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The mincut graph of a graph

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Abstract

In this paper we introduce an intersection graph of a graph G, with vertex set the minimum edge-cuts of G. We find the minimum cut-set graphs of some well-known families of graphs and study the mincut graph as a graph operator. In doing so we follow the research programme on graph operators, as introduced by Prisner in the 1995 monograph "Graph Dynamics". Thus we ask and attempt to answer questions such as 'Which graphs appear as images of graphs?'; 'Which graphs are fixed under the operator?'; 'What happens if the operator is iterated?' We show that every graph is a minimum cut-set graph, henceforth called a *mincut graph*, of infinite depth and with an infinite number of roots.

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1. Introduction

Given a set S and a family $F = \{S_1, S_2 \dots, S_i\}$ of subsets of S, an intersection graph of F is a graph with vertices v_i corresponding to each of the S_i and two vertices v_i and v_j are adjacent if $S_i \cap S_j \neq \emptyset$, see [5, 10]. In 1945 Szpilrajn-Marczewski proved that every graph is an intersection graph, [5]. One of the first classes of intersection graphs to be widely studied was the line graph, generalised as (X, Y)-intersection graphs in [1], while in the 1970's chordal graphs were first characterised in terms of intersection graphs. Other intersection graphs that are studied intensively are interval and circular-arc graphs, competition graphs, p-intersection and tolerance graphs, to name but a few, see [8, 14, 15]. Problems involving intersection graphs often have real world applications in topics like biology, computing, matrix analysis and statistics, see [8, 14].

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In this paper we introduce the intersection graph of a graph G, called a *mincut graph*, with vertex set the minimum edge-cuts of G such that two vertices in the intersection graph are adjacent if their corresponding minimum edge-cut sets have non-empty intersection. We then study some of its properties and characteristics.

Although not many of the operators in [16] operate on the edges of graphs, we recall that the line graph operator also acts on the edges of a graph G, such that edges of G are vertices of $L(G)$ and two vertices are adjacent in $L(G)$ if their corresponding edges share a vertex in G. The mincut graph is related to the line graph in that as the line graph of a graph G reflects the mutual positions of the edges, see [17], so the mincut graph reflects the mutual positions of the minimum edge-cuts.

The cycle graph of a graph has as vertices the simple cycles of G with vertices adjacent if their corresponding cycles intersect. Thus, if we restrict ourselves to planar graphs, then cycles of G correspond to minimal cuts in the dual G^* and the mincut graph would be a subgraph of the cycle graph of the dual.

It is our hope that this new structure can be helpful in examining the effect of certain graph parameters on connectivity and possibly also lead to the identification of some new connectivity parameters. For example, it is known that the maximum number of mincuts has different order of magnitude for graphs with odd and even edge connectivity, see [11]. Let G be a graph with minimum edge-cut number λ and n vertices, with $X = \{X_1, X_2, \ldots, X_i\}$ the set of mincuts of G. In [12], Lehel *et al* determine the following upper bounds for $|X|$, the number of minimum cuts, in simple graphs:

$$|X| \leq \begin{cases} \frac{2n^2}{(\lambda+1)^2} + \frac{(\lambda-1)n}{\lambda+1}, & \text{if } \lambda \geq 4 \text{ and } \lambda \text{ is an even integer,} \\ (1 + \frac{4}{\lambda+5})n, & \text{if } \lambda > 5 \text{ and } \lambda \text{ is an odd integer.} \end{cases}$$

The problem of counting the number of mincuts of a graph is considered by many authors and various data structures are created to represent all these mincuts, see for example [4, 6, 7]. In the mincut graph the bounds on $|X|$ become bounds on the order of the mincut graph. The effect on these upper bounds of certain properties of graphs such as *radius and diameter* or *maximum and minimum degree* are investigated in [2]. Although outside the scope of this paper it would be interesting to know whether these upper bounds are related to or have similar effects on parameters in the mincut graph.

2. The Mincut Graph

Definition 2.1. *Let G be a simple connected graph, then an edge-cut of G is a subset X of $E(G)$, such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a* minimum edge-cut *and this cardinality is the edge-connectivity of G, denoted $\lambda(G)$. We will call such a minimum edge-cut a* mincut *of G.*

Example 2.2. *We illustrate this concept with an example. The diagrams in Figure 1 are the Wheel graph, W_6, and the Peterson graph, both with minimal edge cuts labeled $\{e_1, \ldots, e_5\}$. However, $\{e_1, \ldots, e_5\}$ is not a mincut for either of the graphs, since in each of the graphs the removal of any of the three edges incident on a single vertex of degree* 3 *will also give a disconnected graph. We call a mincut that disconnects a single vertex from the graph a* trivial cut*.*

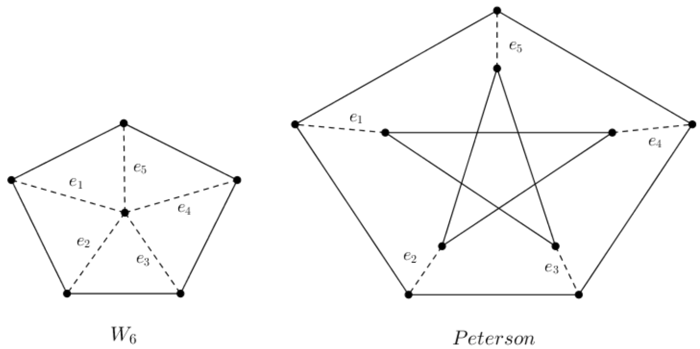

Figure 1. The edge sets $\{e_1, \ldots, e_5\}$ are minimal, but not mincuts.

Definition 2.3. *Let $X = \{X_1, X_2, \ldots, X_i\}$ be the set of all mincuts of a simple connected graph G. Represent each of the X_i with a vertex v_i such that two vertices v_i and v_j are adjacent if $X_i \cap X_j \neq \emptyset$, and call this intersection graph the* mincut graph *of G, denoted by $X(G)$.*

Lemma 2.4. *Let G be a disconnected graph, then $X(G)$ is the null graph, K_0, with no vertices or edges.*

Proof. Since G is disconnected, $\lambda(G) = 0$ and G has no edge-cut. Hence $X(G)$ has no vertices and consequently no edges. □

Theorem 2.5. *Let G be a connected graph with n vertices and minimum edge-cut number $\lambda(G) = k$, $k \geq 1$. Then $X(G)$, the mincut graph of G, is unique.*

Proof. Since $k \geq 1$ we have to remove at least one edge to disconnect the graph. Thus we know there is at least one edge-cut and hence at least one mincut. Thus, by Definition 2.3, $X(G)$ exists and has at least one vertex.

Let $X = \{X_1, X_2, \ldots, X_i\}$ be the set of all mincuts of a simple connected graph G. Since $k \geq 1$ we know that X is non-empty. Let $X'(G)$ and $X''(G)$ be two mincut graphs of G. By the definition of a mincut graph we have $V(X'(G)) = V(X''(G))$ corresponding to X. Since two vertices in the mincut graph of G are adjacent if their corresponding mincuts have non-empty intersection, we must have $v_iv_j \in E(X'(G))$ if and only if $v_iv_j \in E(X''(G))$ for all $v_i, v_j \in V(X'(G))$ and $v_i, v_j \in V(X''(G))$. Therefore $X'(G) \cong X''(G)$ and we conclude that $X(G)$ is unique. □

2.1. Mincut Graphs of Some Families of Graphs

In this section we describe the mincut graphs of some well-known families of graphs.

Proposition 2.6. *Let T_n be a tree on n vertices, then $X(T_n) \cong \overline{K_{n-1}}$, the complement of K_{n-1}.*

Proof. For any tree $\lambda(T_n) = 1$ and every edge of T_n is a bridge. Hence, each of the $n-1$ edges of T_n is a mincut and thus $X(T_n)$ has $n-1$ vertices. But none of the singleton mincuts intersect and thus $X(T_n) \cong \overline{K_{n-1}}$, the complement of K_{n-1}, or the empty graph (no edges) on $n-1$ vertices. □

Proposition 2.7. *Let $K_{m,n}$ be the complete bipartite graph with vertex partition sets of size m and n respectively, such that $n > m > 1$, then $K_{m,n} \cong \overline{K_n}$.*

Proof. The mincuts of $K_{m,n}$ are exactly the m edges incident on each of the n vertices of degree m in the larger of the two vertex partitions. Since $K_{m,n}$ is bipartite none of these incident sets intersect and we have a mincut graph with n vertices and no edges. If $m = 1$ we simply have a tree. □

Proposition 2.8. *Let W_n, $n > 4$, be the wheel on n vertices, then $X(W_n) \cong C_{n-1}$, the cycle on $n - 1$ vertices.*

Proof. For any W_n, $\lambda(W_n) = 3$ and the mincuts are exactly the trivial cuts with edges incident on each of the $n - 1$ vertices v_i on the "rim" of the wheel. By labeling the $n - 1$ vertices on the rim in sequence, we see that every mincut X_i has non-empty intersection with X_{i-1} and X_{i+1}, thus giving the cycle C_{n-1} as the mincut graph. □

Recall that the line graph, $L(G)$, of a graph G has the edges of G as its vertices, such that two vertices in $L(G)$ are adjacent if their corresponding edges in G have a vertex in common.

Proposition 2.9. *Let C_n be the cycle on n vertices then $X(C_n) \cong L(K_n)$, the line graph of K_n, the complete graph on n vertices.*

Proof. The edge connectivity of C_n is $\lambda(C_n) = 2$, and any choice of two edges is a mincut. The set of all mincuts, X, of C_n is the set of all two element subsets of $E(C_n)$, where $|E(C_n)| = n$ and hence $|X| = \binom{n}{2}$. The edge set of K_n is the set of all two element subsets of the n-element vertex set of K_n. Hence the vertex set of $L(K_n)$ is the set of two element subsets of $V(K_n)$ with $|V(L(K_n))| = \binom{n}{2}$. The vertices of both $X(C_n)$ and $L(K_n)$ can be labeled as the combinations of $\binom{n}{2}$ for an n element set, resulting in the same intersections and isomorphism follows. □

Example 2.10. *In Figure 2 we show some examples of the mincut graphs of some of the graphs dealt with in the preceding section.*

3. Super Edge-connected Graphs

In this section we define super-edge connected, or super-λ, graphs and give a sufficient condition for a mincut graph of a graph G to be isomorphic to G.

Definition 3.1. *A graph G is* maximally edge connected *when $\lambda = \delta$, where λ is the cardinality of the minimum edge-cut and δ is the minimum vertex degree of G.*

Definition 3.2. *A maximally edge-connected graph is* super-λ *if every minimum edge-cut set is* trivial*; that is, consists of the edges incident on a vertex of minimum degree.*

The following proposition states five sufficient conditions for a graph to be super-λ, see [9]. The first two conditions were proved by Lesniak in 1974, see [13].

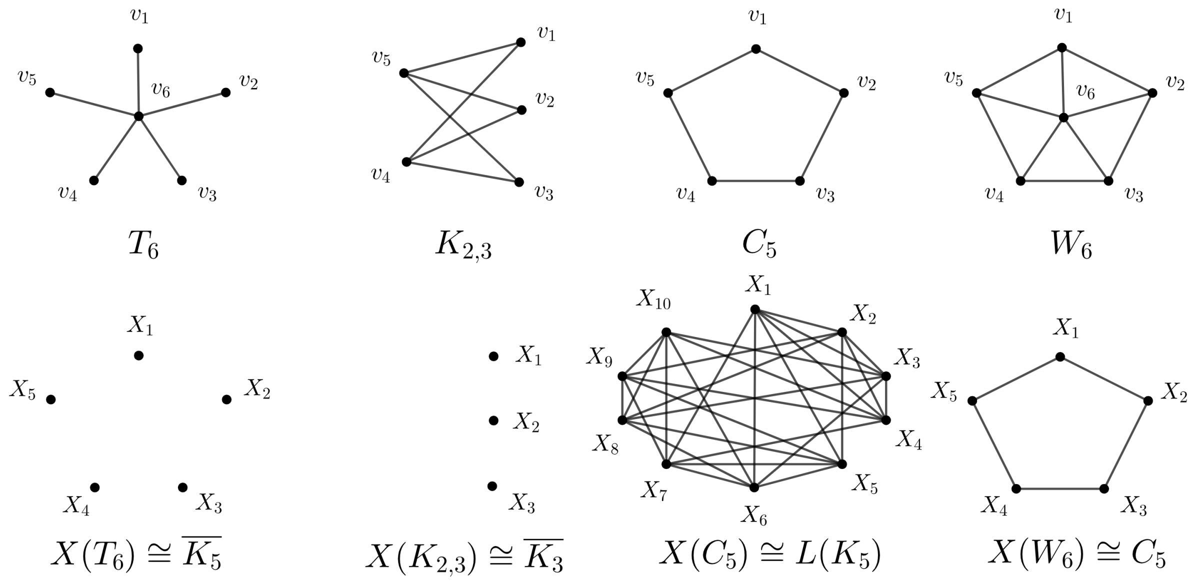

Figure 2. Mincut graphs of some well-known classes of graphs.

Proposition 3.3. *Let G be a graph of order n, minimum degree δ and maximum degree Δ. Then G is super-λ if any of the following conditions are satisfied:*

1. *Let $G \neq K_{n/2} \times K_2$, the cartesian product of $K_{n/2}$ and K_2. If for any non-adjacent $u, v \in V(G)$, deg(u) + deg$(v) \geq n$, then G is super-λ.*
2. *If for any non-adjacent $u, v \in V(G)$, deg(u) + deg$(v) \geq n + 1$, then G is super-λ. This is a corollary to the preceding condition.*
3. *If $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$, then G is super-λ.*
4. *If G has diameter 2 and contains no complete subgraph H on δ vertices with deg$_G v = \delta$ for all $v \in V(H)$, then G is super-λ.*
5. *If G has diameter 2 and $n > 2\delta + \Delta - 1$, then G is super-λ.*

Proposition 3.4. *If G is super-λ and r-regular, $G \not\cong K_2$, then $X(G) \cong G$.*

Proof. If G is r-regular and super-λ, then $V(H) = \{v \in V(G) | deg(v) = \delta(G)\} = V(G)$ and the edge-set incident on *every* vertex is a mincut. Hence, if $v_i v_j \in E(G)$, then $X_i \cap X_j \neq \emptyset$ and $x_i x_j \in E(X(G))$ and isomorphism follows. We exclude K_2 since $K_2 \cong T_2$ and $X(T_2) \cong K_1$ as we saw in Section 2.1. □

Definition 3.5. *A* strongly regular *graph is a regular graph G with parameters $srg(v, r, \lambda, \mu)$, where v is the order and r the regularity of G, such that every two adjacent vertices have λ neighbours in common and every two non-adjacent vertices have μ neighbours in common.*

The following Lemma is well known in the literature.

Lemma 3.6. *The line graph of a complete graph $L(K_n)$ is a strongly regular graph with parameters $srg(\binom{n}{2}, 2(n-2), n-2, 4)$.*

Corollary 3.7. *Let K_n be the complete graph on n vertices, $K_{n,n}$ the complete bipartite graph with equal vertex partitions and $L(K_n)$ the line graph of the complete graph. If $n > 2$, then*

1. $X(K_n) \cong K_n$
2. $X(K_{n,n}) \cong K_{n,n}$
3. $X(L(K_n)) \cong L(K_n)$.

Proof. We exclude $n = 2$, since K_2 is a tree, $K_{2,2} \cong C_4$ and $L(K_2) \cong K_1$. If $n > 2$ all three graphs are regular and super-λ.

For K_n we have $\delta = n - 1 \geq \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 3$ and hence K_n is super-λ by condition 3 of Proposition 3.3.

It is fairly straight forward, although lengthy, to show that $K_{n,n}$ and $L(K_n)$ are super-λ by condition 4 of Proposition 3.3.

The three graphs are respectively $n - 1$, n and $2(n - 2)$ regular and hence isomorphic to their mincut graphs by Proposition 3.4. □

4. When is a Graph a Mincut Graph?

In this section we address the question of which graphs are mincut graphs, that is, which graphs lie in the image of the mincut operator, which we will refer to as the X-operator and denote as $X(\cdot)$. We show that every graph is a mincut graph, by constructing a super-graph G^* from a given graph G such that G is the mincut graph of G^*. Furthermore, we show that every graph G has an infinite number of X-roots, that is, graphs that are mapped to G by $X(\cdot)$. For more on the concepts and definitions on graph operators used here, see Prisner in [16].

Definition 4.1. *A graph operator is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.*

We define the operator recursively, such that, if ϕ is an operator and k a positive integer, then $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$, for $k \geq 2$, see [16, 17]. Since an operator, ϕ, is a mapping on the set of finite graphs into itself, it is natural to ask which graphs lie in the image of ϕ and, given $\phi(G)$, are we able to determine G?

Every graph is an intersection graph, see [5, 10]. We show that every graph is the mincut graph of not just one graph, but more. It is possible to construct an infinite family of non-isomorphic graphs that map to the same mincut graph.

Lemma 4.2. *Let G be a super-λ graph with $H \subseteq G$ the induced subgraph on the set of vertices of G such that $V(H) = \{v \in V(G) | deg(v) = \delta(G)\}$, then $H \cong X(G)$.*

Proof. Let $x_i \in V(X(G))$ be the vertex in $X(G)$ corresponding to the mincut $X_i \in X$, the set of all mincuts in G. Since G is super-λ, the mincuts of G are exactly the incident edge-sets on the vertices $v \in V(H)$ and hence $|X| = |V(H)|$. Thus for every $v_i \in V(H)$ there is exactly one

$x_i \in V(X(G))$. Furthermore, if $v_iv_j \in E(H)$ then $X_i \cap X_j \neq \emptyset$, where X_i and X_j are the incident edge sets on v_i and v_j. If $v_iv_j \notin E(H)$ then $X_i \cap X_j = \emptyset$, since they are the incendent edge-sets on v_i and v_j. Hence $x_ix_j \in E(X(G))$ if and only if $v_iv_j \in E(H)$ and isomorphism follows. □

Example 4.3. *The graph G in Figure 3 is super-λ, by Condition 1 of Proposition 3.3 with vertices of minimum degree v_1, v_2, v_3, and v_4 and $X(G) \cong K_{1,3}$ is the induced subgraph on vertices $v_1 \ldots v_4$.*

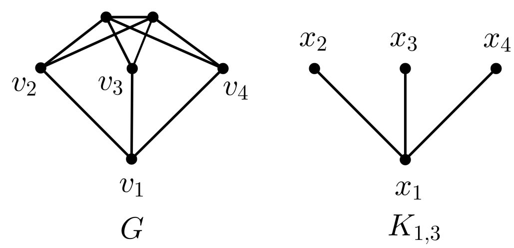

Figure 3. Super-λ graph

Definition 4.4. *The* join $G = G_1 \vee G_2$ *of two graphs G_1 and G_2 has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1),\, v \in V(G_2)\}$.*

In other words, the join of two graphs joins every vertex in the one graph to every vertex in the other graph. Thus, for example, we can say $K_n = K_1 \vee K_{n-1}$ and $K_{m,n} = \overline{K_m} \vee \overline{K_n}$. If one of the graphs is K_1 the operation is also called the *vertex join* of the graph.

Theorem 4.5. *Every graph G is the mincut graph of an infinite family of graphs.*

Proof. Let G be a graph of order p, not necessarily connected. Let K_q be a complete graph of order $q = p+5$ and $G^* \cong G \cup K_q$, the disjoint union of G and K_q. We note that $n = |V(G^*)| = 2p+5$. Add edges between G and K_q until $deg_{G^*}(v) = p+3$ for all $v \in V(G)$.

Now $deg_{G^*}(u) + deg_{G^*}(v) = 2p+6$ for all non-adjacent $u, v \in V(G)$ and $deg_{G^*}(u) + deg_{G^*}(x) \geq p+3+p+4 = 2p+7$ for all $u \in V(G)$ and $x \in V(K_q)$. Thus we have $deg_{G^*}(u) + deg_{G^*}(v) \geq 2p+6 \geq n+1$ for all non-adjacent $u, v \in V(G^*)$ and we conclude that G^* is super-λ by condition (2) of Proposition 3.3. We also have $\delta(G^*) = p+3$ and the vertices of minimum degree in G^* are exactly the vertices of G. Hence, by Lemma 4.2, $G \cong X(G^*)$.

If we now join K_m, $m \geq 1$ and G^*, the order of $K_m \vee G^*$ is $m+2p+5$, the degree of each vertex in G^* increases by m and the degree of each vertex in K_m is now $m+2p+4$. For any non-adjacent $u, v \in V(G)$, $deg_{K_m \vee G^*}(u) + deg_{K_m \vee G^*}(v) = 2m+2p+6 > m+2p+6$ and for any non-adjacent $u \in V(G)$ and $x \in V(K_q)$, $deg_{K_m \vee G^*}(u) + deg_{K_m \vee G^*}(x) \geq m+p+3+m+p+4 = 2m+2p+7 > m+2p+6$. We have $\delta(K_m \vee G^*) = p+3+m$, that is the degree of the vertices of the original graph G in $K_m \vee G^*$. Condition (2) of Proposition 3.3 holds and by Lemma 4.2, $G \cong X(K_m \vee G^*)$, for any choice of m. □

We note that the join of any K_m with just K_q in G^* would also give us a super-λ graph with G the induced subgraph on vertices of minimum degree, but then the sufficient condition 2 from Proposition 3.3 would not necessarily hold.

Example 4.6. *The graph G in Figure 4 is the mincut graph of a family of super-λ graphs $\mathscr{F}(G^*)$ such that $G \subseteq G^*$ and $V(G) = \{v \in V(G^*)|deg(v) = \delta(G^*)\}$. In the first diagram we have G^*, the disjoint union of G, of order 4, and K_9, of order $4 + 5 = 9$. We note that the order of G^* is 13. In the second diagram we add edges between G and K_9 until we have $deg_{G^*}(v) = 7$ for all $v \in V(G)$. Now $deg_{G^*}(v) \geq 8$ for all $v \in V(K_9)$ and $\delta(G^*) = 7$. For any two non-adjacent vertices $u, v \in V(G)$ we have $deg_{G^*}(u) + deg_{G^*}(v) = 14$ and for any two non-adjacent vertices $v \in V(G)$ and $x \in K_9$ we have $deg_{G^*}(v) + deg_{G^*}(x) \geq 15 > 14$. Thus, for any two non-adjacent vertices in G^* we have that the sum of their degrees is greater than or equal to 14, or $n+1$ and we conclude that G^* is super-λ. By Lemma 4.2 we have that G is the mincut graph of G^*. Furthermore, we could now join any K_m to G^* and G would still be the induced subgraph on vertices of minimum degree.*

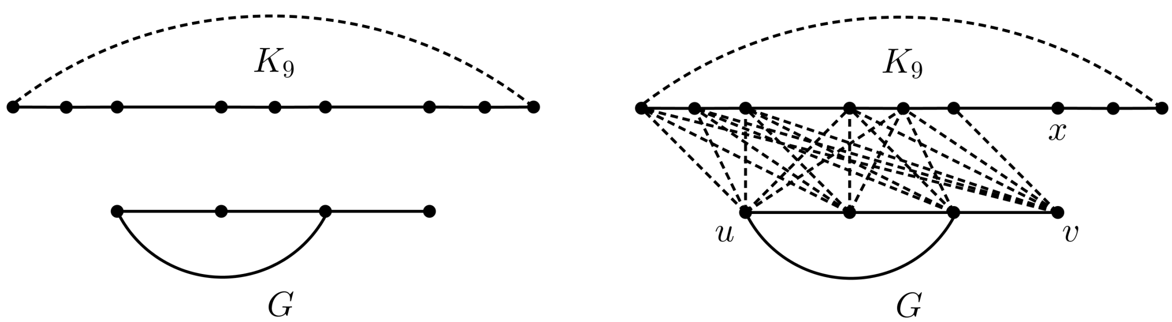

Figure 4. Graphs G and K_9, with G^* constructed such that $X(G^*) \cong G$.

Corollary 4.7. *For any connected graph G there is a non-isomorphic graph H such that $X(G) \cong X(H)$.*

5. Roots and Depth

In this section we start to explore some further ideas around the mincut operator. Since we have shown that every graph is a mincut graph we have some immediate consequences for the *roots* and *depth*, see [16, 17] of any graph under $X(\cdot)$.

Definition 5.1. *For a given graph operator ϕ, a ϕ-*root *of a graph G is any graph H with $\phi(H) \cong G$.*

With respect to roots and iteration we also have the following definition of the depth of a graph under some operator.

Definition 5.2. *For a graph G and an operator ϕ we define the ϕ-depth, $depth(G)$, as the largest integer, d, (if there is one, otherwise ∞) for which there is some graph H such that $\phi^d(H) \cong G$.*

Proposition 5.3. *Every graph G has an infinite number of X-roots.*

Proof. The proof follows directly from the construction in Theorem 4.5. By our construction $X(G^*) \cong G$ and it followed that for any choice of m, $X(K_m \vee G^*) \cong G$. □

Proposition 5.4. *Every graph G has infinite X-depth.*

Proof. By Theorem 4.5, every graph is the mincut graph of a family of graphs. Therefore, given a graph G_0, there is some graph $G_{(-1)}$ such that $X(G_{(-1)}) \cong G_0$, where the value of the subscript is merely to indicate position in a sequence. By the same reasoning, there is a graph $G_{(-2)}$ such that $X(G_{(-2)}) \cong G_{(-1)}$. Thus we can construct a set $\{\dots, G_{(-3)}, G_{(-2)}, G_{(-1)}, G_0\}$ where every $X(G_{(-n)}) \cong G_{(-n+1)}$ for all $n \geq 1$ and $X^n(G_{(-n)}) \cong G_0$. □

Example 5.5. *We refer to the graphs in Figure 5. Each sequence represents a number of iterations under $X(\cdot)$. We know that $X(P_4) \cong \overline{K_3}$ since it is basically a tree on four vertices, and since $\overline{K_3}$ is disconnected $X(\overline{K_3}) \cong K_0$ by Lemma 2.4. The graph G_1 was obtained from P_4 by the construction in Theorem 4.5 using $P_4 \cup K_2$. We note that any graph with exactly three mincuts that do not intersect is also a root of $\overline{K_3}$ and any disconnected graph is a root of K_0. In the second iteration sequence we have C_4 as a root of $L(K_4)$. We could have constructed a root using our construction method from Theorem 4.5 or, since line graphs of complete graphs are fixed under $X(\cdot)$ we could simply have $L(K_4)$. W_5 is a root of C_4 since $W_5 \cong K_1 \vee C_4$ and C_4 is 2-regular with $n = 4 > 2 + 1$. We could apply our construction technique again to G_1 and W_5 respectively to continue each sequence in reverse and generate an infinite sequence of graphs such that each is an X-root of the next. Admittedly our graphs will become very large very soon but it is possible in principle.*

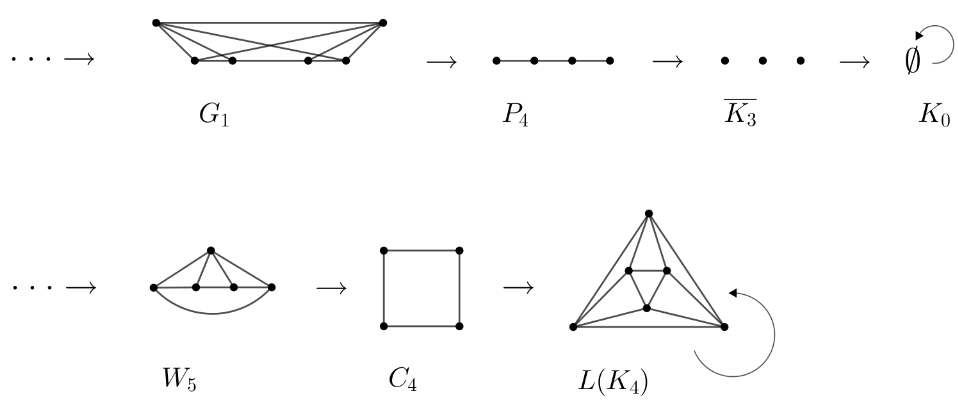

Figure 5. Graphs with some of their X-roots.

6. Further Discussion

In this section we conclude by introducing and exploring further topics and questions raised by the properties and characteristics of the mincut graph.

We also look at the possible implications of certain induced subgraphs in $X(G)$ for a root if that root is not necessarily obtained from the construction process in Theorem 4.5.

6.1. Determination Problem

We know from Whitney that a graph is uniquely determined by its line graph up to isolated vertices, unless it is C_3 or $K_{1,3}$, see [17]. Although our construction in Theorem 4.5 gives the interesting result that every graph is a mincut graph, it unfortunately does not shed much light on what the mincut graph $X(G)$ tells us about the mincuts of G if G is not a graph of the family obtained by our construction. We explore this problem in a little more detail.

Lemma 6.1. *Let $X = \langle A,\ \bar{A}\rangle$ and $Y = \langle B,\ \bar{B}\rangle$ be two non-trivial mincuts of a graph G with $A,\ \bar{A}$ and $B,\ \bar{B}$ the vertex sets of the components of $G - X$ and $G - Y$ respectively. If $X \cap Y \neq \emptyset$ then either $A \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.*

We note that the converse to Lemma 6.1 is not necessarily true.

We know, see [2, 12], that if $X = \langle A,\ \bar{A}\rangle$ and $Y = \langle B,\ \bar{B}\rangle$ are two mincuts of G and $A \cap B \neq \emptyset$ then either $A \subset B$ (or $B \subset A$), that is they are *nested*, or they *overlap*. If X and Y overlap (also called *crossing mincuts*), then $A \cap B$, $\bar{A} \cap B$ and $A \cap \bar{B}$ are *non-empty*. We also note from [12] that two mincuts can overlap only if λ, the minimum edge connectivity number of the graph, is even.

Proposition 6.2 (Lehel et al, [12])**.** *Let $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ be two overlapping mincuts of a connected graph G. Then $|\langle \bar{A} \cap \bar{B}, A \cap \bar{B}\rangle| = |\langle \bar{A} \cap \bar{B}, \bar{A} \cap B\rangle| = |\langle A \cap B, A \cap \bar{B}\rangle| = |\langle A \cap B, \bar{A} \cap B\rangle| = \dfrac{\lambda}{2}$. Consequently $A \cup B$, $A \cap B$, $A \cap \bar{B}$, $\bar{A} \cap B$ are mincuts. Moreover, $|\langle \bar{A} \cap \bar{B}, A \cap B\rangle| = |\langle \bar{A} \cap B, A \cap \bar{B}\rangle| = 0$.*

Lemma 6.3 (Chandran & Ram, [2])**.** *If $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ are a pair of crossing mincuts, then $X \cap Y = \emptyset$.*

Let $X = \langle A,\ \bar{A}\rangle$ and $Y = \langle B,\ \bar{B}\rangle$ be overlapping mincuts of a graph G with non-trivial mincuts $W = \langle A \cap B,\ \bar{A} \cup \bar{B}\rangle$, $Z = \langle A \cup B, \bar{A} \cap \bar{B}\rangle$ as per Proposition 6.2. Suppose that $W \cap X,\ Y$ and $X,\ Y \cap Z$ are non-empty, then we have a corresponding induced cycle C_4 in $X(G)$. Given what we know from Proposition 6.2, it would seem that we should at least be able to construct an induced subgraph in G corresponding to the cycle in $X(G)$ without resorting to the construction in Theorem 4.5.

Similarly, let $X = \langle A,\ \bar{A}\rangle$, $Y = \langle B,\ \bar{B}\rangle$ and $Z = \langle C,\ \bar{C}\rangle$ be non-trivial nested mincuts of a graph G with $A \subset B \subset C$. If the induced subgraph in $X(G)$ corresponding to the mincuts X, Y and Z is connected then we should have a cycle C_3 or at least an induced path P_3 in $X(G)$.

It would be interesting to know whether we can have induced cycles larger than C_4 in $X(G)$ such that the corresponding mincuts in G are not of the form as per our construction in Theorem 4.5.

6.2. Periodicity of the Mincut Operator

Definition 6.4. *Let ϕ be an operator on a class of graphs Γ. A graph $G \in \Gamma$ is* periodic *in ϕ if there is some natural number n with $G \cong \phi^n(G)$. The smallest such number is called the* period *of G.*

In Corollary 3.4 we showed that it is sufficient for $G \cong X(G)$ if G is super-λ and r-regular. That is, the graph is fixed under the operator, or has periodicity 1. In fact, it can be shown that this condition is also necessary.

Furthermore, we know that there are 2-periodic graphs under $X(\cdot)$. For example, the cartesian product $K_m \times K_2$ has mincut graph $X(K_m \times K_2) \cong K_1 \vee (K_m \times K_2)$, the join of itself with K_1. But the mincut graph of $K_1 \vee (K_m \times K_2)$ is again $K_m \times K_2$ which means that iteration of the $X(\cdot)$ operator leads to a cycle of two graphs. A natural question to ask is whether there are other graphs of this periodicity and if so what would be their characteristics. Furthermore, do we have graphs of a higher X-periodicity?

6.3. Convergence or Divergence

In [18] Wilf and Van Rooij showed that unless a graph is $K_{1,3}$, C_n or P_n, repeated application of the line graph operator eventually leads to a steadily increasing number of vertices, that is, it *diverges* under $L(\cdot)$.

We know that the maximum number of mincuts in a graph of order n is $\binom{n}{2}$ and this bound is tight for simple graphs when $G \cong C_n$, see [4, 12]. We also have other upper bounds on the number of mincuts of a graph depending on whether λ is even or odd as mentioned in Section 1.

Intuitively, if a graph is to diverge under $X(\cdot)$ both the number of vertices and edges need to increase. If only the number of vertices increases the edge density of the graph decreases and the likelihood of every mincut intersecting with at least one other decreases, converging to the null graph in the end. If only the number of edges increases we eventually end up with a complete graph which is fixed. But for $|V(X(G))|$ and $|E(X(G))|$ to increase and $X(G)$ to remain connected we must have the edge density increasing. Thus we would expect such a graph to eventually become fixed (or at least periodic) under the X-operator.

Hence, given the upper bounds on the number of mincuts, that we expect graphs that "grow" to eventually become fixed or periodic, that non-intersecting mincuts are disconnected and that the mincut graph of a disconnected graph is the null graph, K_0, we have the following two conjectures:

Conjecture 1 (Convergence)**.** *Let G be a simple connected graph and $X(\cdot)$ the mincut operator. Then $X^n(G)$ converges for sufficiently large n.*

Conjecture 2 (Convergence to null graph)**.** *Let G be a simple connected graph and $X(\cdot)$ the mincut operator. Then $X^k(G) \to K_0$ for sufficiently large k, except under a finite number of conditions.*

6.4. Connectivity of $X(G)$

$X(G)$ will be connected if every mincut has non-empty intersection with at least one other mincut. What are the characteristics of a connected graph that will imply that its mincut graph is connected? Is it possible to find minimum bounds on $\kappa(G)$, $\lambda(G)$, and $\delta(G)$, (the vertex connectivity, edge connectivity and minimum degree values of G) that will guarantee a connected $X(G)$? Intuitively there should be some relationship between the vertices and the way the edges are distributed (some kind of "edge density" function?) and $\lambda(G)$ that will ensure connectivity of $X(G)$.

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