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On the signed 2-independence number of graphs

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Abstract

In this paper, we study the signed 2-independence number in graphs and give new sharp upper and lower bounds on the signed 2-independence number of a graph by a simple uniform approach. In this way, we can improve and generalize some known results in this area.

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1. Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use [13] as a reference for terminology and notation which are not defined here. The *open neighborhood* of a vertex v is denoted by $N(v)$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The minimum and maximum degree of G are respectively denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$.

Let $S \subseteq V$. For a real-valued function $f : V \to R$ we define $f(S) = \sum_{v\in S} f(v)$. Also, $f(V)$ is the weight of f. A *signed* 2*-independence function*, abbreviated S2IF, of G is defined in [14] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \leq 1$, for every $v \in V$. The *signed* 2*-independence number*, abbreviated S2IN, of G is $\alpha_s^2(G) = \max\{f(V)|f$ is a S2IF of $G\}$. This concept was

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defined in [14] as a certain dual of the signed domination number of a graph [3] and has been studied by several authors including [8, 10, 11, 12].

A set $S \subseteq V$ is a *dominating set* if each vertex in $V \backslash S$ has at least one neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set [7]. A subset $B \subseteq V$ is a 2-*packing* in G if for every pair of vertices $u, v \in B$, $d(u, v) \geq 3$. The 2-*packing number* (or *packing number*) $\rho(G)$ is the maximum cardinality of a 2-packing in G.

Gallant et al. [5] introduced the concept of *limited packing* in graphs. They exhibited some real-world applications of it to network security, NIMBY, market saturation and codes. In this paper we exhibit an application of it to signed 2-independence number in graphs. In fact as it is defined in [5], a set of vertices $B \subseteq V$ is called a k-*limited packing* in G provided that for all $v \in V$, we have $|N[v] \cap B| \leq k$. The *limited packing number*, denoted $L_k(G)$, is the largest number of vertices in a k-limited packing set. It is easy to see that $L_1(G) = \rho(G)$. In [6], Harary and Haynes introduced the concept of *tuple domination* in graphs. A set $D \subseteq V$ is a k-*tuple dominating set* in G if $|N[v] \cap D| \geq k$, for all $v \in V(G)$. The k-*tuple domination number*, denoted $\gamma_{\times k}(G)$, is the smallest number of vertices in a k-tuple dominating set. When $k = 2$, D is called a *double dominating set* and the 2-tuple domination number is called the *double domination number* and is denoted by $dd(G)$. In fact the authors showed that every graph G with $\delta \geq k - 1$ has a k-tuple dominating set and hence a k-tuple domination number.

By a simple uniform approach, we derive many new sharp bounds on $\alpha_s^2(G)$ in terms of several different graph parameters. Some of our results improve some known bounds on the S2IN of graphs in [8, 11, 12].

The authors noted that most of the existing bounds on $\alpha_s^2(G)$ are lower bounds. In section 2, we prove that $\alpha_s^2(G) \geq 2\lfloor \frac{\delta+2\rho(G)}{2} \rfloor - n$, for a graph G of order n. Also in section 3, by a simple connection between the concepts of limited packing and tuple domination, we obtain the exact value of the signed 2-independence numbers of regular graphs. In particular, we bound the signed 2-independence numbers of cubic graphs from below and above just in terms of order as, $-\frac{n}{3} \leq \alpha_s^2(G) \leq 0$.

2. Main results

At this point we are going to present some sharp upper bounds on $\alpha_s^2(G)$. First, let us introduce some notation. Let $f : V \longrightarrow \{-1, 1\}$ be a maximum S2IF of G. We define $V_+ = \{v \in V | f(v) = 1\}$, $V_- = \{v \in V | f(v) = -1\}$, $G_+ = G[V_+]$ and $G_- = G[V_-]$ where G_+ and G_- are the subgraphs of G induced by V_+ and V_-, respectively. For convenience, let $[V_+, V_-]$ be the set of edges having one end point in V_+ and the other in V_-. Finally, $deg_{G_+}(v) = |N(v) \cap V_+|$ and $deg_{G_-}(v) = |N(v) \cap V_-|$. Obviously, $|V_+| = \frac{n+\alpha_s^2(G)}{2}$ and $|V_-| = \frac{n-\alpha_s^2(G)}{2}$.

Theorem 2.1. *Let G be a graph of order n. Then*

$$\alpha_s^2(G) \leq \left(\frac{\lfloor \frac{\Delta}{2} \rfloor - \lceil \frac{\delta}{2} \rceil + 1}{\lfloor \frac{\Delta}{2} \rfloor + \lceil \frac{\delta}{2} \rceil + 1}\right)n$$

and this bound is sharp.

Proof. Let f be a maximum S2IF of G. Let $v \in V_+$. Since $f(N[v]) \leq 1$, the vertex v has at least $\lceil \frac{deg(v)}{2} \rceil \geq \lceil \frac{\delta}{2} \rceil$ neighbors in V_-. Therefore $|[V_+, V_-]| \geq \lceil \frac{\delta}{2} \rceil |V_+|$. Now let $v \in V_-$. Since f is a S2IF, the vertex v has at most $\lfloor \frac{deg(v)}{2} \rfloor + 1 \leq \lfloor \frac{\Delta}{2} \rfloor + 1$ neighbors in V_+. Therefore $|[V_+, V_-]| \leq (\lfloor \frac{\Delta}{2} \rfloor + 1)|V_-|$. In fact

$$\lceil \frac{\delta}{2} \rceil |V_+| \leq |[V_+, V_-]| \leq (\lfloor \frac{\Delta}{2} \rfloor + 1)|V_-|.$$

Using $|V_+| = \frac{n+\alpha_s^2(G)}{2}$ and $|V_-| = \frac{n-\alpha_s^2(G)}{2}$, we obtain the desired upper bound. For sharpness it is sufficient to consider the complete graph K_n. □

In [8] the author established a relationship between the signed 2-independence number and the domination number of a graph as follows.

Theorem 2.2. ([8]) *If G is a connected graph of order $n \geq 2$, then $\alpha_s^2(G) + 2\gamma(G) \leq n$, and this bound is sharp.*

Now we are going to improve Theorem 2.2. We shall need the following result, which can be found implicit in [4] and explicit in [2] as Corollary 81.

Theorem 2.3. ([2],[4]) *If G is a graph with $\delta \geq k-1$, then $\gamma_{\times k}(G) \geq \gamma(G) + k - 1$.*

Theorem 2.4. *If G is a connected graph of order n, then $\alpha_s^2(G) + 2\gamma(G) \leq n - 2\lceil \frac{\delta}{2} \rceil + 2$, and this bound is sharp.*

Proof. Let f be a maximum S2IF of G. We have shown that $|N[v] \cap V_-| \geq \lceil \frac{\delta}{2} \rceil$ for all $v \in V_+$. On the other hand, if $v \in V_-$, then $deg_{G_-}(v) \geq \lceil \frac{deg(v)}{2} \rceil - 1 \geq \lceil \frac{\delta}{2} \rceil - 1$. Therefore $|N[v] \cap V_-| \geq \lceil \frac{\delta}{2} \rceil$. This shows that V_- is a $\lceil \frac{\delta}{2} \rceil$-tuple dominating set in G. This implies, $|V_-| \geq \gamma_{\times \lceil \frac{\delta}{2} \rceil}(G)$ and hence $\alpha_s^2(G) \leq n - 2\gamma_{\times \lceil \frac{\delta}{2} \rceil}(G)$. Now by Theorem 2.3, we have $\alpha_s^2(G) \leq n - 2(\gamma(G) + \lceil \frac{\delta}{2} \rceil - 1)$. Therefore $\alpha_s^2(G) + 2\gamma(G) \leq n - 2\lceil \frac{\delta}{2} \rceil + 2$. For sharpness it is sufficient to consider the complete graph K_n. □

By the concept of limited packing we can present a sharp lower bound on $\alpha_s^2(G)$ that involves the packing number.

Theorem 2.5. *Let G be a connected graph of order n. Then*

$$\alpha_s^2(G) \geq 2\lfloor \frac{\delta + 2\rho(G)}{2} \rfloor - n$$

and this bound is sharp.

Proof. Let B be a $\lfloor \frac{\delta}{2} \rfloor$-limited packing set in G. Obviously, $L_{\lfloor \frac{\delta}{2} \rfloor}(G) \leq L_{\lfloor \frac{\delta}{2}+1 \rfloor}(G)$. We claim that $B \neq V$. If $B = V$ and $v \in V$ such that $deg(v) = \Delta$, then $\Delta + 1 = |N[v] \cap B| \leq \lfloor \frac{\delta}{2} \rfloor \leq \Delta$, a contradiction. Now let $u \in V - B$. It is easy to check that $|N[v] \cap (B \cup \{u\})| \leq \lfloor \frac{\delta}{2} \rfloor + 1$, for all $v \in V(G)$. Therefore $B \cup \{u\}$ is a ($\lfloor \frac{\delta}{2} \rfloor + 1$)-limited packing set in G. Hence

$$L_{\lfloor \frac{\delta}{2} \rfloor + 1}(G) \geq |B \cup \{u\}| = |B| + 1 = L_{\lfloor \frac{\delta}{2} \rfloor}(G) + 1.$$

Repeating these inequalities, we have

$$L_{\lfloor \frac{\delta}{2} \rfloor + 1}(G) \geq L_{\lfloor \frac{\delta}{2} \rfloor}(G) + 1 \geq ... \geq L_1(G) + \lfloor \frac{\delta}{2} \rfloor = \rho(G) + \lfloor \frac{\delta}{2} \rfloor. \tag{1}$$

Now let B be a maximum $(\lfloor \frac{\delta}{2} \rfloor + 1)$-limited packing set in G. We define $f : V \to \{-1, 1\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v \in B \\ -1 & \text{if } v \in V - B. \end{cases}$$

We deduce that

$$\begin{aligned} f(N[v]) &= |N[v] \cap B| - |N[v] \cap (V - B)| \\ &= 2|N[v] \cap B| - |N[v]| \leq 2\lfloor \tfrac{\delta}{2} \rfloor - \delta + 1 \leq 1, \end{aligned}$$

for all $v \in V$. Therefore, f is a S2IF of G. This implies

$$\alpha_s^2(G) \geq f(V) = |B| - |V - B| = 2|B| - n = 2L_{\lfloor \frac{\delta}{2} \rfloor + 1}(G) - n.$$

Now (1) implies

$$\alpha_s^2(G) \geq 2L_{\lfloor \frac{\delta}{2} \rfloor + 1}(G) - n \geq 2(\rho(G) + \lfloor \frac{\delta}{2} \rfloor) - n,$$

as desired. Considering the graph K_n we can see that this bound is sharp. □

Volkmann in [11] proved that if G is a graph of order n, then $2 - n \leq \alpha_s^2(G)$. Moreover if $n \geq 3$, then $4 - n \leq \alpha_s^2(G)$. Obviously, the lower bound in Theorem 2.5 is an improvement of the first inequality and when $\delta \geq 2$ this improves the second, as well.
At the end of this section we exhibit a short comment about signed 2-independence number of bipartite graphs. The following upper bound on $\alpha_s^2(G)$ of a bipartite graph was obtained by Wang [12].

Theorem 2.6. ([12]) *If G is a bipartite graph of order $n \geq 2$, then*

$$\alpha_s^2(G) \leq n + 6 - 2\sqrt{2n + 9}.$$

Furthermore, the bound is sharp.

We now improve the bound in the previous theorem.

Theorem 2.7. *Let G be a bipartite graph of order n. Then*

$$\alpha_s^2(G) \leq n + 2(2 + \lceil \frac{\delta}{2} \rceil) - 2\sqrt{(2 + \lceil \frac{\delta}{2} \rceil)^2 + 2\lceil \frac{\delta}{2} \rceil n}$$

and this bound is sharp.

Proof. Let f be a maximum S2IF of G. Let X and Y be the partite sets of G. For convenience we define $X_+ = X \cap V_+$, $X_- = X \cap V_-$ and let Y_+ and Y_- be defined, analogously. Obviously, $V_+ = X_+ \cup Y_+$ and $V_- = X_- \cup Y_-$.
Since every vertex in X_+ has at least $\lceil \frac{\delta}{2} \rceil$ neighbors in Y_-, by the pigeonhole principle, there exists a vertex v in Y_- that is joined to at least $\frac{\lceil \frac{\delta}{2} \rceil |X_+|}{|Y_-|}$ vertices in X_+. This implies

$$\frac{\lceil \frac{\delta}{2} \rceil |X_+|}{|Y_-|} - |X_-| - 1 \leq |N[v] \cap X_+| - |N[v] \cap X_-| - 1 = f(N[v]) \leq 1,$$

and hence

$$\lceil \frac{\delta}{2} \rceil |X_+| \leq |Y_-|(|X_-| + 2). \tag{2}$$

A similar argument shows that

$$\lceil \frac{\delta}{2} \rceil |Y_+| \leq |X_-|(|Y_-| + 2). \tag{3}$$

Using inequalities (2) and (3) we have

$$\lceil \frac{\delta}{2} \rceil |V_+| \leq 2|X_-||Y_-| + 2|V_-| \leq \frac{1}{2}(|X_-| + |Y_-|)^2 + 2|V_-| = \frac{1}{2}|V_-|^2 + 2|V_-|.$$

Using $|V_+| = n - |V_-|$, we obtain

$$|V_-|^2 + (4 + 2\lceil \frac{\delta}{2} \rceil)|V_-| - 2|V_-|n \geq 0.$$

This yields to $|V_-| \geq \frac{-4-2\lceil \frac{\delta}{2} \rceil + \sqrt{(4+2\lceil \frac{\delta}{2} \rceil)^2 + 8\lceil \frac{\delta}{2} \rceil n}}{2}$. Now, by using the value of $|V_-|$ we derive the desired bound. $\square$

Using calculus we can see that $g(x) = n + 2(x+2) - 2\sqrt{(x+2)^2 + 2nx}$ is a decreasing function for $x \geq 0$. So, for $\delta \geq 1$, $\lceil \frac{\delta}{2} \rceil \geq 1$ implies that

$$n + 2(2 + \lceil \frac{\delta}{2} \rceil) - 2\sqrt{(2 + \lceil \frac{\delta}{2} \rceil)^2 + 2\lceil \frac{\delta}{2} \rceil n} \leq n + 6 - 2\sqrt{2n+9}$$

and therefore Theorem 2.7 is an improvement of Theorem 2.6.

3. Remarks on signed 2-independence in regular graphs

Zelinka [14] obtained the following sharp upper bound on $\alpha_s^2(G)$ for regular graphs G.

Theorem 3.1. ([14]) *If G is an r-regular graph of order n, then $\alpha_s^2(G) \leq \frac{n}{r+1}$ when r is even and $\alpha_s^2(G) \leq 0$ when r is odd.*

We note that the bound in Theorem 2.1 implies the previous result. The authors in [9] proved the following result.

Lemma 3.1. ([9]) *Let G be a graph. Then the following statements hold.*
(i) *Let $\delta \geq k-1$. If $B \subseteq V$ is a k-limited packing set, then $V-B$ is a $(\delta-k+1)$- tuple dominating set in G.*
(ii) *Let $\delta \geq k$. If $D \subseteq V$ is a k-tuple dominating set, then $V-D$ is a $(\Delta-k+1)$-limited packing set in G.*

Now, by the above lemma we are able to obtain the exact value of the signed 2-independence number of regular graphs, first in terms of order and limited packing number, second in terms of order and tuple domination number. At the end we bound $\alpha_s^2(G)$ of a cubic graph G from above and below, just in terms of the order. First we need the following lemma.

Lemma 3.2. *Let G be a graph of order n, then*
(i) $2L_{\lfloor\frac{\delta}{2}\rfloor+1}(G)-n \leq \alpha_s^2(G) \leq 2L_{\lfloor\frac{\Delta}{2}\rfloor+1}(G)-n$,
(ii) $n-2\gamma_{\times\lceil\frac{\Delta}{2}\rceil}(G) \leq \alpha_s^2(G) \leq n-2\gamma_{\times\lceil\frac{\delta}{2}\rceil}(G)$.

Proof. (i) In the proof of Theorem 2.5 we have seen that $2L_{\lfloor\frac{\delta}{2}\rfloor+1}(G)-n \leq \alpha_s^2(G)$.
Now let f be a maximum S2IF of G. In the proof of Theorem 2.1 we have shown that $|N[v]\cap V_+| \leq \lfloor\frac{\Delta}{2}\rfloor+1$, for all $v \in V_-$. On the other hand, if $v \in V_+$, then $deg_{G_+}(v) \leq \lfloor\frac{deg(v)}{2}\rfloor \leq \lfloor\frac{\Delta}{2}\rfloor$. Therefore V_+ is a $(\lfloor\frac{\Delta}{2}\rfloor+1)$-limited packing set in G. This implies $|V_+| \leq L_{\lfloor\frac{\Delta}{2}\rfloor+1}(G)$ and hence $\alpha_s^2(G) \leq 2L_{\lfloor\frac{\Delta}{2}\rfloor+1}(G)-n$.
(ii) According to the proof of Theorem 2.4, we have $\alpha_s^2(G) \leq n-2\gamma_{\times\lceil\frac{\delta}{2}\rceil}(G)$.
Now let D be a minimum $\lceil\frac{\Delta}{2}\rceil$-tuple dominating set in G. We define $f: V \rightarrow \{-1,1\}$ by

$$f(v)=\begin{cases} -1 & \text{if } v \in D \\ \ \ 1 & \text{if } v \in V-D. \end{cases}$$

By the previous lemma, we conclude that $f(N[v]) = |N[v]\cap(V-D)| - |N[v]\cap D| \leq \Delta - \lceil\frac{\Delta}{2}\rceil + 1 - \lceil\frac{\Delta}{2}\rceil \leq 1$. Therefore f is a S2IF of G. This implies $\alpha_s^2(G) \geq f(V) = |V-D|-|D| = n-2|D| = n-2\gamma_{\times\lceil\frac{\Delta}{2}\rceil}(G)$. □

Considering regular graphs, by the previous lemma, we have the following corollary.

Corollary 3.1. *Let G be an r-regular graph of order n. Then*
(i) $\alpha_s^2(G) = 2L_{\lfloor\frac{r}{2}\rfloor+1}(G)-n$.
(ii) $\alpha_s^2(G) = n-2\gamma_{\times\lceil\frac{r}{2}\rceil}(G)$.

As an immediate result of the previous corollary we obtain the following.

Corollary 3.2. *If G is a cubic graph of order n, then*
(i) $\alpha_s^2(G) = 2L_2(G)-n$.
(ii) $\alpha_s^2(G) = n-2dd(G)$.

In [1], the authors showed that if G is a cubic graph of order n, then $\frac{n}{3} \leq L_2(G)$. Moreover, the upper bound $L_2(G) \leq \frac{n}{2}$ was presented in [5] for a cubic graph G. Therefore Corollary 3.2 leads to

$$-\frac{n}{3} \leq \alpha_s^2(G) \leq 0$$

for cubic graphs.

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