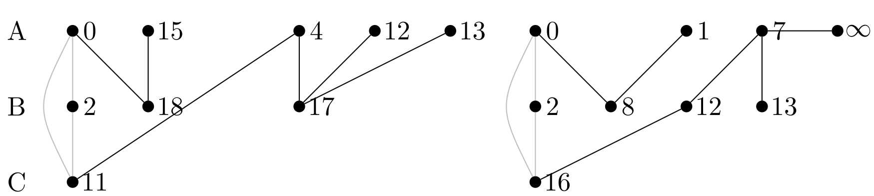

Figure 156. $G_3(3; 4, 2, 0)$

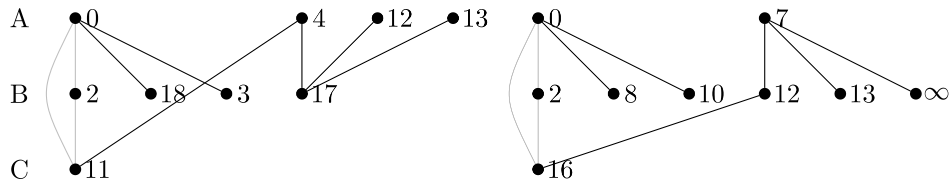

Figure 157. $G_4(3;4,2,0)$

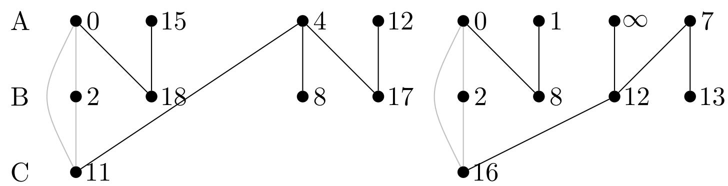

Figure 158. $G_5(3;4,2,0)$

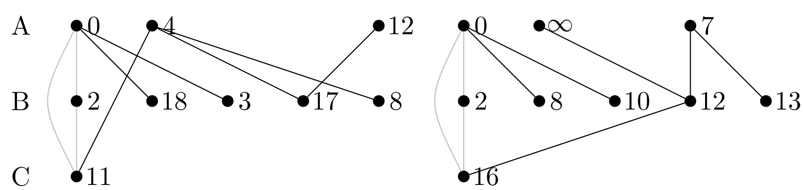

Figure 159. $G_6(3;4,2,0)$

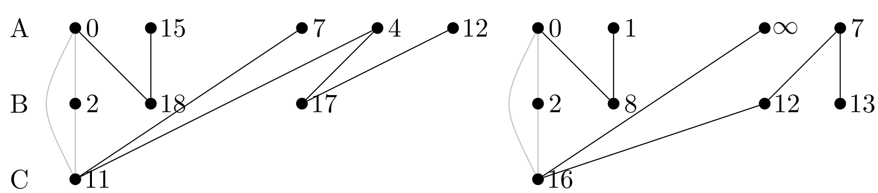

Figure 160. $G_7(3;4,2,0)$

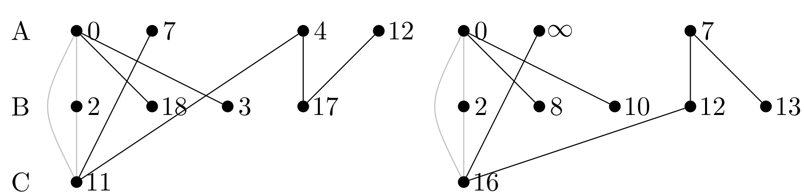

Figure 161. $G_8(3;4,2,0)$

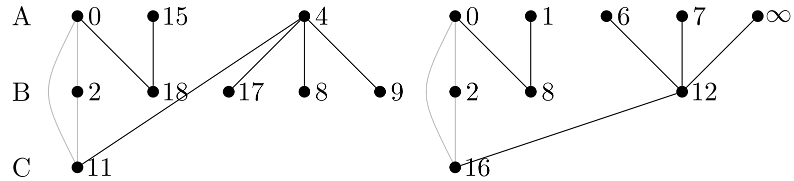

Figure 162. $G_9(3;4,2,0)$

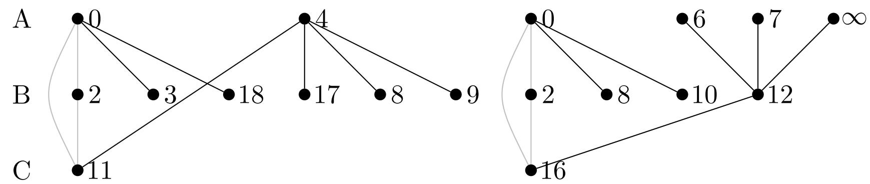

Figure 163. $G_{10}(3;4,2,0)$

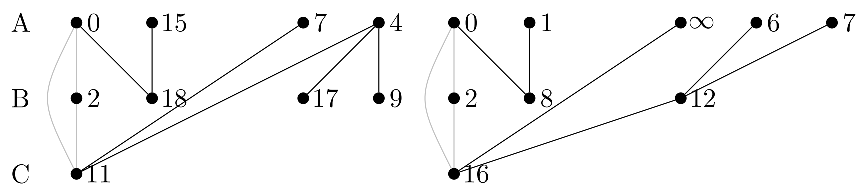

Figure 164. $G_{11}(3;4,2,0)$

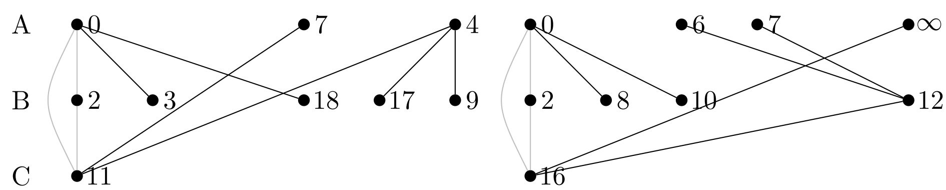

Figure 165. $G_{12}(3;4,2,0)$

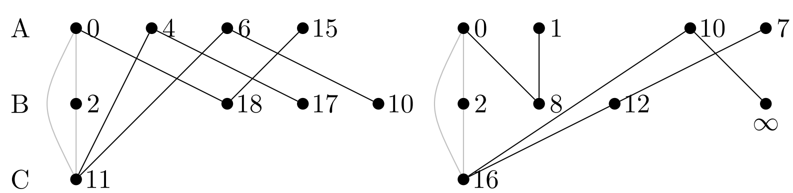

Figure 166. $G_{13}(3;4,2,0)$

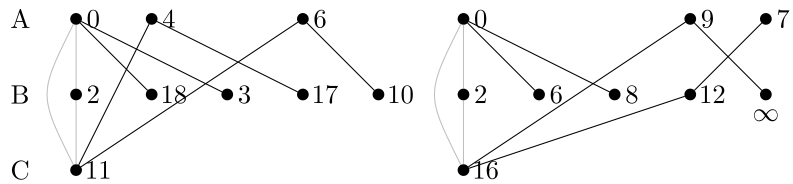

Figure 167. $G_{14}(3; 4, 2, 0)$

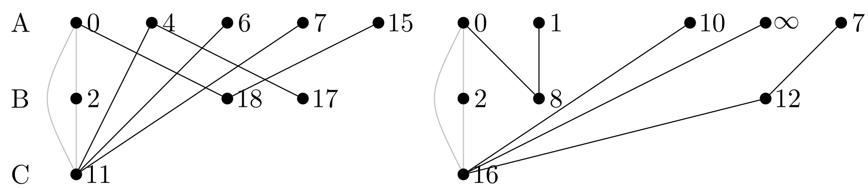

Figure 168. $G_{15}(3; 4, 2, 0)$

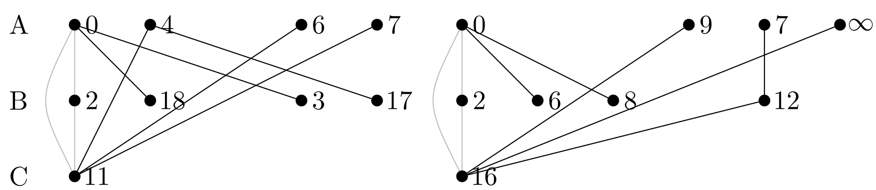

Figure 169. $G_{16}(3; 4, 2, 0)$

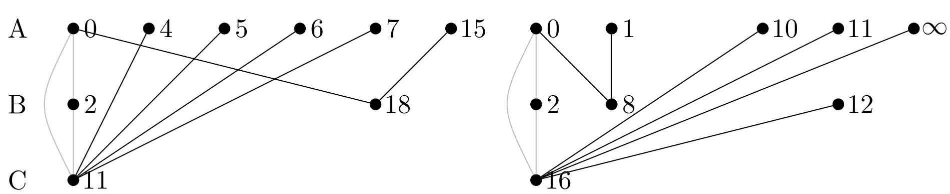

Figure 170. $G_{17}(3; 4, 2, 0)$

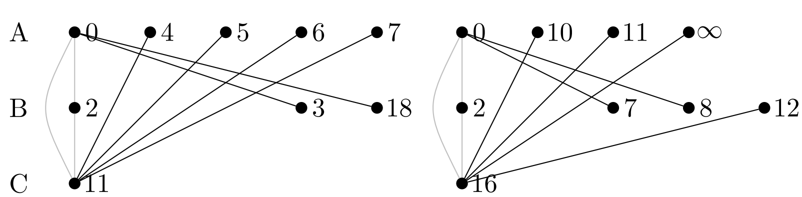

Figure 171. $G_{18}(3; 4, 2, 0)$

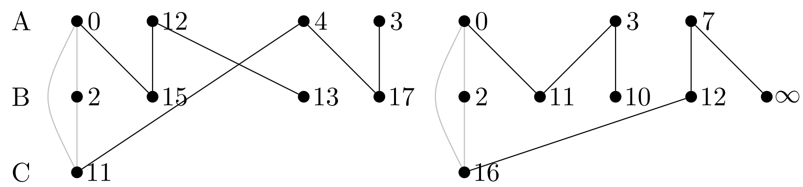

Figure 172. $G_1(3;3,3,0)$

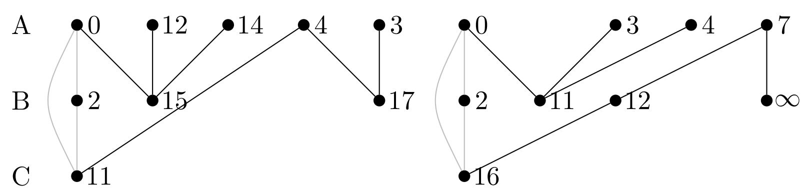

Figure 173. $G_2(3;3,3,0)$

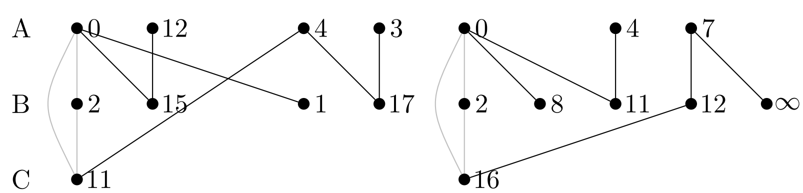

Figure 174. $G_3(3;3,3,0)$

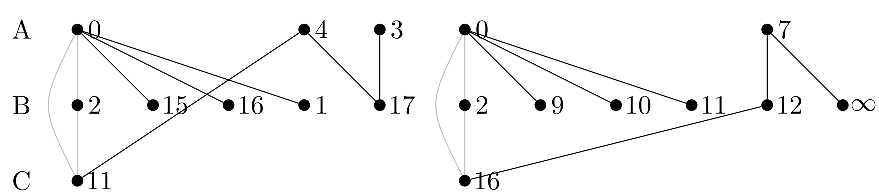

Figure 175. $G_4(3;3,3,0)$

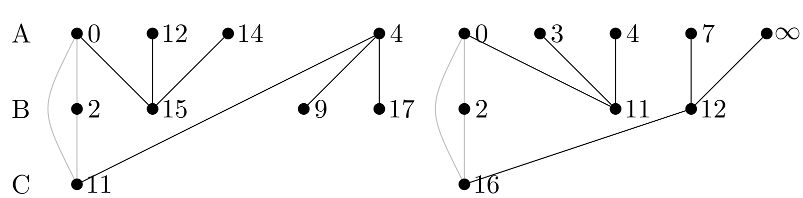

Figure 176. $G_5(3;3,3,0)$

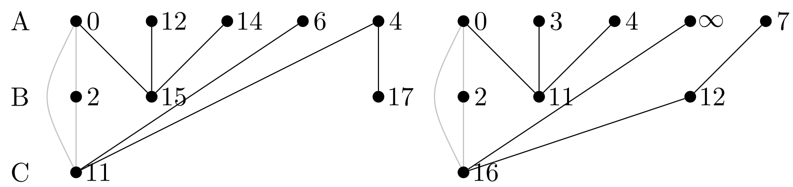

Figure 177. $G_6(3; 3, 3, 0)$

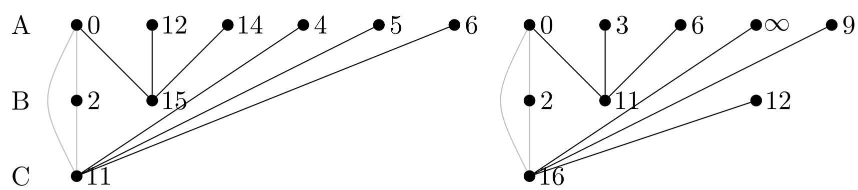

Figure 178. $G_7(3; 3, 3, 0)$

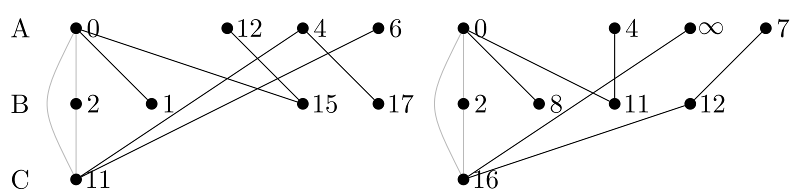

Figure 179. $G_8(3; 3, 3, 0)$

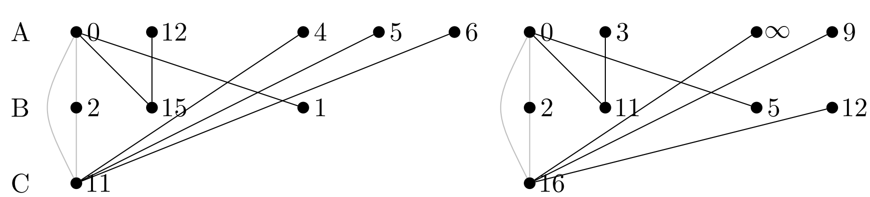

Figure 180. $G_9(3; 3, 3, 0)$

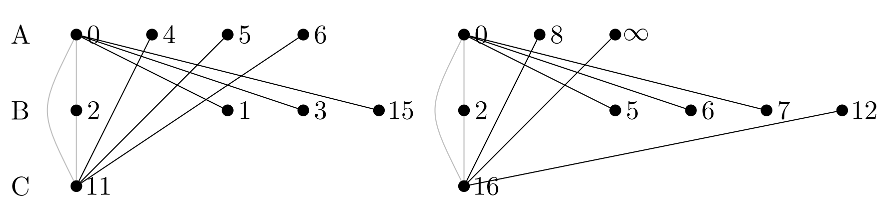

Figure 181. $G_{10}(3; 3, 3, 0)$

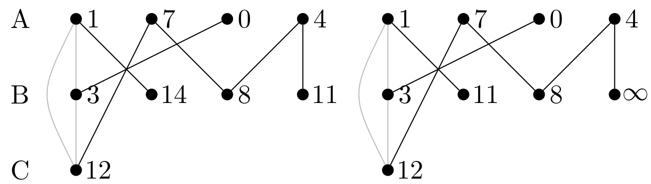

Figure 182. $G_1(3; 4, 1, 1)$

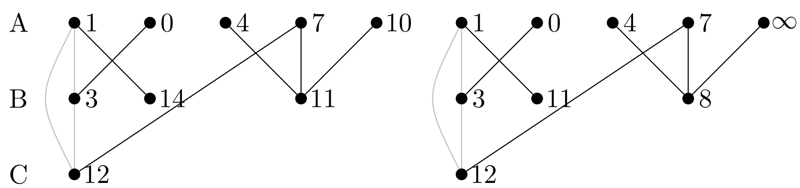

Figure 183. $G_2(3; 4, 1, 1)$

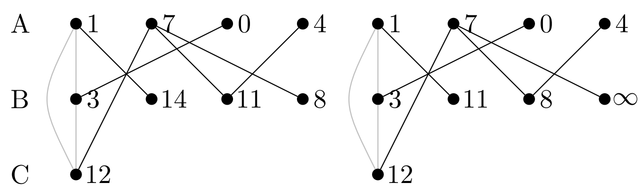

Figure 184. $G_3(3; 4, 1, 1)$

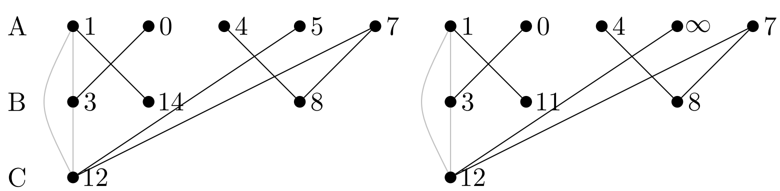

Figure 185. $G_4(3; 4, 1, 1)$

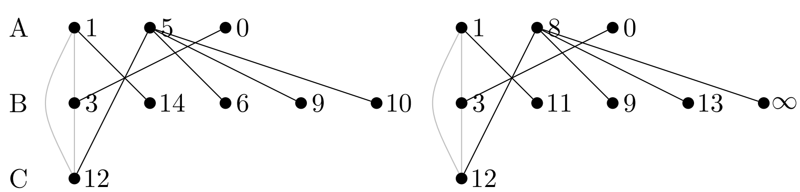

Figure 186. $G_5(3; 4, 1, 1)$

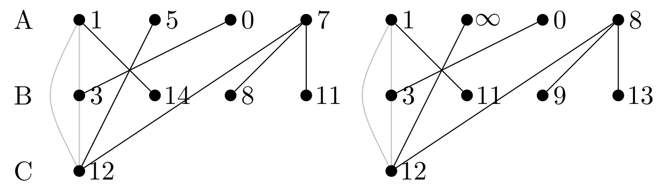

Figure 187. $G_6(3; 4, 1, 1)$

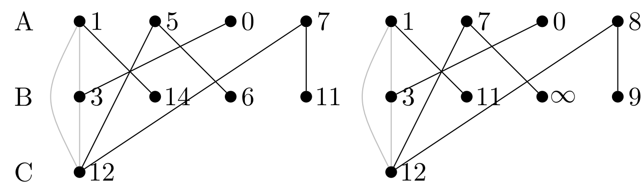

Figure 188. $G_7(3; 4, 1, 1)$

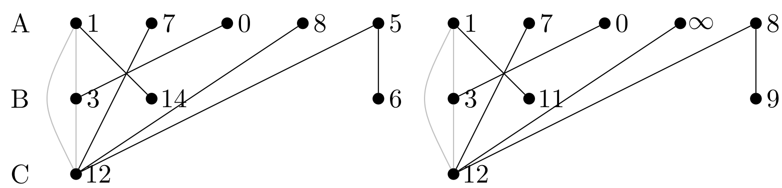

Figure 189. $G_8(3; 4, 1, 1)$

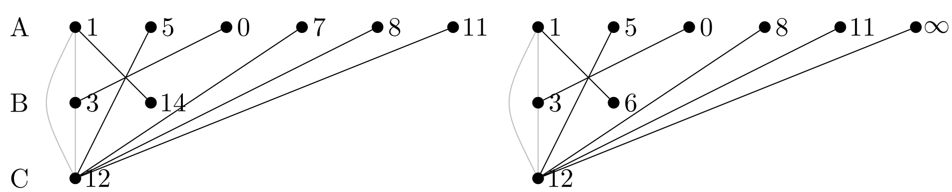

Figure 190. $G_9(3; 4, 1, 1)$

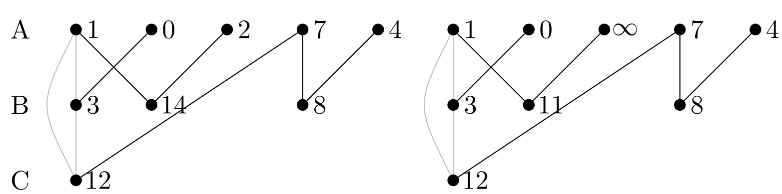

Figure 191. $G_1(3; 3, 2, 1)$

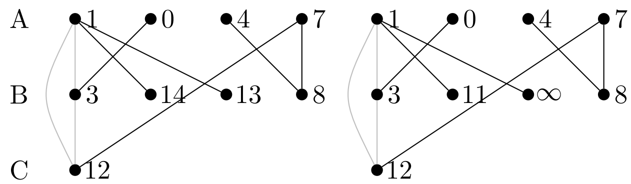

Figure 192. $G_2(3;3,2,1)$

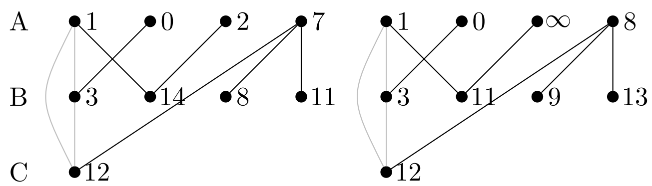

Figure 193. $G_3(3;3,2,1)$

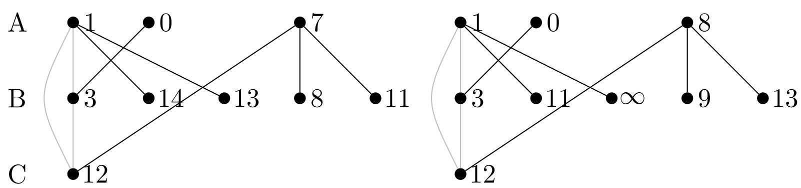

Figure 194. $G_4(3;3,2,1)$

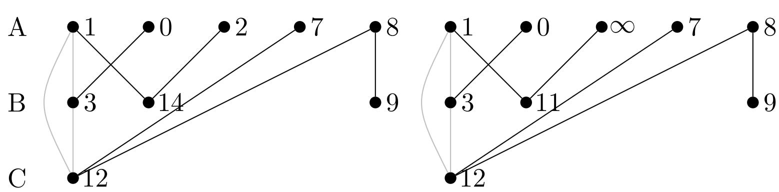

Figure 195. $G_5(3;3,2,1)$

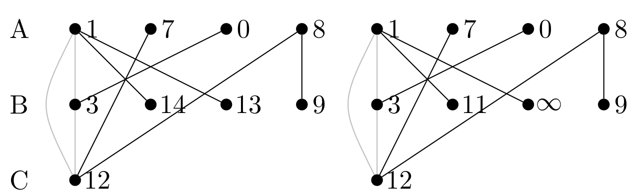

Figure 196. $G_6(3;3,2,1)$

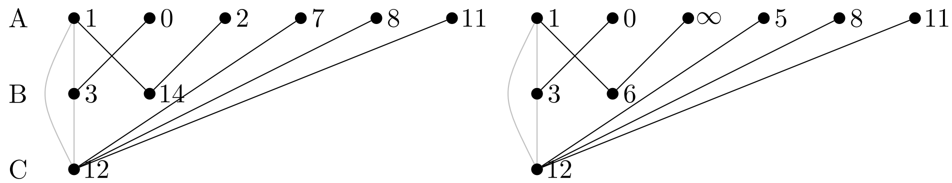

Figure 197. $G_7(3;3,2,1)$

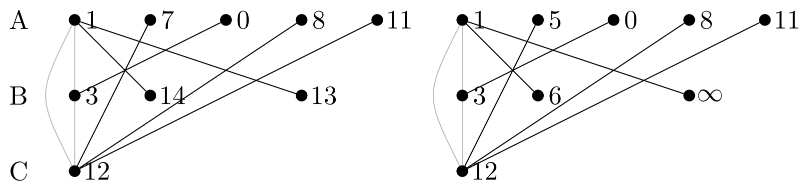

Figure 198. $G_8(3;3,2,1)$

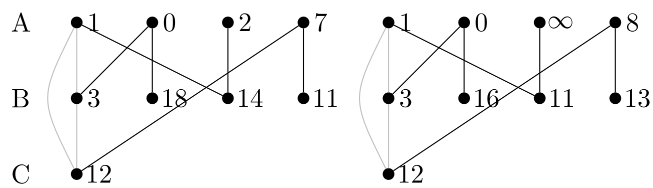

Figure 199. $G_1(3;2,2,2)$

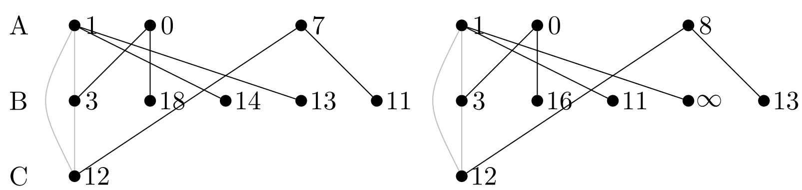

Figure 200. $G_2(3;2,2,2)$

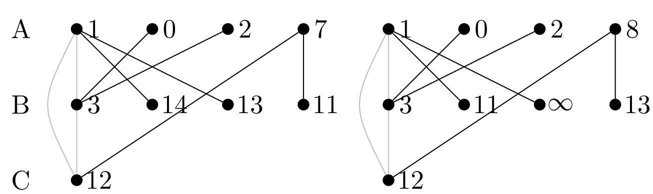

Figure 201. $G_3(3;2,2,2)$

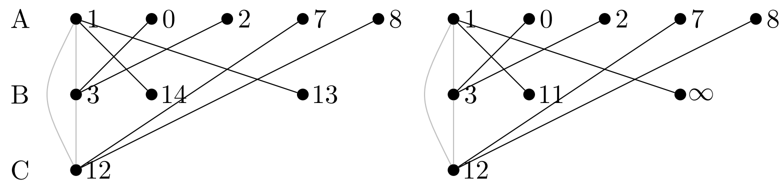

Figure 202. $G_4(3; 2, 2, 2)$

5.2.2. Graphs containing a 5-cycle

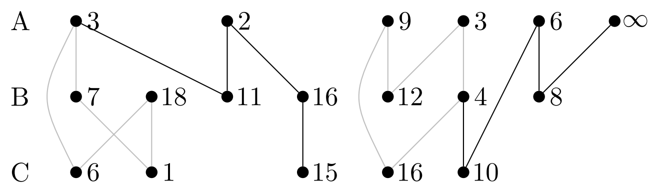

Figure 203. $G_1(5; 4, 0, 0, 0, 0)$

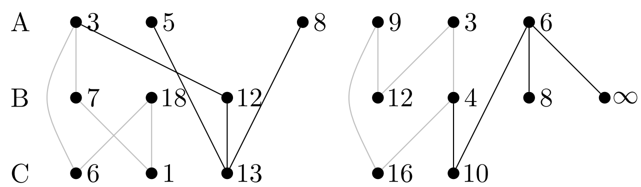

Figure 204. $G_2(5; 4, 0, 0, 0, 0)$

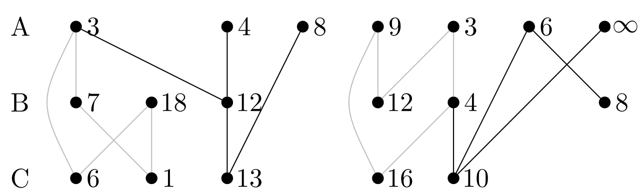

Figure 205. $G_3(5; 4, 0, 0, 0, 0)$

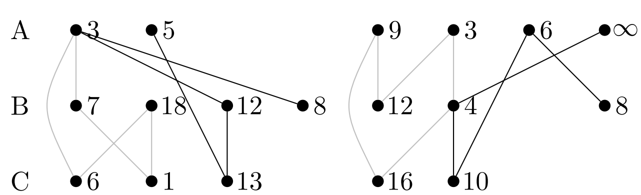

Figure 206. $G_4(5; 4, 0, 0, 0, 0)$

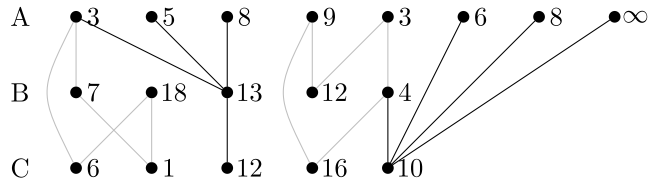

Figure 207. $G_5(5;4,0,0,0,0)$

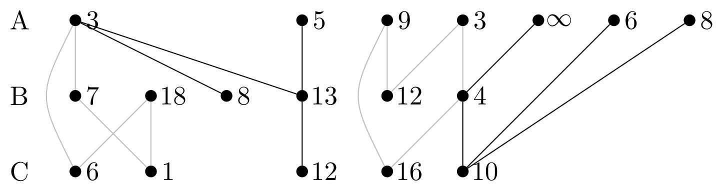

Figure 208. $G_6(5;4,0,0,0,0)$

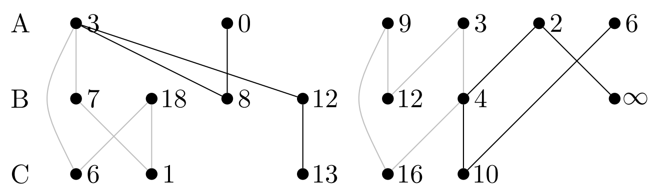

Figure 209. $G_7(5;4,0,0,0,0)$

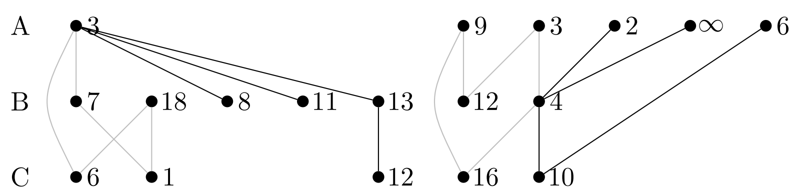

Figure 210. $G_8(5;4,0,0,0,0)$

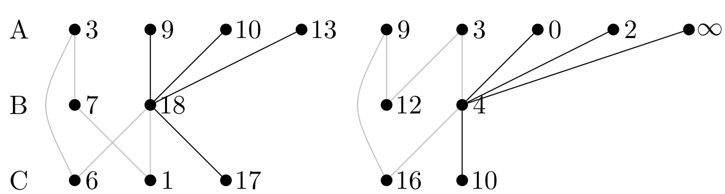

Figure 211. $G_9(5;4,0,0,0,0)$

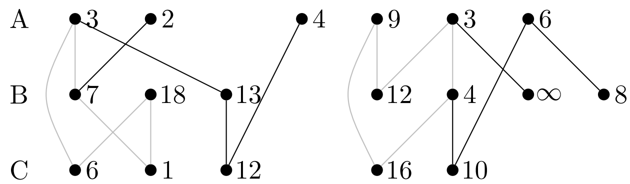

Figure 212. $G_1(5; 3, 1, 0, 0, 0)$

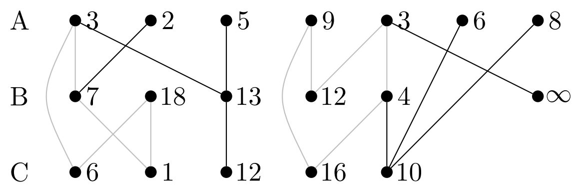

Figure 213. $G_2(5; 3, 1, 0, 0, 0)$

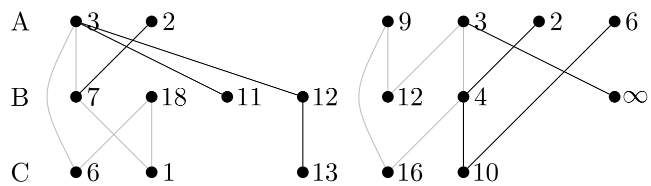

Figure 214. $G_3(5; 3, 1, 0, 0, 0)$

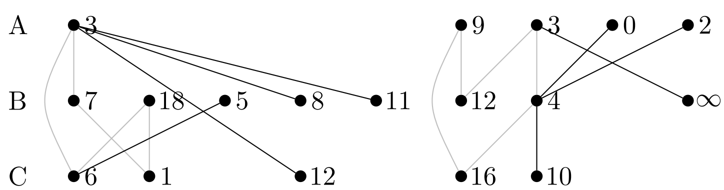

Figure 215. $G_4(5; 3, 1, 0, 0, 0)$

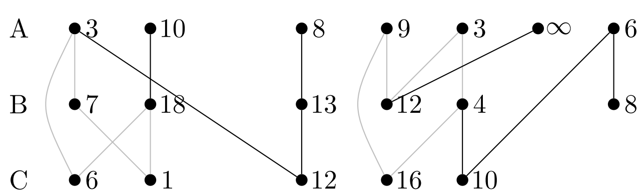

Figure 216. $G_1(5; 3, 0, 1, 0, 0)$

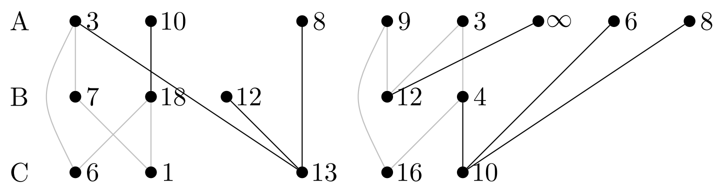

Figure 217. $G_2(5;3,0,1,0,0)$

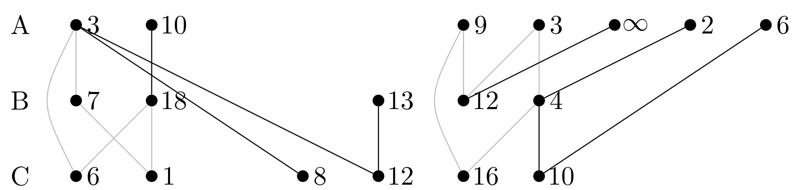

Figure 218. $G_3(5;3,0,1,0,0)$

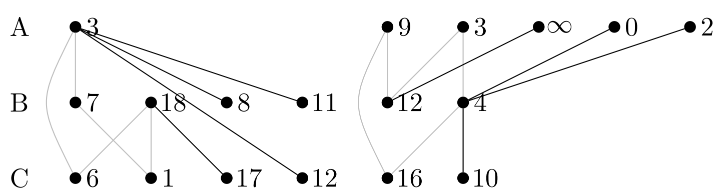

Figure 219. $G_4(5;3,0,1,0,0)$

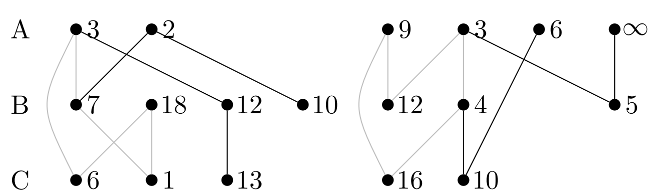

Figure 220. $G_1(5;2,2,0,0,0)$

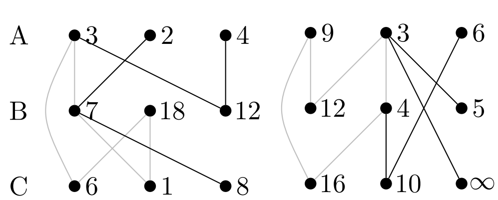

Figure 221. $G_2(5;2,2,0,0,0)$

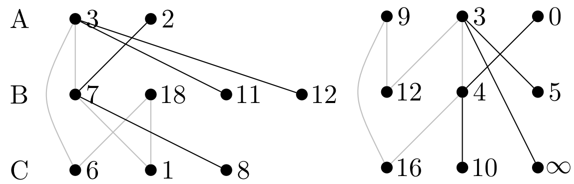

Figure 222. $G_3(5; 2, 2, 0, 0, 0)$

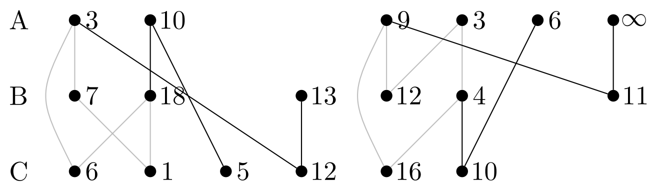

Figure 223. $G_1(5; 2, 0, 2, 0, 0)$

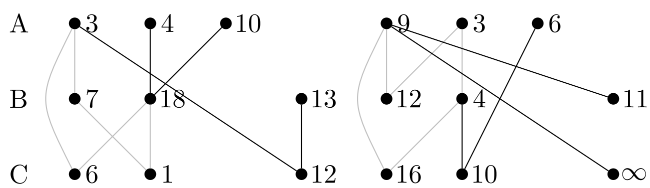

Figure 224. $G_2(5; 2, 0, 2, 0, 0)$

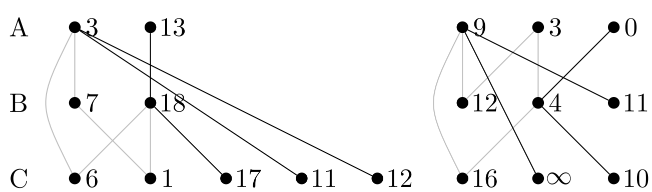

Figure 225. $G_3(5; 2, 0, 2, 0, 0)$

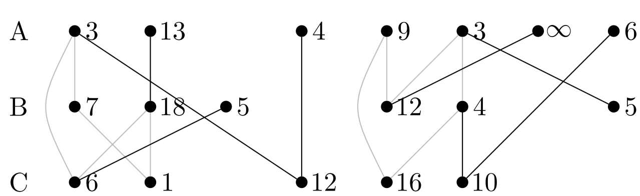

Figure 226. $G_1(5; 2, 1, 1, 0, 0)$

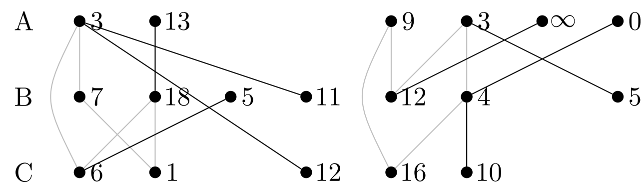

Figure 227. $G_2(5; 2, 1, 1, 0, 0)$

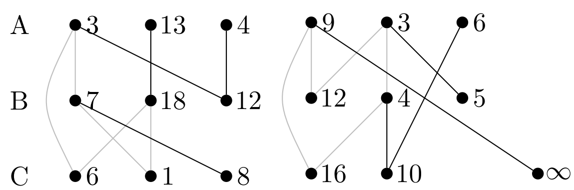

Figure 228. $G_1(5; 2, 1, 0, 1, 0)$

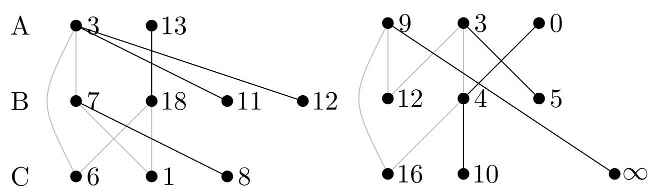

Figure 229. $G_2(5; 2, 1, 0, 1, 0)$

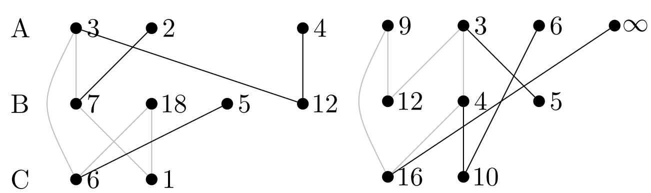

Figure 230. $G_1(5; 2, 1, 0, 0, 1)$

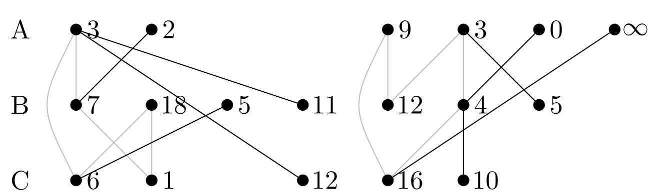

Figure 231. $G_2(5; 2, 1, 0, 0, 1)$

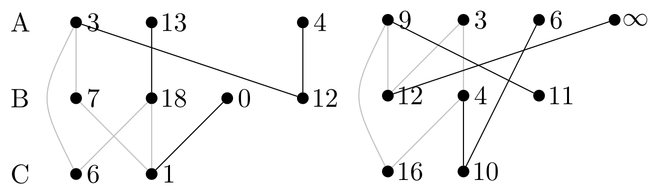

Figure 232. $G_1(5; 2, 0, 1, 1, 0)$

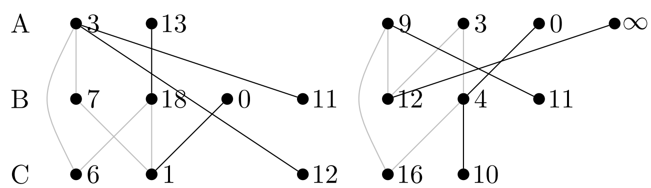

Figure 233. $G_2(5; 2, 0, 1, 1, 0)$

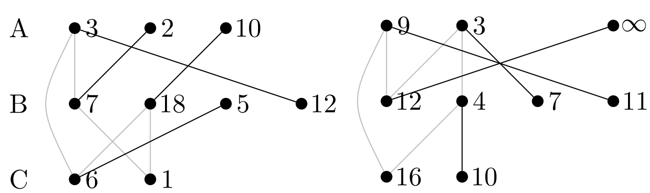

Figure 234. $G_1(5; 1, 1, 1, 1, 0)$

5.2.3. Graphs containing a 7-Cycle

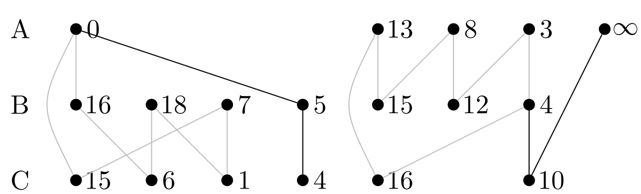

Figure 235. $G_1(7; 2, 0, 0, 0, 0, 0, 0)$

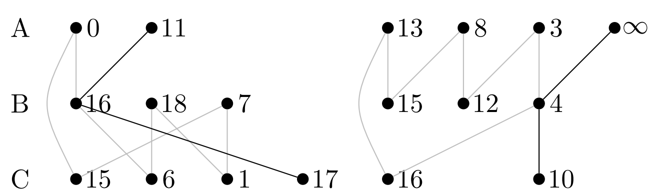

Figure 236. $G_2(7; 2, 0, 0, 0, 0, 0, 0)$

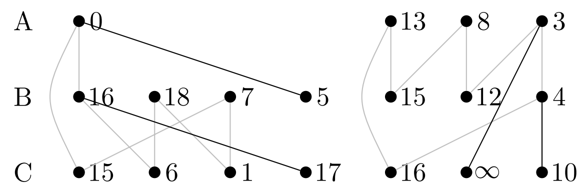

Figure 237. $G_1(7; 1, 1, 0, 0, 0, 0, 0)$

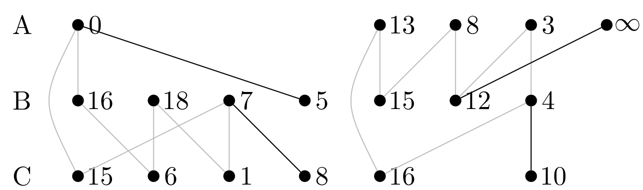

Figure 238. $G_1(7; 1, 0, 1, 0, 0, 0, 0)$

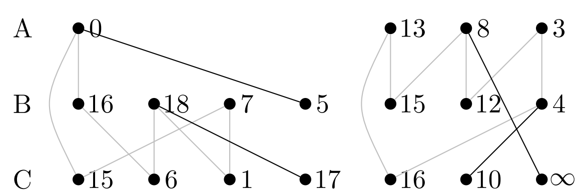

Figure 239. $G_1(7; 1, 0, 0, 1, 0, 0, 0)$

5.2.4. The graph C_9

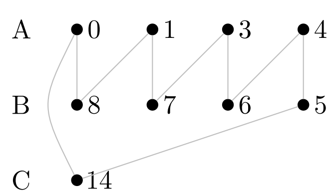

Figure 240. C_9