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On the construction of super edge-magic total graphs

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Abstract

Suppose $G = (V, E)$ be a simple graph with p vertices and q edges. An *edge-magic total labeling* of G is a bijection $f : V \cup E \to \{1, 2, \dots, p + q\}$ where there exists a constant r for every edge xy in G such that $f(x) + f(y) + f(xy) = r$. An edge-magic total labeling f is called a *super edge-magic total labeling* if for every vertex $v \in V(G)$, $f(v) \leq p$. The *super edge-magic total graph* is a graph which admits a super edge-magic total labeling. In this paper, we consider some families of super edge-magic total graph G. We construct several graphs from G by adding some vertices and edges such that the new graphs are also super edge-magic total graphs.

Keywords: edge-magic total labeling, super edge-magic total graph, super edge-magic total labeling
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1. Introduction

We assume that all graphs in this paper are simple and finite. Let $G = (V, E)$ be a graph with p vertices and q edges. Let f be a bijection function defined as $f : V \cup E \to \{1, 2, \dots, p+q\}$. Ringel and Llado [16] provided the definition that the function f is called an *edge-magic total labeling* if there exists a constant r for every edge xy in G such that the weight of the edge $f(x) + f(y) + f(xy) = r$. We can say the constant r as a *magic constant* of f. Wallis [18] then called a graph G admitting an edge-magic labeling as an *edge-magic total* graph.

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The edge-magic total concept was introduced by Kotzig and Rosa [10, 11]. They proved that complete bipartite graphs $K_{m,n}$ $(m, n \geq 1)$ and cycles C_n $(n \geq 3)$ are edge-magic total graphs. They also proved that a complete graph K_n is edge-magic total graph if and only if $n = 1, 2, 3, 4, 5$ or 6; and the disjoint union of n copies of P_2 has an edge-magic total labeling if and only if n is odd. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 8, 14, 16, 17].

In this paper, we consider an edge-magic total labeling of G where the p smallest labels are given to $V(G)$. Enomoto et al. [3] defined this version of edge-magic total labeling as a *super edge-magic total labeling*. If there exists a super edge-magic total labeling in a graph G, then G is called as a *super edge-magic total graph*.

Enomoto et al. [3] proved that caterpillars are super edge-magic total. They also determined that a complete graph K_n is super edge-magic total if and only if $n = 1, 2,$ or 3; and a complete bipartite graph $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$. Enomoto et al. also proved that odd cycles are super edge-magic total. Some other results on super edge-magic total graph can be seen in [4, 5, 6, 7, 8, 9, 15].

The following properties are useful to show whether a graph G is super edge-magic total or not. A graph $G = (V, E)$ is super edge-magic total graph if there exists a vertex labeling that causes a consecutive labeling.

Lemma 1.1. *[2, 6] A graph G is super edge-magic total if and only if there is a vertex labeling f such that $f(V(G))$ and $\{f(u) + f(v) \mid uv \in E(G)\}$ are both consecutive.*

In this case, in order to show that graph G is super edge-magic total graph, it is simply indicated by taking a bijection of vertex labeling $f : V \to \{1, 2, \dots, p\}$ where $\{f(u) + f(v) \mid uv \in E(G)\}$ is consecutive. The vertex labeling f can be extended to be a total labeling by defining $f(uv) = p + q + \min\{f(u) + f(v) \mid uv \in E(G)\} - f(u) - f(v)$ for every edge $uv \in E(G)$. So that, the total labeling f is a super edge-magic total labeling of G.

In this paper, we will construct some families of super edge-magic total graph which obtained from a known super edge-magic total graph. We obtain four results. First theorem is related to a path P_n. Lee and Lee [12] have provided a construction on a path P_2 such that a new graph is super edge-magic total. In this paper, we generalized such construction on a path P_{2n} $(n \geq 1)$.

The second result is related to disjoint union graph and joint product graphs. For graphs G and H, a *disjoint union* graph $G \cup H$ is a graph with vertex set $V(G) \cup V(H)$ and an edge set $E(G) \cup E(H)$. A *joint product* graph of G and H, denoted by $G + H$, is a graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G),\ v \in V(H)\}$. For any super edge-magic total graphs G, we construct a new graph from G by using disjoint union and joint product with some graphs, such that a new graph is also super edge-magic total.

For the third result, we define graph $G(+)P_m(+)H$ where $m \geq 2$ as a graph obtained by taking one copy of the graphs G and H and a path P_m, then connect an end point of P_m to all vertices of G and the other end point of P_m to all vertices of H. For any super edge-magic total graphs G, we provide some graphs H such that $G(+)P_m(+)H$ is also super edge-magic total. The last result is a construction of a super edge-magic total graph from a super edge-magic total graph by considering a super edge-magic labeling of the origin graph.

2. Main Results

In this section, we provide some constructions to obtain a new super edge-magic total graph which obtained from a super edge-magic total graph.

First, we consider a path P_n $(n \geq 2)$. López et al. [13] have proved that paths are super edge-magic total. Now, we define a graph $(P_n \cup hK_1)(+)2K_1$ $(h \geq 1)$, which is a graph obtained by taking one copy of a path P_n, h copies of K_1, and two isolated vertices $(2K_1)$, then connect all end points of P_n and all vertices of h copies of K_1 to both two vertices of $2K_1$. We can say that $V((P_n \cup hK_1)(+)2K_1) = V(P_n) \cup V(hK_1) \cup V(2K_1)$ and $E((P_n \cup hK_1)(+)2K_1) = E(P_n) \cup \{uv \mid u \in V(hK_1)$ or u is an end point of P_n; $v \in V(2K_1)\}$. In [12], Lee and Lee have proved that $(P_2 \cup hK_1)(+)2K_1$ $(h \geq 1)$ are super edge-magic total. In the following theorem, we generalize Lee and Lee construction on a path P_{2n} $(n \geq 1)$.

Theorem 2.1. *For integers $h, n \geq 1$, graphs $(P_{2n} \cup hK_1)(+)2K_1$ are super edge-magic total.*

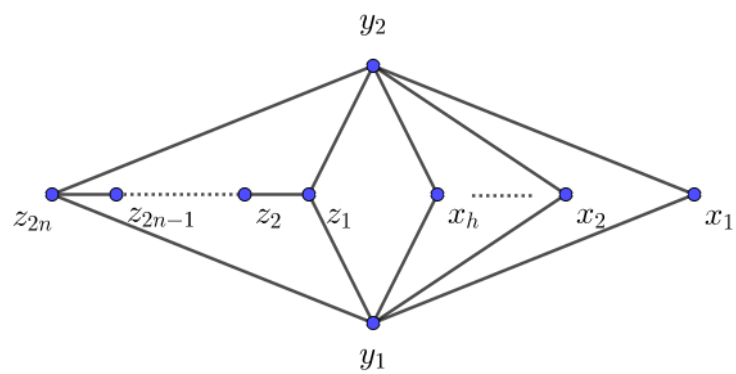

Figure 1. Graph $(P_{2n} \cup hK_1)(+)2K_1$.

Proof of Theorem 2.1. Let $V(hK_1) = \{x_i \mid 1 \leq i \leq h\}$, $V(2K_1) = \{y_1, y_2\}$, $V(P_{2n}) = \{z_i \mid 1 \leq i \leq 2n\}$, and $E(P_{2n}) = \{z_i z_{i+1} \mid 1 \leq i \leq 2n-1\}$. It is easy to verify that $(P_{2n} \cup hK_1)(+)2K_1$ has $2n + h + 2$ vertices and $2n + 2h + 3$ edges.

Now, we define a vertex labeling $f : V((P_{2n} \cup hK_1)(+)2K_1) \to \{1, 2, \ldots, 2n + h + 2\}$ where for $v \in V((P_{2n} \cup hK_1)(+)2K_1)$,

$$f(v) = \begin{cases} 1, & \text{if } v = y_1, \\ 2n + h + 2, & \text{if } v = y_2, \\ 1 + i, & \text{if } v = z_{2i} \text{ with } 1 \leq i \leq n, \\ n + 1 + h + i, & \text{if } v = z_{2i-1} \text{ with } 1 \leq i \leq n, \\ n + 1 + i, & \text{if } v = x_i \text{ with } 1 \leq i \leq h. \end{cases}$$

By the labeling above, we obtain that for $uv \in E((P_{2n} \cup hK_1)(+)2K_1)$:

- If $u = y_1$, since v is an end point of P_{2n} or an element of $V(hK_1)$ then $\{f(u) + f(v)\} = \{1 + f(v)\} = \{n + 2, n + 3, \ldots, n + h + 3\}$.

- If $u, v \in V(P_{2n})$, then $\{f(u)+f(v)\} = \{f(z_{2i-1})+f(z_{2i}) \mid 1 \leq i \leq n\} \cup \{f(z_{2i})+f(z_{2i+1}) \mid 1 \leq i \leq n-1\} = \{n+h+4, n+h+6, \ldots, 3n+h+2\} \cup \{n+h+5, n+h+7, \ldots, 3n+h+1\} = \{n+h+4, n+h+5, \ldots, 3n+h+2\}$.

- If $u = y_2$, since v is an end point of P_{2n} or an element of $V(hK_1)$ then $\{f(u)+f(v)\} = \{2n+h+2+f(v)\} = \{3n+h+3, 3n+h+4, \ldots, 3n+2h+4\}$.

Therefore, $\{f(u)+f(v) \mid uv \in E((P_{2n} \cup hK_1)(+)2K_1)\}$ is a consecutive sequence. By Lemma 1.1, the graph $(P_{2n} \cup hK_1)(+)2K_1$ is a super edge-magic total graph. □

Before we continue to the next constructions, we need to show the following property of a super edge-magic total labeling.

Lemma 2.1. *Let G be a connected graph with $m \geq 2$ vertices. Let f be a super edge-magic total labeling of G. Then $\max\{f(u)+f(v) \mid uv \in E(G)\} \geq m+1$.*

Proof. Suppose that $\max\{f(u)+f(v) \mid uv \in E(G)\} \leq m$. Since G is connected, a vertex u with $f(u) = m$ will be adjacent to another vertex v. So, $f(u)+f(v) = m+f(v) \geq m+1$, a contradiction. □

In the following theorem, we give a construction of a super edge-magic total graph obtained from any super edge-magic total graphs by applying disjoint union and joint product to an origin graph.

Theorem 2.2. *Let G_m be a connected graph with $m \geq 3$ vertices. Let f be a super edge-magic total labeling of G_m. If $k = \max\{f(u)+f(v) \mid uv \in E(G_m)\}$, then $(G_m \cup (k-m-1)K_1) + K_1$ is a super edge-magic total graph.*

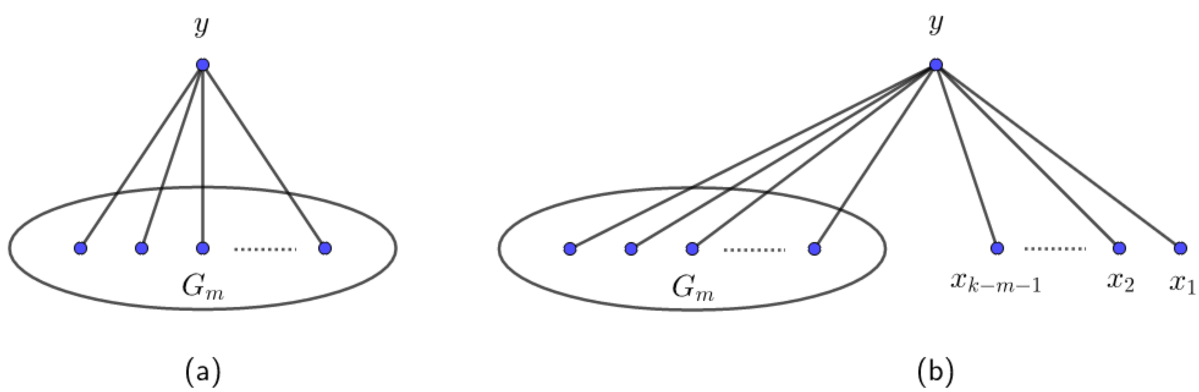

Figure 2. Graph $(G_m \cup (k-m-1)K_1) + K_1$ where: (a) $k = m+1$; (b) $k \geq m+2$.

Proof. Let $H = (G_m \cup (k-m-1)K_1) + K_1$. By considering Lemma 2.1, we obtain $k-m-1 \geq 0$. In case $k-m-1 = 0$, we have $H = (G_m \cup (k-m-1)K_1) + K_1 = G_m + K_1$. We define $V((k-m-1)K_1) = \{x_i \mid 1 \leq i \leq k-m-1\}$. Note that $(k-m-1)K_1$ is a graph without edges. Thus, we can say that $V(H) = V(G_m) \cup V((k-m-1)K_1) \cup \{y\}$. Meanwhile, $E(H) = E(G_m) \cup \{uy \mid u \in V(G_m \cup (k-m-1)K_1)\}$. It is easy to see that $|V(H)| = k$ and $|E(H)| = |E(G_m)| + k - 1$.

Let f be a super edge-magic labeling of G_m where $k = \max\{f(u) + f(v) \mid uv \in E(G_m)\}$. Note that for $v \in V(G_m)$, $f(v) \in \{1, 2, \ldots, m\}$. Now, we define a vertex labeling $g : V(H) \to \{1, 2, \ldots, k\}$ where for $v \in V(H)$,

$$g(v) = \begin{cases} f(v), & \text{if } v \in V(G_m), \\ k, & \text{if } v = y, \\ m+i, & \text{if } v = x_i \text{ with } 1 \leq i \leq k-m-1. \end{cases}$$

By the labeling above, we obtain that for $uv \in E(H)$:

- If $u, v \in V(G_m)$, since f is a super edge-magic labeling of G_m, then $\{g(u) + g(v)\} = \{f(u) + f(v)\}$ is a consecutive sequence, whose greatest element is k.
- if $u \in V(G_m)$ and $v = y$, then $\{g(u) + g(v)\} = \{g(u) + k\} = \{k+1, k+2, \ldots, k+m\}$.
- if $u \in V((k-m-1)K_1)$ and $v = y$, then $\{g(u) + g(v)\} = \{g(u) + k\} = \{k+m+1, k+m+2, \ldots, 2k-1\}$.

Therefore, $\{g(u) + g(v) \mid uv \in E(H)\}$ is a consecutive sequence. By Lemma 1.1, the graph H is a super edge-magic total graph. □

Now, let us consider the graph $G(+)P_m(+)H$ where $m \geq 2$. Let u and v be two end points of the path P_m. Then we can write $V(G(+)P_m(+)H) = V(G) \cup V(P_m) \cup V(H)$ and $E(G(+)P_m(+)H) = E(G) \cup E(P_m) \cup E(H) \cup \{ux, vy \mid x \in V(G); y \in V(H)\}$. Thus, $|V(G(+)P_m(+)H)| = |V(G)| + |V(P_m)| + |V(H)|$ and $|E(G(+)P_m(+)H)| = |E(G)| + |E(P_m)| + |E(H)| + |V(G)| + |V(H)|$.

Theorem 2.3. *Let G_m be a connected graph with $m \geq 3$ vertices. Let f be a super edge-magic total labeling of G_m and $m + k = \max\{f(u) + f(v) \mid uv \in E(G_m)\}$. Then for $k \geq 2$ and $n \geq 1$, the graph $G_m(+)P_{2k-2}(+)nK_1$ is a super edge-magic total graph.*

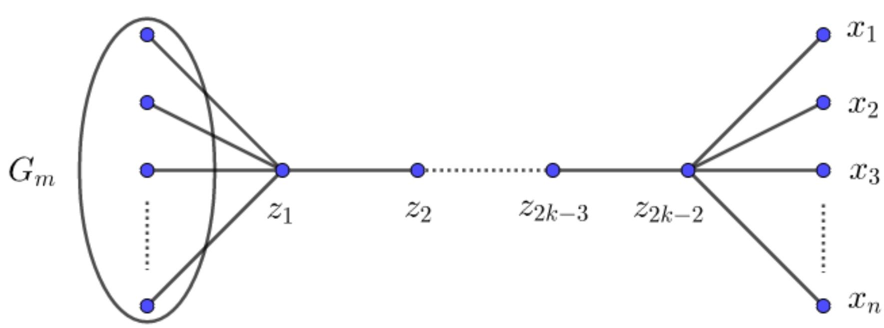

Figure 3. Graph $G_m(+)P_{2k-2}(+)nK_1$.

Proof of Theorem 2.3. Let $H = G_m(+)P_{2k-2}(+)nK_1$ where $n \geq 1$. It is easy to see that $|V(H)| = m+n+2k-2$ and $|E(H)| = |E(G_m)|+m+n+2k-3$. We define $V(nK_1) = \{x_i \mid 1 \leq i \leq n\}$. Note that nK_1 is a graph without edges. Let $V(P_{2k-2}) = \{z_i \mid 1 \leq i \leq 2k-2\}$ and $E(P_{2k-2}) = \{z_i z_{i+1} \mid 1 \leq i \leq 2k-3\}$. We assume that z_1 and z_{2k-2} is adjacent to all vertices of G_m and nK_1, respectively.

Let f be a super edge-magic labeling of G_m. By considering Lemma 2.1, we have $\max\{f(u)+f(v) \mid uv \in E(G_m)\} \geq m+1$. Now, we assume that $\max\{f(u)+f(v) \mid uv \in E(G_m)\} = m+k \geq m+2$. Note that for $v \in V(G_m)$, $f(v) \in \{1, 2, \ldots, m\}$. Define a vertex labeling $g : V(H) \to \{1, 2, \ldots, m+n+2k-2\}$ where for $v \in V(H)$,

$$g(v) = \begin{cases} f(v), & \text{if } v \in V(G_m), \\ m+i, & \text{if } v = z_{2i} \text{ where } 1 \leq i \leq k-1, \\ m+k+i, & \text{if } v = z_{2i+1} \text{ where } 0 \leq i \leq k-2, \\ m+2k-2+i, & \text{if } v = x_i \text{ with } 1 \leq i \leq n. \end{cases}$$

By the labeling above, we obtain that for $uv \in E(H)$:

- If $u, v \in V(G_m)$, since f is a super edge-magic labeling of G_m, then $\{g(u)+g(v)\} = \{f(u)+f(v)\}$ is a consecutive sequence, whose greatest element is $m+k$.
- if $u \in V(G_m)$ and $v = z_1$, then $\{g(u)+g(v)\} = \{g(u)+(m+k)\} = \{m+k+1, m+k+2, \ldots, 2m+k\}$.
- if $u, v \in P_{2k-2}$, then $\{g(u)+g(v)\} = \{2m+k+1, 2m+k+2, \ldots, 2m+3k-3\}$.
- if $u \in V(nK_1)$ and $v = z_{2k-2}$, then $\{g(u)+g(v)\} = \{g(u)+(m+k-1)\} = \{2m+3k-2, 2m+2k-1, \ldots, 2m+3k-3+n\}$.

Therefore, $\{g(u)+g(v) \mid uv \in E(H)\}$ is a consecutive sequence. By Lemma 1.1, the graph H is a super edge-magic total graph. □

In the last theorem below, we will construct a super edge-magic total graph from a super edge-magic total graph by considering a super edge-magic labeling of the origin graph.

Theorem 2.4. *Let G_m be a connected graph with $m \geq 3$ vertices. Let f be a super edge-magic total labeling of G_m. Let $F = \{f(u)+f(v) \mid uv \in E(G_m)\}$. For $ab \in E(G_m)$, let $f(a)+f(b) = \min(F)$ where $f(a) < f(b)$, $\max(F) = m+k$, and for $c \in V(G_m)$, $f(c) = k$.*

1. *For $f(a) = 1$, let G_m^* be a graph obtained by taking one copies of G_m and nK_1 where $n \geq 1$, then connect all vertices of nK_1 to b. Then G_m^* is a super edge-magic total graph.*
2. *Let G_m^{**} be a graph obtained by taking one copies of G_m and nK_1 where $n \geq 1$, then connect all vertices of nK_1 to c. Then G_m^{**} is a super edge-magic total graph.*

Proof. Let f be a super edge-magic labeling of G_m. Let $F = \{f(u)+f(v) \mid uv \in E(G_m)\}$. For $ab \in E(G_m)$, let $f(a)+f(b) = \min(F)$ where $f(a) < f(b)$. By considering Lemma 2.1, let $\max(F) = m+k$ and for $c \in V(G_m)$, $f(c) = k$.

We define $V(nK_1) = \{x_i \mid 1 \le i \le n\}$. Note that nK_1 is a graph without edges. Let $H \in \{G_m^*, G_m^{**}\}$. So, $V(H) = V(G_m) \cup V(nK_1)$. It is easy to see that $|V(H)| = m + n$. In the other hand, $E(G_m^*) = E(G_m) \cup \{bu \mid u \in V(nK_1)\}$ and $E(G_m^{**}) = E(G_m) \cup \{cu \mid u \in V(nK_1)\}$. Thus, we can verify that $|E(H)| = |E(G_m)| + n$. We distinguish two cases.

Case 1. $H = G_m^*$

So, $f(a) = 1$. Now, we define a vertex labeling $g : V(H) \to \{1, 2, \ldots, m + n\}$ where for $v \in V(H)$,

$$g(v) = \begin{cases} i, & \text{if } v = x_i, \\ f(v) + n, & \text{if } v \in V(G_m). \end{cases}$$

By the labeling above, we obtain that for $uv \in E(H)$:

- If $u \in V(nK_1)$ and $v = b$, then $\{g(u) + g(v)\} = \{g(u) + (f(b) + n)\} = \{f(b) + n + 1, f(b) + n + 2, \ldots, f(b) + 2n\}$.
- If $u, v \in V(G_m)$, since f is a super edge-magic labeling of G_m, then $\{g(u) + g(v)\} = \{(f(u) + n) + (f(v) + n)\} = \{f(u) + f(v) + 2n\}$ is a consecutive sequence, whose least element is $f(b) + 2n + 1$.

Therefore, $\{g(u) + g(v) \mid uv \in E(H)\}$ is a consecutive sequence. By Lemma 1.1, the graph H is a super edge-magic total graph.

Case 2. $H = G_m^{**}$

Now, we define a vertex labeling $h : V(H) \to \{1, 2, \ldots, m + n\}$ where for $v \in V(H)$,

$$h(v) = \begin{cases} f(v), & \text{if } v \in V(G_m), \\ m + i, & \text{if } v = x_i. \end{cases}$$

By the labeling above, we obtain that for $uv \in E(H)$:

- If $u, v \in V(G_m)$, since f is a super edge-magic labeling of G_m, then $\{g(u) + g(v)\} = \{f(u) + f(v)\}$ is a consecutive sequence, whose greatest element is $m + k$.
- If $u \in V(nK_1)$ and $v = c$, then $\{g(u) + g(v)\} = \{g(u) + k\} = \{m + k + 1, m + k + 2, \ldots, m + k + n\}$.

Therefore, $\{g(u) + g(v) \mid uv \in E(H)\}$ is a consecutive sequence. By Lemma 1.1, the graph H is a super edge-magic total graph. □

An illustration of graphs G_m^* and G_m^{**} of a super edge-magic total graph G_m with $m \ge 3$ vertices can seen in Figure 4 below. Let G_m be a super edge-magic total graph with $m \ge 3$ vertices where $V(G_m) = \{z_i \mid 1 \le i \le m\}$ and f be a super edge-magic labeling of G_m. Let $F = \{f(u) + f(v) \mid uv \in E(G_m)\}$. In figure below, we assume that $f(z_p) + f(z_q) = \min(F)$ where $f(z_p) < f(z_q)$. Thus, it is clear that $a = z_p$ and $b = z_q$. Let $\max(F) = m+k$ and $f(z_r) = k$. Therefore, we have $c = z_r$. Note that it is possible to have either vertex $c = a$ or $c = b$, where $k = 1$ or $k = f(z_q)$, respectively.

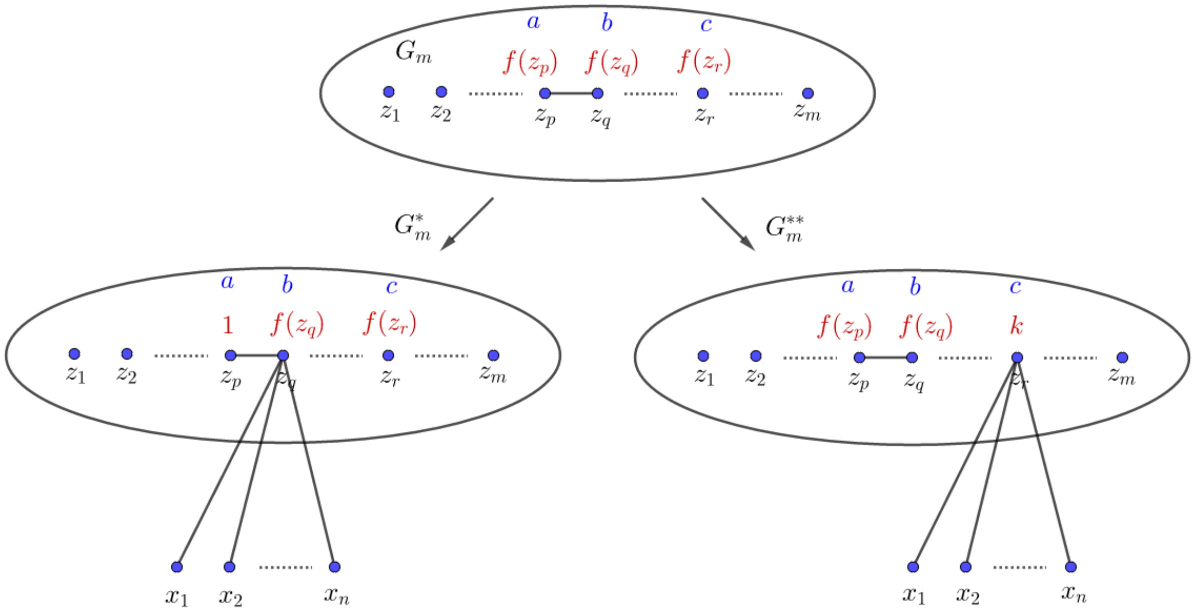

Figure 4. Graphs G_m^* (left) and G_m^{**} (right).

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