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Lower and upper bounds on independent double Roman domination in trees

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Abstract

For a graph $G = (V, E)$, a double Roman dominating function (DRDF) $f : V \to \{0, 1, 2, 3\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a neighbor $u \in N(v)$, with $f(u) = 3$, or at least two neighbors $x, y \in N(v)$ having $f(x) = f(y) = 2$, and every vertex with value 1 under f has at least a neighbor with value 2 or 3. The weight of a DRDF is the sum $f(V) = \sum_{v\in V} f(v)$. A DRDF f is an independent double Roman dominating function (IDRDF) if the vertices with weight at least two form an independent set. The independent double Roman domination number $i_{dR}(G)$ is the minimum weight of an IDRDF on G. In this paper, we show that for every tree T with diameter at least three, $i(T) + i_R(T) - \frac{s(T)}{2} + 1 \le i_{dR}(T) \le i(T) + i_R(T) + s(T) - 2$, where $i(T), i_R(T)$ and $s(T)$ are the independent domination number, the independent Roman domination number and the number of support vertex of T, respectively.

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1. Introduction

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood* is $N(v) \cup \{v\}$. The *degree* of a vertex v denoted by $\deg_G(v)$ is the cardinality of its open neighborhood. The *maximum degree* of a graph G is denoted by $\Delta = \Delta(G)$. A *leaf* of a tree T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves. We denote the set of leaves and support of G by $L(G)$ and $S(G)$, respectively. The *distance* between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Also, the depth of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* T_v at v is the subtree of T induced by $D[v]$. A *double star* $DS_{p,q}$ is a tree containing exactly two vertices that are not leaves, where one of which is adjacent to p leaves and the other is adjacent to q leaves. A *healthy spider* is a tree obtained from the star $K_{1,k}$ for $k \geq 2$ by subdividing each edge once, while a *wounded spider* $S_{k,t}$ is obtained from a star $K_{1,k}$ by subdividing t edges exactly once, where $1 \leq t \leq k - 1$.

A set $S \subseteq V$ is a *dominating set* of G if every vertex $V - S$ has a neighbor in S. The *independent domination number* $i(G)$ is the minimum cardinality of a set that is both independent and dominating.

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of an RDF f is $f(V(G)) = \sum_{u \in V(G)} f(u)$. Roman domination was introduced by Cockayne et al. in [14], and has been intensively studied in recent years [2, 3, 6, 11, 15, 19].

An *independent Roman dominating function* (IRDF) on G is an RDF such that the set $\{u \in V(G) \mid f(u) \geq 1\}$ is independent set. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an IRDF on G. The concept of independent Roman dominating function was first defined in [14] and studied by several authors, see [12, 13].

In [10], Beeler et al. introduced double Roman domination defined as follows. A *double Roman dominating function* (DRDF) on G is a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex v has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then vertex v has at least one neighbor w with $f(w) \geq 2$. The *double Roman domination number* $\gamma_{dR}(G)$ is the minimum weight of a DRDF on G. For a DRDF f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2, 3$. Since these four sets determine f, we can equivalently write $f = (V_0, V_1, V_2, V_3)$ (or $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ to refer f). We note that $\omega(f) = |V_1| + 2|V_2| + 3|V_3|$. Double Roman domination is studied for example in [1, 4, 5, 8, 9, 16, 18, 21, 22, 23], and elsewhere.

A DRDF $f = (V_0, V_1, V_2, V_3)$ is an *independent double Roman dominating function* (IDRDF) if $V_2 \cup V_3$ is an independent set. The *independent double Roman domination number* $i_{dR}(G)$ is the minimum weight of an IDRDF on G. Clearly, for all G we have the following,

$$\gamma_{dR}(G) \leq i_{dR}(G). \tag{1}$$

In this paper, we prove that for any tree T with diameter at least three,

$$i(T) + i_R(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i(T) + i_R(T) + s(T) - 2.$$

We make use of the following results in this paper.

Proposition A ([17]). *Let G be a graph. There exists an i_{dR}-function $f = (V_0, V_1, V_2, V_3)$ such that $V_1 = \emptyset$.*

By Proposition A, we assume no vertex needs to be assigned the value 1 for any $i_{dR}(G)$-function f.

Proposition B ([17]). *Let T be a tree of order $n \geq 3$. Then*

(i) *T has an $i_{dR}(T)$-function $f = (V_0, \emptyset, V_2, V_3)$ such that $L(T) \cap V_3 = \emptyset$.*

(ii) *For any IDRDF $f = (V_0, \emptyset, V_2, V_3)$ of T, $V_2 \cap S(T) = \emptyset$.*

Proposition C ([20]). *Let T be a tree of order at least three. Then*

(i) *T has an $i_R(T)$-function $f = (V_0, V_1, V_2)$ such that $L(T) \cap V_2 = \emptyset$.*

(ii) *For any IRDF $f = (V_0, V_1, V_2)$ of T, $V_1 \cap S(T) = \emptyset$.*

Proposition D. *Let G be a graph of order $n \geq 4$. Then $i_R(G) = 3$ if and only if (a) $\Delta(G) = n - 2$ or (b) $n = 3$ and $\Delta(G) \leq 1$.*

Proposition E ([7]). *For any graph G, $i(G) \leq i_R(G) \leq 2i(G)$, with equality in lower bound if and only if $G = \overline{K_n}$.*

The next result is easy to establish, and so we omit the proof.

Proposition 1.1. *For any graph G, $i_R(G) \leq i_{dR}(G)$.*

2. Trees

In this section, we present bounds on independent double Roman domination of a tree in terms of the sum its independent domination and independent Roman domination numbers. We start with the following lemmas.

Lemma 2.1. *Let r, s, t, ℓ be non-negative integers and let T be a tree and T' a subtree of T.*

1. *If $i_{dR}(T) \leq i_{dR}(T') + 3s + 2t - \ell$, $i_R(T') + 2s + t - \ell \leq i_R(T)$, $i(T') + s + t - r \leq i(T)$, $s(T') \leq s(T) - r$, and $i_{dR}(T') - i_R(T') - s(T') + 2 \leq i(T')$, then $i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T)$.*
2. *If $i_{dR}(T) \geq i_{dR}(T') + 3s + 2t - \ell$, $i_R(T') \geq i_R(T) - 2s - t + \ell$, $i(T') \geq i(T) - s - t - r$, $s(T') \leq s(T) - 2r$, and $i(T') \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} - 1$, then $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.*

Proof. (1) By the assumptions we have

$$\begin{aligned}
i(T) &\geq i(T') + s + t - r \\
&\geq i_{dR}(T') - i_R(T') - s(T') + 2 + s + t - r \\
&\geq (i_{dR}(T) - 3s - 2t + \ell) - (i_R(T) - 2s - t + \ell) - (s(T) - r) + 2 + s + t - r \\
&\geq i_{dR}(T) - i_R(T) - s(T) + 2.
\end{aligned}$$

(2) By the assumptions we obtain

$$\begin{aligned}
i(T) &\leq i(T') + s + t + r \\
&\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + s + t + r - 1 \\
&\leq (i_{dR}(T) - 3s - 2t + \ell) - (i_R(T) - 2s - t + \ell) + \frac{s(T) - 2r}{2} + s + t + r - 1 \\
&< i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1
\end{aligned}$$

□

Lemma 2.2. *Let T be a tree. Then*
(i) $i_{dR}(T) = i_R(T) + 1$ if and only if T is a star.
(ii) $i_{dR}(T) = i_R(T) + 2$ if and only if T is a wounded spider with only one foot or T is a tree obtained from a double star by subdividing its central edge once or twice.

Proof. (i) If T is a star, then clearly $i_{dR}(T) = 3$ and $i_R(T) = 2$ and we are done. Let $i_{dR}(T) = i_R(T) + 1$. We show that T is a star. Let $f = (V_0, \emptyset, V_2, V_3)$ be an i_{dR}-function of T such that $|V_3|$ is as large as possible. We consider two cases.

Case 1. $V_3 \neq \emptyset$.
Let $v \in V_3$. If $T = N_T[v]$, then T is a star and we are done. Suppose $T \neq N_T[v]$ and let $T' = T - N_T[v]$. Assume $T_1, T_2, \ldots, T_q$ $(q \geq 1)$ are the components of T'. Clearly, the function f, restricted to T' is an IDRDF of T' and hence

$$i_{dR}(T') = i_{dR}(T_1) + i_{dR}(T_2) + \cdots + i_{dR}(T_q) \leq i_{dR}(T) - 3. \tag{2}$$

On the other hand, any i_{dR}-function of T' can be extended to an IDRDF of T by assigning a 3 to v and a 0 to vertices in $N_T(v)$ and so

$$i_{dR}(T) \leq i_{dR}(T') + 3 = i_{dR}(T_1) + i_{dR}(T_2) + \ldots + i_{dR}(T_q) + 3. \tag{3}$$

By (2) and (3), we have

$$i_{dR}(T) = i_{dR}(T_1) + i_{dR}(T_2) + \ldots + i_{dR}(T_q) + 3. \tag{4}$$

Similarly, we have

$$i_R(T) = i_R(T_1) + i_R(T_2) + \ldots + i_R(T_q) + 2 \tag{5}$$

and

$$i(T) = i(T_1) + i(T_2) + \ldots + i(T_q) + 1 = i(T') + 1. \quad (6)$$

By (4), (5) and Proposition 1.1, we obtain $i_{dR}(T) - i_R(T) \geq \sum_{i=1}^{q}(i_{dR}(T_i) - i_R(T_i)) + 1 \geq q + 1$ which contradicts the assumption $i_{dR}(T) = i_R(T) + 1$.

Case 2. $V_3 = \emptyset$.
Then all leaves of T are assigned 2 under f. Since $V_3 = \emptyset$, $\mathrm{diam}(T) = 3$ is impossible. So, let $\mathrm{diam}(T) \geq 4$ and u, v be two leaves at distance $\mathrm{diam}(T)$, then the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(u) = g(v) = 1$ and $g(x) = f(x)$ for $x \in V(T) - \{u, v\}$, is an IRDF of T of weight at most $i_{dR}(T) - 2$ which is a contradiction. Therefore $\mathrm{diam}(T) \leq 2$ and so T is a star.

(ii) Let $i_{dR}(T) = i_R(T) + 2$. Assume that $f = (V_0, \emptyset, V_2, V_3)$ is an i_{dR}-function of T such that $|V_3|$ is as large as possible. First let $V_3 \neq \emptyset$. As above, we have

$$i_{dR}(T) - i_R(T) \geq \sum_{i=1}^{q}(i_{dR}(T_i) - i_R(T_i)) + 1 \geq q + 1.$$

We deduce from the assumption $i_{dR}(T) - i_R(T) = 2$ that $q = 1$ and $i_{dR}(T') - i_R(T') = 1$, that is T' is a star (by (i)). Using (6) we obtain

$$2 = i_{dR}(T) - i_R(T) = i_{dR}(T') - i_R(T') + 1 = i(T') + 1 = i(T).$$

It follows from Proposition E that $3 \leq i_R(T) \leq 4$. If $i_R(T) = 3$, then by Proposition D, we have $\Delta(G) = n - 2$ and so T is a wounded spider with only one foot. Assume that $i_R(T) = 4$. Then $i_R(T) = 2i(T)$ and using the constructive characterization given by Chellali and Jafari Rad [13] we can see that the only trees satisfying $i_{dR}(T) - i_R(T) = 2$ are trees obtained from a double star by subdividing its central edge once or twice. □

Theorem 2.1. *Let T be a tree with $s(T) \geq 2$ support vertices. Then*

$$i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2.$$

Proof. It is enough to prove $i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$. The proof is by induction on $t = i_{dR}(T) - i_R(T)$. Since T is not a star, we have $t > 1$ by Lemma 2.2 (item (i)). If $t = 2$, then the result holds by Lemma 2.2 (item (ii)). Assume that $t \geq 3$ and statement holds for each tree T' with $i_{dR}(T') - i_R(T') < t$. Let T be a tree with $t = i_{dR}(T) - i_R(T)$. It follows from Lemma 2.2 (item (i)) that $\mathrm{diam}(T) \geq 3$. If $\mathrm{diam}(T) = 3$, then $T = DS_{p,q}$ $(q \geq p \geq 1)$ and hence $i_{dR}(T) = 3 + 2p$, $i_R(T) = 2 + p$ and $i(T) = 1 + p$, and clearly the inequalities hold. Assume that $\mathrm{diam}(T) \geq 4$ and $v_1 v_2 \ldots v_k$ $(k \geq 5)$ is a diametral path in T such that $\deg(v_2)$ is as large as possible. We consider the following cases.

Case 1. $\deg(v_2) \geq 3$ and v_3 is not a support vertex and has a child a with depth 1 and degree 2
.
Let $v_3 a a'$ be a path in T and let $T' = T - \{a, a', v_1\}$. First we show that $i_{dR}(T) - 4 \leq i_{dR}(T') \leq i_{dR}(T) - 3$. To proved the left side, suppose that $f = (V_0, \emptyset, V_2, V_3)$ is an $i_{dR}(T')$-function such

that $V_3 \cap L(T') = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v_3) = 0$ and the function $g : V(T) \to \{0, 1, 2, 3\}$ define by $g(a) = 3$, $g(x) = 0$ for $x \in \{v_1, a'\}$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of T yielding $i_{dR}(T) \leq i_{dR}(T') + 3$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and the function $g : V(T) \to \{0, 1, 2, 3\}$ define by $g(v_1) = g(a') = 2$, $g(a) = 0$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of T and we have $i_{dR}(T) \leq i_{dR}(T') + 4$. To proved the right side, suppose that $f = (V_0, \emptyset, V_2, V_3)$ is an $i_{dR}(T)$-function such that $V_3 \cap L(T) = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v_3) = 0$ and $f(a) + f(a') = 3$ and the function f restricted to T' is an IDRDF of T and we have $i_{dR}(T) \geq i_{dR}(T') + 3$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and $f(v_1) = f(a') = 2$ and the function f restricted to T' is an IDRDF of T and we have $i_{dR}(T) \geq i_{dR}(T') + 4$.

Using Proposition C and a similar argument we can see that $i_R(T') = i_R(T) - 2$. Now we show that $i(T) = i(T') + 1$. To show $i(T') + 1 \geq i(T)$, let S be an $i(T')$-set. If $v_3 \notin S$, then we may assume $v_2 \in S$ and clearly $S \cup \{a'\}$ is an IDS of T and so $i(T) \leq i(T') + 1$. Assume that $v_3 \in S$. If $N_{T'}(v_4) \cap S \neq \{v_3\}$, then $(S - N_{T'}(v_2)) \cup \{v_2\}$ is an independent dominating set of T' smaller than S which is a contradiction. Hence, $N_{T'}(v_4) \cap S = \{v_3\}$. Now $(S - N_{T'}(v_2)) \cup \{v_2, v_4, a\}$ is an independent dominating set of T which implies that $i(T) \leq i(T') + 1$. To prove $i(T) \geq i(T') + 1$, let S be an $i(T)$-set. Clearly $|S \cap \{a, a'\}| = 1$ and either $v_2 \in S$ or $L_{v_2} \subseteq S$. In both cases, $(S - (\{a, a'\} \cup L_{v_3})) \cup \{v_2\}$ is an IDS of T' and so $i(T) \geq i(T') + 1$. Thus $i(T) = i(T') + 1$. Therefore

$$i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 3 - (i_R(T) - 2) = i_{dR}(T) - i_R(T) - 1 \leq t - 1.$$

Using the induction hypothesis on T' and setting $s = t = r = \ell = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and using the induction hypothesis on T' and setting $s = 1, t = r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Case 2. $\deg(v_2) \geq 3$ and v_3 is not a support vertex and any child of v_3 has degree at least 3 . Let $T' = T - T_{v_3}$. Clearly, $s(T') \leq s(T)$ and any $i_{dR}(T')$-function (resp. $i_R(T')$-function) can be extended to an IDRDF (resp. IRDF) of T by assigning a 3 (resp. a 2) to each child of v_3 and a 0 to remaining vertices and hence $i_{dR}(T) \leq i_{dR}(T') + 3|C(v_3)|$ and $i_R(T) \leq i_R(T') + 2|C(v_3)|$. Likewise we have $i(T) \leq i(T') + |C(v_3)|$. Now we show that $i_{dR}(T) \geq i_{dR}(T') + 3|C(v_3)|$. Let f be an $i_{dR}(T')$-function. By Proposition B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v_3) = 0$ and f must assign a 3 to each child of v_3 and the function f restricted to T' is an IDRDF of T' implying that $i_{dR}(T) \geq i_{dR}(T') + 3|C(v_3)|$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and f assigns 2 to each leaf of T_{v_3}. If $N(v_4) \cap ((V_2 \cup V_3) - \{v_3\}) \neq \emptyset$ and $z \in N(v_4) \cap ((V_2 \cup V_3) - \{v_3\})$, then the function $g : V(T') \to \{0, 1, 2, 3\}$ defined by $g(z) = 3$ and $g(x) = f(x)$ otherwise, is an IDRDF of T' implying that $i_{dR}(T) \geq i_{dR}(T') + 1 + 4|C(v_3)|$ and if $N(v_4) \cap ((V_2 \cup V_3) - \{v_3\}) = \emptyset$, then the function $g : V(T') \to \{0, 1, 2, 3\}$ defined by $g(v_4) = 3$ and $g(x) = f(x)$ otherwise, is an IDRDF of T' yielding $i_{dR}(T) \geq i_{dR}(T') + 4|C(v_3)|$. Thus $i_{dR}(T) = i_{dR}(T') + 3|C(v_3)|$. Similarly we can see that $i_R(T) = i_R(T') + 2|C(v_3)|$ and $i(T) = i(T') + |C(v_3)|$. It follows that

$$i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 3|C(v_3)| - i_R(T) + 2|C(v_3)| = i_{dR}(T) - i_R(T) - |C(v_3)| \leq t - 1.$$

Applying the induction hypothesis on T' and setting $s = 1$ and $t = r = \ell = 0$, Proposition 2.1 leads to $i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Case 3. $\deg(v_2) \geq 3$ and v_3 is a support vertex.
Let $v' \in L_{v_3}$. We distinguish the following subcases.

Subcase 3.1. $|L_{v_3}| \geq 2$.
Let $T' = T - \{v_1, v'\}$. Obviously $s(T) = s(T')$. Now we show that $i_{dR}(T') = i_{dR}(T) - 2$. Let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T')$-function such that $L(T') \cap V_3 = \emptyset$. By Proposition B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$ then f can be extended to an IDRDF of T by assigning a 2 to v' and a 0 to v_1, and if $f(v_2) = 0$ then to double Roman dominate v_2 and the leaf adjacent to v_2 and nothing that f is a $i_{dR}(T')$-function, we must have $f(v_3) = 3$, and f can be extended to an IDRDF of T by assigning a 2 to v_1 and a 0 to v', and hence $i_{dR}(T) \leq i_{dR}(T') + 2$. To prove the inverse inequality, let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function such that $L(T) \cap V_3 = \emptyset$. As above $f(v_2) = 3$ and $f(v_3) = 0$ or $f(v_2) = 0$ and $f(v_3) = 3$. In each case, the function f restricted to T' is an IDRDF of T' of weight $i_{dR}(T) - 2$ and so $i_{dR}(T) \geq i_{dR}(T') + 2$. Thus $i_{dR}(T) = i_{dR}(T') + 2$. Similarly, we can verify that $i_R(T) = i_R(T') + 1$ and $i(T) = i(T') + 1$. It follows that $i_{dR}(T') - i_R(T') = i_{dR}(T) - 2 - i_R(T) + 1 = i_{dR}(T) - i_R(T) - 1 = t - 1$. Applying the induction hypothesis on T' and setting $t = 1$ and $s = r = \ell = 0$, Proposition 2.1 leads to $i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Subcase 3.2. $|L_{v_3}| = 1$.
Let $T' = T - \{v_1, v'\}$. Obviously, $s(T') = s(T) - 1$ and as above we can see that $i_{dR}(T') \leq i_{dR}(T) - 2$, $i_R(T') \leq i_R(T) - 1$ and $i(T') = i(T) - 1$. Next we show that $i_{dR}(T) \leq i_{dR}(T') + 3$. Suppose that $f = (V_0, \emptyset, V_2, V_3)$ is an $i_{dR}(T')$-function such that $V_3 \cap L(T') = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then as in Subcace 3.1, we can see that $i_{dR}(T) \leq i_{dR}(T') + 2$. If $f(v_2) = 0$, then $f(v_3) \geq 2$ and the function $g : V(T) \to \{0, 1, 2, 3\}$ define by $g(v_1) = 2$, $g(v') = 0$, $g(v_3) = 3$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of T and so $i_{dR}(T) \leq i_{dR}(T') + 3$. Hence $i_{dR}(T') + 2 \leq i_{dR}(T) \leq i_{dR}(T') + 3$.

Likewise, we can see that $i_R(T) \leq i_R(T') + 1$ and so $i_R(T) = i_R(T') + 1$. Hence

$$i_{dR}(T') - i_R(T') = i_{dR}(T) - 2 - i_R(T) + 1 = i_{dR}(T) - i_R(T) - 1 \leq t - 1.$$

Using the induction hypothesis on T' and setting $s = 0, t = 2, r = \ell = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and using the induction hypothesis on T' and setting $t = 1, s = r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Considering Cases 1, 2, and 3 we may assume that $\deg(v_2) = 2$ and by the choice of diametral path any child of v_3 whit depth one will be of degree two. We proceed with further cases.

Case 4. $\deg(v_2) = 2$.
Let $T' = T - T_{v_3}$. Clearly $s(T') \leq s(T) - 1$ and any $i_{dR}(T')$-function (resp. $i_R(T')$-function) can be extended to an IDRDF of T by assigning a 3 (resp. a 2) to v_2 and a 0 to remaining vertices and so $i_{dR}(T) \leq i_{dR}(T') + 3$ and $i_R(T) \leq i_R(T') + 2$. Also any $i(T')$-set can be extended to an IDS of T by adding v_2 and so $i(T) \leq i(T') + 1$. Now let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function. By Proposition B we have $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then the function f restricted to T' is an IDRDF of T' yielding $i_{dR}(T) \geq i_{dR}(T') + 3$. Assume that $f(v_2) = 0$. Then $f(v_1) = 2$ and $f(v_3) \geq 2$. If $f(v_3) = 3$, then clearly $(N(v_4) - \{v_3\}) \cap (V_2 \cup V_3) = \emptyset$ and the function $g : V(T') \to \{0, 1, 2, 3\}$ defined by $g(v_4) = 2$ and $g(x) = f(x)$ is an IDRDF of T' yielding $i_{dR}(T) \geq i_{dR}(T') + 3$, and if $f(v_3) = 2$, then clearly $(N(v_4) - \{v_3\}) \cap (V_2 \cup V_3) \neq \emptyset$ and the

function $g : V(T') \to \{0,1,2,3\}$ defined by $g(z) = 3$ for some $z \in (N(v_4) - \{v_3\}) \cap (V_2 \cup V_3)$ and $g(x) = f(x)$ is an IDRDF of T' implying that $i_{dR}(T) \geq i_{dR}(T') + 3$. Hence $i_{dR}(T) \geq i_{dR}(T') + 3$ and thus $i_{dR}(T) = i_{dR}(T') + 3$. Likewise we have $i_R(T) = i_R(T') + 2$ and $i(T) = i(T') + 1$. Hence $i_{dR}(T') - i_R(T') = t - 1$.

Applying the induction hypothesis on T' and setting $s = 1$ and $t = r = \ell = 0$, Proposition 2.1 leads to $i_{dR}(T) - i_R(T) - s(T) + 2 \leq i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Case 5. v_3 is a support vertex and v_3 has two children a and b with depth 1 and degree 2. Suppose v_3aa' and v_3bb' are paths in T. Let $T' = T - \{a, a', b'\}$. It is easy to verify that $s(T') = s(T) - 2$, $i_{dR}(T') = i_{dR}(T) - 4$, $i_R(T') + 2 \leq i_R(T) \leq i_R(T') + 3$ and $i(T') + 1 \leq i(T) \leq i(T') + 2$. Hence $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 4 - i_R(T) + 2 = i_{dR}(T) - i_R(T) - 2 \leq t - 1$.

Using the induction hypothesis on T' and setting $s = \ell = 0, t = 2, r = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and using the induction hypothesis on T' and setting $s = 1, t = r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Case 6. v_3 is a support vertex and v_3 has exactly one child with depth 1 and degree 2. First let $\deg(v_4) = 2$. Suppose $T' = T - T_{v_4}$. If T' is a star, then the result can be seen easily. Let T' is not a star. Clearly $s(T') \leq s(T) - 1$ and as above we can see that $i_{dR}(T) = i_{dR}(T') + 5$, $i_R(T) = i_R(T') + 3$, $i(T) = i(T') + 2$. Hence $i_{dR}(T') - i_R(T') = t - 1$. Using the induction hypothesis on T' and setting $s = t = r = 1, t = 0$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and using the induction hypothesis on T' and setting $s = t = 1, r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Assume now that $\deg(v_4) \geq 3$ and $v' \in L_{v_3}$. Consider the following subcases.

Subcase 6.1. v_4 has a child a with depth 1 and degree 2. Suppose v_4aa' is a path in T and let $T' = T - \{v_1, v_2, a, a'\}$. Clearly, $s(T) = s(T') - 2$ and it is easy to verify that $i_{dR}(T) = i_{dR}(T') + 5$, $i_R(T) = i_R(T') + 3$, $i(T) = i(T') + 2$. Hence $i_{dR}(T') - i_R(T') \leq t - 1$ and using the induction hypothesis on T' and setting $s = t = 1, r = t = 0$, Proposition 2.1 leads to $i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2$.

Subcase 6.2. v_4 is a strong support vertex. First let $|L_{v_3}| \geq 2$. Suppose that $w \in L_{v_4}$. Suppose that $T' = T - \{v', w\}$. Clearly, $s(T) = s(T')$ and one can easily see that $i_{dR}(T') = i_{dR}(T) + 2$, $i_R(T) = i_R(T') + 1$, $i(T) = i(T') + 1$. Hence

$i_{dR}(T') - i_R(T') \leq t - 1$ and using the induction hypothesis on T' and setting $t = 1, s = r = \ell = 0$, Proposition 2.1 leads to $i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2$.

Now, let $|L_{v_3}| = 1$. Assume that $T' = T - T_{v_3}$. Clearly $s(T') = s(T) - 2$ and any $i_{dR}(T')$-function (resp. $i_R(T')$-function) can be extended to an IDRDF of T by assigning a 3 (resp. a 2) to v_3, a 2 (resp. a 1) to v_1 and a 0 to remaining vertices and so $i_{dR}(T) \leq i_{dR}(T') + 5$ and $i_R(T) \leq i_R(T') + 2$. Also any $i(T')$-set can be extended to an IDS of T by adding v_2, v' and so $i(T) \leq i(T') + 2$. Now let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function such that $L(T) \cap V_3 = \emptyset$. By Proposition B, we have $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then $f(v') = 2$ and the function f restricted to T' is an IDRDF of T' yielding $i_{dR}(T) \geq i_{dR}(T') + 5$. Assume that $f(v_2) = 0$. Then $f(v_1) = 2$ and $f(v_3) = 3$ since v_3 is a support vertex and so $f(x) = 2$ for each $x \in L_{v_4}$. Hence the function f restricted to T' is an IDRDF of T' yielding $i_{dR}(T) \geq i_{dR}(T') + 5$. Thus $i_{dR}(T) = i_{dR}(T') + 5$. Likewise we have $i_R(T) = i_R(T') + 3$ and $i(T) = i(T') + 2$. It follows that $i_{dR}(T') - i_R(T') = t - 1$ and using the induction hypothesis on T' and setting $s = t = 1, r = \ell = 0$, Proposition 2.1 leads to $i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2$.

Subcase 6.3. v_4 is adjacent to at most one leaf, any child of v_4 with depth 1 is of degree at least 3 and for any child y of v_4 with depth 2 we have $T_y = DS_{1,\deg(y)-1}$ where $\deg(y) \geq 3$ or T_y is a healthy spider. We consider the following.

- $|L_{v_3}| = 1$.
  Let $T' = T - T_{v_3}$. Clearly, $s(T') = s(T) - 2$, $i_{dR}(T') + 4 \leq i_{dR}(T) \leq i_{dR}(T') + 5$, $i_R(T') + 2 \leq i_R(T) \leq i_R(T') + 3$ and $i(T') + 1 \leq i(T) \leq i(T') + 2$.

  It follows that $i_{dR}(T') - i_R(T') \leq t - 1$ and using the induction hypothesis on T' and setting $t = 3, \ell = 1, r = 2, s = 0$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and using the induction hypothesis on T' and setting $s = t = 1, r = \ell = 0$, Proposition 2.1 leads to $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

$$\begin{aligned} i(T) &\leq i(T') + 2 \\ &\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 1 \\ &\leq i_{dR}(T) - 4 - i_R(T) + 3 + \frac{s(T) - 2}{2} + 1 \\ &= i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1. \end{aligned}$$

- $|L(v_3)| \geq 2$
  Let $T' = T - T_{v_4}$. If T' is a star, then the result is immediate. Assume T' is not a star. Suppose that A is the set of children of v_4 of depth 1, B is the set of children of v_4 of depth 2 and C is the set of vertices $x \in D(v_4) \cap L(T)$ satisfying $d(v_4, x) = 3$. Let $B_1 = B \cap s(T)$ and $B_2 = B - B_1$. Clearly, $s(T') \leq s(T) - 2$, and it is not hard to see that $i_{dR}(T') = i_{dR}(T) - 3|A| - 3|B_1| - 2|B_2| - 2|C| - 2|L_{v_4}|$, $i_R(T') = i_R(T) - 2|A| - 2|B_1| - 2|B_2| - |C| - |L_{v_4}|$ and $i(T') = i(T) - |A| - |B_1| - |C| - |L_{v_4}|$. Hence

$$\begin{aligned} i_{dR}(T') - i_R(T') &\leq i_{dR}(T) - 3|A| - 3|B_1| - 2|B_2| - 2|C| - 2|L_{v_4}| \\ &\qquad -(i_R(T) - 2|A| - 2|B_1| - 2|B_2| - |C| - |L_{v_4}|) \\ &= i_{dR}(T) - i_R(T) - (|A| + |B| + |C| + |L_{v_4}|) \leq t - 1. \end{aligned}$$

  By the induction hypothesis we have

$$\begin{aligned} i(T) &= i(T') + |A| + |B_1| + |C| + |L_{v_4}| \\ &\geq i_{dR}(T') - i_R(T') - s(T') + 2 + |A| + |B_1| + |C| + |L_{v_4}| \\ &> i_{dR}(T) - i_R(T) - s(T) + 2, \end{aligned}$$

and

$$\begin{aligned} i(T) &= i(T') + |A| + |B_1| + |C| + |L_{v_4}| \\ &\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + |A| + |B_1| + |C| + |L_{v_4}| - 1 \\ &< i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1. \end{aligned}$$

Case 7. $\deg(v_3) \geq 3$ and v_3 is not a support vertex.
Then T_{v_3} is a healthy spider and by that choice of diametral path and considering above cases we may assume that the maximal subtree at any child of v_4 with depth two is a healthy spider with at least two feet. We distinguish the following situations.

Subcase 7.1. $\deg(v_3) \geq 4$.
First let $\deg(v_4) = 2$ and let $T' = T - T_{v_4}$. If T' is a star then the results can be verified easily. Let t' is not a star. Clearly, $s(T') \leq s(T) - 2$, $i_{dR}(T') + 2 + 2|C(v_3)| \leq i_{dR}(T) \leq i_{dR}(T') + 3 + 2|C(v_3)|$, $i_R(T) = i_R(T') + 2 + |C(v_3)|$ and $i(T') + |C(v_3)| \leq i(T) \leq i(T') + |C(v_3)| + 1$. Hence

$$i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 2|C(v_3)| - 2 - i_R(T) + 2 + |C(v_3)| = i_{dR}(T) - i_R(T) - |C(v_3)| \leq t - 1,$$

and by the induction hypothesis on T' and setting $t = |C(v_3)|, \ell = 0, r = 1, s = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$. On the other hand, by the induction hypothesis on T', we obtain

$$\begin{aligned} i(T) &\leq i(T') + |C(v_3)| + 1 \\ &\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + |C(v_3)| \\ &\leq i_{dR}(T) - 2 - 2|C(v_3)| - i_R(T) + 2 + |C(v_3)| + \frac{(s(T) - 2)}{2} + |C(v_3)| \\ &= i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1 \end{aligned}$$

Now let $\deg(v_4) \geq 3$. Considering above cases and subcases, we may assume that any child of v_4 with depth 2, is the center of a healthy spider. Assume $a, b \in C(v_3) - \{v_2\}$ and let v_3aa' and v_3bb' be paths in T. We distinguish the following.

- v_4 has a child w with depth 1 and degree 2.
  Suppose v_4ww' is a path in T. Let $T' = T - \{v_1, a, a', w, w'\}$. Obviously, $s(T') = s(T) - 2$. We show that $i_{dR}(T) = i_{dR}(T') + 6$. To prove $i_{dR}(T) \leq i_{dR}(T') + 6$, let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T')$-function such that $L(T) \cap V_3 = \emptyset$. By Lemma B, $f(v_3) = 3$ or $f(v_3) = 0$. If $f(v_3) = 3$, then $f(v_4) = f(v_2) = 0$ and the function $g : V(T) \rightarrow \{0, 1, 2, 3\}$ define by $g(w) = 3, g(v_1) = g(v_3) = g(a') = 2$, $g(a) = g(w') = 0$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of T, and so $i_{dR}(T) \leq i_{dR}(T') + 6$. If $f(v_3) = 0$, then $f(v_2) = 2$

and $f(v_4) \geq 2$ and the function $g : V(T) \to \{0,1,2,3\}$ define by $g(a) = 3, g(w') = 2$, $g(a') = g(w) = g(v_1) = 0$, $g(v_2) = 3$ and $g(x) = f(x)$ for $x \in V(T')$, is an IDRDF of T, and we have $i_{dR}(T) \leq i_{dR}(T') + 6$. To prove $i_{dR}(T) \geq i_{dR}(T') + 6$, let $f = (V_0, \emptyset, V_2, V_3)$ be an $i_{dR}(T)$-function such that $L(T') \cap V_3 = \emptyset$. By Lemma B, $f(v_2) = 3$ or $f(v_2) = 0$. If $f(v_2) = 3$, then we may assume $f(a) = f(b) = 3$ and that $f(w) + f(w') \geq 2$ and the function g defined on T' by $g(v_2) = 2$ and $g(x) = f(x)$ otherwise, is an IDRDF of T' of weight $i_{dR}(T) - 6$, and if $f(v_2) = 0$, then $f(v_1) = f(a') = 2$, $f(v_3) \geq 2$, $f(w) + f(w') = 3$ and the function g defined on T' by $g(v_3) = 2$ and $g(x) = f(x)$ otherwise, is an IDRDF of T' of weight $i_{dR}(T) - 6$ and so $i_{dR}(T) \geq i_{dR}(T') + 6$. Thus $i_{dR}(T) = i_{dR}(T') + 6$. Likewise, we can see that $i_R(T') = i_R(T) - 4$ and $i(T') = i(T) - 2$. It follows that $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 6 - i_R(T) + 4 = i_{dR}(T) - i_R(T) - 2 \leq t - 1$. Using the induction hypothesis on T' and setting $s = 2, t = \ell = r = 0$, Proposition 2.1 leads to $i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T) \leq i_R(T) + i(T) + s(T) - 2$.

- v_4 is a strong support vertex.
  Let $w \in L_{v_4}$, $T' = T - \{v_1, a, a', b, b', w\}$. Clearly $s(T') = s(T) - 2$, and it is easy to verify that $i_{dR}(T') = i_{dR}(T) - 7$, $i_R(T') + 4 \leq i_R(T) \leq i_R(T') + 5$, $i(T) - 3 \leq i(T') \leq i(T) - 2$ and this implies that

  $i_{dR}(T') - i_R(T') \leq t - 1$. Using the induction hypothesis on T' and setting $s = 1, t = 2, \ell = 0, r = 1$, Proposition 2.1 leads to $i_R(T) + i(T) - \frac{s(T)}{2} + 1 \leq i_{dR}(T)$ and also we have

$$
\begin{aligned}
i(T) &\leq i(T') + 3 \\
&\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 2 \\
&\leq i_{dR}(T) - 7 - i_R(T) + 5 + \frac{(s(T) - 2)}{2} + 2 \\
&= i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1
\end{aligned}
$$

- v_4 is adjacent to at most one leaf, any child of v_4 with depth 1 is of degree at least 3 and for child y of v_4 with depth 2 is the center of a healthy spider with at least two feet.
  Suppose that $T' = T - T_{v_4}$. If T' is a star, then the result can be seen immediately. Assume T' is not a star. Let A, B and C be defined as in the Subcase 6.3. Clearly, $s(T') \leq s(T) - 2|B|$ and it is not hard to verify that $i_{dR}(T') = i_{dR}(T) - 3|A| - 2|B| - 2|C| - 2|L_{v_4}|$, $i_R(T') = i_R(T) - 2|A| - 2|B| - |C| - |L_{v_4}|$, $i(T) - |A| - |C| - |L_{v_4}| - 1 \leq i(T') \leq i(T) - |A| - |C| - |L_{v_4}|$.

  These imply that

$$
\begin{aligned}
i_{dR}(T') - i_R(T') \leq& i_{dR}(T) - 3|A| - 2|B| - 2|C| - 2|L_{v_4}| \\
&- (i_R(T) - 2|A| - 2|B| - |C| - |L_{v_4}|) \\
=& i_{dR}(T) - i_R(T) - (|A| + |C| + |L_4|) \leq t - 1.
\end{aligned}
$$

Using the induction hypothesis on T' and setting $s = |A| + |B|, t = |C| + |L_{v_4}|, \ell = 0, r = 2|B|$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and also we have

$$\begin{aligned} i(T) &\leq i(T') + |A| + |C| + |L_{v_4}| + 1 \\ &\leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + |A| + |C| + |L_{v_4}| \\ &\leq i_{dR}(T) - i_R(T) + \frac{(s(T) - 2|B|)}{2} \\ &\leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1. \end{aligned}$$

Subcase 7.2. $\deg(v_3) = 3$ and $\deg(v_4) \geq 3$.
Assume that $T' = T - T_{v_3}$. If T' is a star, then one can check the result easily. Suppose T' is not star. Obviously, $s(T') = s(T) - 2$ and one can see that $i_{dR}(T') + 5 \leq i_{dR}(T) \leq i_{dR}(T') + 6$, $i_R(T') + 3 \leq i_R(T) \leq i_R(T') + 4$ and $i(T) = i(T') + 2$. Hence $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 5 - i_R(T) + 3 = i_{dR}(T) - i_R(T) - 2 \leq t - 1$. Using the induction hypothesis on T' and setting $s = \ell = 0, t = 3, r = 1$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$ and also we have $i(T) = i(T') + 2 \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 1 \leq i_{dR}(T) - 5 - i_R(T) + 4 + \frac{s(T)-2}{2} + 1 = i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$.

Subcase 7.3. $\deg(v_3) = 3$ and $\deg(v_4) = 2$.
Assume that $T' = T - T_{v_4}$. If T' is a star, then we can can check the result easily. Suppose T' is not star. Obviously, $s(T') \leq s(T) - 1$ and $i_{dR}(T') + 6 \leq i_{dR}(T) \leq i_{dR}(T') + 7$, $i_R(T) = i_R(T') + 4$ and $i(T') + 2 \leq i(T) \leq i(T') + 3$. Hence $i_{dR}(T') - i_R(T') \leq i_{dR}(T) - 6 - i_R(T) + 3 \leq t - 1$. Applying the induction hypothesis on T' and setting $s = r = 1, t = 2, \ell = 0$, Proposition 2.1 leads to $i(T) \geq i_{dR}(T) - i_R(T) - s(T) + 2$. On the other hand, by the induction hypothesis we have $i(T) \leq i(T') + 3 \leq i_{dR}(T') - i_R(T') + \frac{s(T')}{2} + 2 \leq i_{dR}(T) - 6 - i_R(T) + 4 + \frac{s(T)-1}{2} + 2 = i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1/2$ and this implies $i(T) \leq i_{dR}(T) - i_R(T) + \frac{s(T)}{2} - 1$ because $i(T)$ is an integer. This completes the proof.

□

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