Electronic Journal of Graph Theory and Applications **10** (1) (2022), 275–288

Electronic Journal of
Graph Theory and Applications

Total domination number of middle graphs

Farshad Kazemnejad[a], Behnaz Pahlavsay[b], Elisa Palezzato[b], Michele Torielli*[c]

[a]*Faculty of Basic Sciences, Department of Mathematics, Ilam University, P.O.Box 69315-516, Ilam, Iran*
[b]*Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo 060-0810, Japan*
[c]*Department of Mathematics, GI-CoRE GSB, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo 060-0810, Japan*

kazemnejad.farshad@gmail.com, pahlavsayb@gmail.com, palezzato@math.sci.hokudai.ac.jp, torielli@math.sci.hokudai.ac.jp

**corresponding author*

Abstract

A *total dominating set* of a graph G with no isolated vertices is a subset S of the vertex set such that every vertex of G is adjacent to a vertex in S. The *total domination number* of G is the minimum cardinality of a total dominating set of G. In this paper, we study the total domination number of middle graphs. Indeed, we obtain tight bounds for this number in terms of the order of the graph G. We also compute the total domination number of the middle graph of some known families of graphs explicitly. Moreover, some Nordhaus-Gaddum-like relations are presented for the total domination number of middle graphs.

Keywords: total domination number, middle graph, Nordhaus-Gaddum-like relation
Mathematics Subject Classification: 05C69
DOI: 10.5614/ejgta.2022.10.1.19

1. Introduction

The concept of total domination in graph theory was first introduced by Cockayne, Dawes and Hedetniemi in [3] and it has been studied extensively by many researchers in the last years, see for example [5], [6], [7], [13], [8], [10], [12] and [14]. The literature on this subject has been surveyed and detailed in the recent book [7]. We refer to [2] as a general reference on graph theory.

Received: 30 September 2020, Revised: 6 April 2022, Accepted: 11 April 2022.

Let G be a graph with vertex set $V(G)$ of *order* n and edge set $E(G)$ of *size* m. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V(G)$ are $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. For a connected graph G, the *degree* of a vertex v is $d_G(v) = |N_G(v)|$. The *distance* $d_G(v, w)$ in G of two vertices $v, w \in V(G)$ is the length of the shortest path connecting the two vertices. The *diameter* $\operatorname{diam}(G)$ of G is the shortest distance between any two vertices in G.

A *dominating set* of a graph G is a set $S \subseteq V(G)$ such that $N_G[v] \cap S \neq \emptyset$, for any vertex $v \in V(G)$. The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$.

Definition 1.1. *Let G be a graph with no isolated vertices. A* total dominating set *of G is a set $S \subseteq V(G)$ such that $N_G(v) \cap S \neq \emptyset$, for any vertex $v \in V(G)$. The* total domination number *of G is the minimum cardinality of a total dominating set of G and is denoted by $\gamma_t(G)$.*

Example 1.2. *Consider the path P_3 with vertex set $\{v_1, v_2, v_3\}$ and edge set $\{v_1v_2, v_2v_3\}$. Then the set $S = \{v_1, v_2\}$ is a total dominating set of P_3.*

For any non-empty $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced on S. For any $v \in V(G)$, we denote by $G \setminus v$ the subgraph of G induced on $V(G) \setminus \{v\}$.

The *complement* $\overline{G}$ of G is a graph with vertex set $V(G)$ such that for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$.

The *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, where vertices x and y are adjacent in $L(G)$ if and only if edges x and y share a common vertex in G.

In [4], the authors introduced the notion of the middle graph $M(G)$ of G as an intersection graph on $V(G)$.

Definition 1.3. *The* middle graph $M(G)$ *of a graph $G = (V, E)$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds:*

1. *x, y are in $E(G)$ and x, y are adjacent in G.*
2. *x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.*

Example 1.4. *Consider the graph P_3, then the middle graph $M(P_3)$ is the one in Figure 1.*

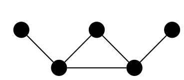

Figure 1. The middle graph $M(P_3)$.

It is easy to see that $M(G)$ contains the line graph $L(G)$ as induced subgraph, and that if G is a graph of order n and size m, then $M(G)$ has order $n + m$ and size $2m + |E(L(G))|$, and it is obtained by subdividing each edge of G exactly once and joining all the adjacent edges of G in $M(G)$.

In order to avoid confusion, we fix a "standard" notation for $V(M(G))$ and $E(M(G))$. Fix $V(G) = \{v_1, v_2, \ldots, v_n\}$, then $V(M(G)) = V(G) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{ij} \mid v_iv_j \in E(G)\}$ and $E(M(G)) = \{v_im_{ij}, v_jm_{ij} \mid v_iv_j \in E(G)\} \cup E(L(G))$.

In this article, we continue our study from [9] on domination of middle graphs. The paper proceeds as follows. In Section 2, we describe explicitly the total domination number of the middle graph of several known families of graphs and some upper and lower bounds for $\gamma_t(M(G))$ in terms of the order of G. In Section 3, we describe bounds for the total domination number of the middle graph of trees. In Section 4, we obtain the same type of results for $\gamma_t(M(G \circ K_1))$, $\gamma_t(M(G \circ P_2))$ and $\gamma_t(M(G + K_p))$. We conclude the paper discussing some Nordhaus-Gaddum like relations for the total domination number of middle graphs.

2. Middle graph of known graphs and their total domination number

We start our study on total domination with two easy Lemmas.

Lemma 2.1. *Let G be a connected graph of order $n \geq 3$ and S a total dominating set of $M(G)$. Then there exists $S' \subseteq E(G)$ a total dominating set of $M(G)$ with $|S'| \leq |S|$.*

Proof. If $S \subseteq E(G)$, then take $S' = S$. We can then assume that there exists $v \in S \cap V(G)$. If all edges adjacent to v are already in S, then take $S_1 = S \setminus \{v\}$. Otherwise, let $e \in E(G) \setminus S$ be an edge adjacent to v, and consider $S_1 = (S \cup \{e\}) \setminus \{v\}$. Since S is a finite set, then this process must terminate after a finite number of steps, and hence we obtain the described S'. □

Lemma 2.2. *Let G be a connected graph of order $n \geq 2$ and $v \in V(G)$ a vertex not adjacent to any vertex of degree* 1*. Then*

$$\gamma_t(M(G \setminus v)) \leq \gamma_t(M(G)) \leq \gamma_t(M(G \setminus v)) + 1.$$

Proof. Let S be a total dominating set of $M(G\setminus v)$. This implies that for every $w \in N_G(v)$, $w \in S$ or there exists an edge of the form $ww_0 \in E(G\setminus v)$ such that $ww_0 \in S$. As a consequence, $S \cup \{vw\}$ is a total dominating set of $M(G)$, for any $w \in N_G(v)$, and hence $\gamma_t(M(G)) \leq \gamma_t(M(G \setminus v)) + 1$.

On the other hand, let S be a minimal total dominating set of $M(G)$. By Lemma 2.1, we can assume that $S \subseteq E(G)$. Consider $S_v = N_{M(G)}(v) \cap S$. Since S is a minimal total dominating set, $|S_v| \geq 1$. Assume $S_v = \{e_1, \ldots, e_k\}$. For any $1 \leq i \leq k$, e_i is an edge of G of the form w_iv. By the assumption on v, $N_{M(G)}(w_1) = \{e_1, e_{11}, \ldots, e_{1p}\}$ with $p \geq 1$, for some $e_{1j} \in E(G \setminus v)$. If $S \cap \{e_{11}, \ldots, e_{1p}\} \neq \emptyset$, then consider $S_1 = (S \setminus e_1) \cup \{w_1\}$, otherwise $S_1 = (S \setminus e_1) \cup \{e_{11}\}$. By applying the same construction for each e_i, we obtain S_k a total dominating set of $M(G \setminus v)$ with $|S_k| = |S|$, and hence $\gamma_t(M(G \setminus v)) \leq \gamma_t(M(G))$. □

We are now ready to describe explicitly the total dominating number of the middle graph of several known families of graphs.

Proposition 2.3. *For any star graph $K_{1,n}$ on $n+1$ vertices, with $n \geq 2$, we have*

$$\gamma_t(M(K_{1,n})) = n.$$

Proof. Fix $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$ and $E(K_{1,n}) = \{v_0v_1, v_0v_2, \ldots, v_0v_n\}$. Then, $V(M(K_{1,n})) = V(K_{1,n}) \cup \mathcal{M}$, where $\mathcal{M} = \{m_i \mid 1 \leq i \leq n\}$.

If $S = \mathcal{M}$, then S is a total dominating set of $M(K_{1,n})$ with $|S| = n$, and hence $\gamma_t(M(K_{1,n})) \leq n$. On the other hand, using [9, Proposition 3.1], $n = \gamma(M(K_{1,n})) \leq \gamma_t(M(K_{1,n}))$. □

Proposition 2.4. *For any double star graph $S_{1,n,n}$ on $2n+1$ vertices, with $n \geq 1$, we have*

$$\gamma_t(M(S_{1,n,n})) = 2n.$$

Proof. Fix $V(S_{1,n,n}) = \{v_0, v_1, \ldots, v_{2n}\}$ and $E(S_{1,n,n}) = \{v_0v_i, v_iv_{n+i} \mid 1 \leq i \leq n\}$. Then $V(M(S_{1,n,n})) = V(S_{1,n,n}) \cup \mathcal{M}$, where $\mathcal{M} = \{m_i, m_{i(n+i)} \mid 1 \leq i \leq n\}$.

Since $S = \mathcal{M}$ is a total dominating set of $M(S_{1,n,n})$ with $|S| = 2n$, then $\gamma_t(M(S_{1,n,n})) \leq 2n$.

On the other hand, let S be a total dominating set $M(S_{1,n,n})$. By Lemma 2.1, we can assume that $S \subseteq \mathcal{M}$. Since, for every $1 \leq i \leq n$, $N_{M(S_{1,n,n})}(v_{n+i}) = \{m_{i(n+i)}\}$, then $m_{i(n+i)} \in S$ for every $1 \leq i \leq n$. Similarly, for every $1 \leq i \leq n$, $N_{M(S_{1,n,n})}(m_{i(n+i)}) = \{m_i, v_i, v_{n+i}\}$ implies that $m_i \in S$ for every $1 \leq i \leq n$, and hence $\mathcal{M} \subseteq S$. This implies that $\gamma_t(M(S_{1,n,n})) \geq 2n$. □

Proposition 2.5. *For any path P_n of order $n \geq 3$,*

$$\gamma_t(M(P_n)) = \lceil \frac{2n}{3} \rceil.$$

Proof. Fix $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$. Then $V(M(P_n)) = V \cup \mathcal{M}$ where $V = V(P_n)$ and $\mathcal{M} = \{m_{i(i+1)} \mid 1 \leq i \leq n-1\}$.

If $n \equiv 0 \mod 3$, then consider

$$S = \{m_{12}, m_{23}, m_{45}, m_{56}, \ldots, m_{(n-2)(n-1)}, m_{(n-1)n}\}.$$

We have that S is a total dominating set of $M(P_n)$ with $|S| = \frac{2n}{3}$. If $n \equiv 1 \mod 3$, then consider

$$S = \{m_{12}, m_{23}, m_{45}, m_{56}, \ldots, m_{(n-3)(n-2)}, m_{(n-2)(n-1)}\} \cup \{m_{(n-1)n}\}.$$

We have that S is a total dominating set of $M(P_n)$ with $|S| = \lceil \frac{2n}{3} \rceil$. If $n \equiv 2 \mod 3$, then consider

$$S = \{m_{12}, m_{23}, m_{45}, m_{56}, \ldots, m_{(n-4)(n-3)}, m_{(n-3)(n-2)}\} \cup \{m_{(n-2)(n-1)}, m_{(n-1)n}\}.$$

We have that S is a total dominating set of $M(P_n)$ with $|S| = \lceil \frac{2n}{3} \rceil$. This implies $\gamma_t(M(P_n)) \leq \lceil \frac{2n}{3} \rceil$.

On the other hand, let S be a total dominating set for $M(P_n)$. For every $i = 1, \ldots, n-2$, let $G_i = P_n[v_i, v_{i+1}, v_{i+2}]$. Since S dominates all vertices of the graph $M(G_i)$, $|S \cap V(M(G_i))| \geq 2$. This implies that $|S| \geq \lceil \frac{2n}{3} \rceil$. □

Since if we delete a vertex from a complete graph K_{n+1} we obtain a graph isomorphic to K_n, Lemma 2.2 gives us the following result.

Lemma 2.6. *For any $n \geq 3$, we have*

$$\gamma_t(M(K_n)) \leq \gamma_t(M(K_{n+1})) \leq \gamma_t(M(K_n)) + 1.$$

Proposition 2.7. *Let K_n be the complete graph on $n \geq 2$ vertices. Then*

$$\gamma_t(M(K_n)) = \lceil \frac{2n}{3} \rceil$$

Proof. If $2 \leq n \leq 4$, a direct computation shows that $\gamma_t(M(K_n)) = \lceil \frac{2n}{3} \rceil$. Assume now $n \geq 5$. The graph K_n has several subgraphs isomorphic to P_n, and hence $M(K_n)$ has subgraphs isomorphic to $M(P_n)$. Fix one of those and consider S a total dominating set of $M(P_n)$. Since S is also a total dominating set for $M(K_n)$, we have $\gamma_t(M(K_n)) \leq \lceil \frac{2n}{3} \rceil$.

We will prove the opposite inequality by induction. Assume that we have equality for $\gamma_t(M(K_n))$ and we want to prove it for $\gamma_t(M(K_{n+1}))$. If $n \equiv 2 \mod 3$, then $n+1 \equiv 0 \mod 3$, and hence, $\gamma_t(M(K_n)) = \lceil \frac{2n}{3} \rceil = \lceil \frac{2(n+1)}{3} \rceil$. On the other hand, by Lemma 2.6, $\gamma_t(M(K_n)) \leq \gamma_t(M(K_{n+1}))$. This fact, together with the first part of the proof, implies that $\gamma_t(M(K_{n+1})) = \lceil \frac{2(n+1)}{3} \rceil$. If $n \equiv 0, 1 \mod 3$, by Lemma 2.6 and the first part of the proof, it is enough to show that $\gamma_t(M(K_n)) < \gamma_t(M(K_{n+1}))$. As a contradiction, assume that $\gamma_t(M(K_n)) = \gamma_t(M(K_{n+1}))$. If $n \equiv 0 \mod 3$, then $n-1 \equiv 2 \mod 3$, and hence this would implies $\gamma_t(M(K_{n-1})) = \gamma_t(M(K_n)) = \gamma_t(M(K_{n+1}))$. Similarly, if $n \equiv 1 \mod 3$, then $n+1 \equiv 2 \mod 3$, and hence $\gamma_t(M(K_n)) = \gamma_t(M(K_{n+1})) = \gamma_t(M(K_{n+2}))$. Hence we need to show that $\gamma_t(M(K_n)) < \gamma_t(M(K_{n+2}))$, when $n \geq 4$. Let S be a total dominating set of $M(K_{n+2})$. To fix the notation, assume $V(K_{n+2}) = \{v_1, \dots, v_{n+2}\}$ and $V(M(K_{n+2})) = V(K_{n+2}) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{ij} \mid 1 \leq i < j \leq n+2\}$. By Lemma 2.1, we can assume that $S \subseteq \mathcal{M}$. After possibly relabeling $V(K_{n+2})$, we can assume that $m_{(n+1)(n+2)} \in S$. Since S is a total dominating set of $M(K_{n+2})$, then it contains at least one element of the form $m_{i(n+1)}$ or $m_{i(n+2)}$, for some $i = 1, \dots, n$. By construction, $M(K_n)$ is isomorphic to $M(K_{n+2}[v_1, \dots, v_n])$, this implies that, similarly to the proof of Lemma 2.6, we can construct S' a total dominating set of $M(K_n)$ by exchanging a vertex of the form $m_{i(n+1)}$ or $m_{i(n+2)}$ with one of the form m_{ij} and just discarding $m_{(n+1)(n+2)}$. This implies that $|S'| < |S|$, and hence $\gamma_t(M(K_n)) < \gamma_t(M(K_{n+2}))$. □

Theorem 2.8. *Let G be any graph of order n. Then*

$$\lceil \frac{2n}{3} \rceil \leq \gamma_t(M(G)) \leq n-1$$

Proof. From G we can obtain graph isomorphic to K_n by adding all the necessary edges. This implies that we can see G as a subgraph of K_n, and hence $M(G)$ as a subgraph of $M(K_n)$. Since any total dominating set of $M(G)$ is also a total dominating set for $M(K_n)$, this implies that $\gamma_t(M(G)) \geq \gamma_t(M(K_n))$. We obtain the left inequality by Proposition 2.7.

Consider T a spanning tree of G and S a minimal total dominating set of $M(T)$. By Lemma 2.1, we can assume that $S \subseteq E(T)$. This implies that $|S| \leq |E(T)| = n-1$. Since S is also a total dominating set of $M(G)$, then $\gamma_t(M(G)) \leq n-1$. □

Remark 2.9. *By Propositions 2.3 and 2.5, the inequalities of Theorem 2.8 are all sharp.*

Theorem 2.10. *If G is a graph with order n and there exists a subgraph of G isomorphic to P_n, then*

$$\gamma_t(M(G)) = \lceil \frac{2n}{3} \rceil.$$

Proof. Since G has a subgraph isomorphic to P_n, then $M(G)$ has a subgraph isomorphic to $M(P_n)$. Moreover, any total dominating set of $M(P_n)$ is also a total dominating set for $M(G)$. By Proposition 2.5, this implies that $\gamma_t(M(K_n)) \leq \lceil \frac{2n}{3} \rceil$. We conclude by Theorem 2.8. □

Directly from Theorem 2.10, we obtain the following result.

Corollary 2.11. *For any $n \geq 3$,*

$$\gamma_t(M(P_n)) = \gamma_t(M(C_n)) = \gamma_t(M(W_n)) = \gamma_t(M(K_n)) = \lceil \frac{2n}{3} \rceil.$$

Proposition 2.12. *Let F_n be the friendship graph with $n \geq 2$. Then*

$$\gamma_t(M(F_n)) = 2n.$$

Proof. Fix $V(F_n) = \{v_0, v_1, \ldots, v_{2n}\}$ and $E(F_n) = \{v_0v_1, v_0v_2, \ldots, v_0v_{2n}\} \cup \{v_1v_2, v_3v_4, \ldots, v_{2n-1}v_{2n}\}$. Then $V(M(F_n)) = V(F_n) \cup \mathcal{M}$, where $\mathcal{M} = \{m_i \mid 1 \leq i \leq 2n\} \cup \{m_{i(i+1)} \mid 1 \leq i \leq 2n-1 \text{ and } i \text{ is odd}\}$.

Since $S = \{m_{i(i+1)} \mid 1 \leq i \leq 2n-1 \text{ and } i \text{ is odd}\} \cup \{v_i \mid i \text{ is odd}\}$ is a total dominating set for $M(F_n)$ with $|S| = 2n$, then $\gamma_t(M(F_n)) \leq 2n$.

On the other hand, since F_n is obtained by joining n copies of C_3 at v_0, any total dominating set S of $M(F_n)$ induces a total dominating set of $M(C_3)$ as subgraph of $M(F_n)$. By Corollary 2.11, $\gamma_t(M(C_3)) = 2$. This fact together with the fact that any two distinct copies of $M(C_3)$ in $M(F_n)$ share only v_0 implies that $|S| \geq 2n$. As a consequence, $\gamma_t(M(F_n)) \geq 2n$. □

Using Theorem 2.8, we can describe the total domination number of the middle graph of a complete bipartite graph.

Proposition 2.13. *Let K_{n_1,n_2} be the complete bipartite graph with $n_2 \geq n_1 \geq 2$. Then*

$$\gamma_t(M(K_{n_1,n_2})) = \begin{cases} n_2 + \lceil \frac{2n_1-n_2}{3} \rceil, & \textit{if } n_1 \leq n_2 \leq 2n_1 - 1, \\ n_2, & \textit{if } n_2 \geq 2n_1. \end{cases}$$

Proof. Fix $V(K_{n_1,n_2}) = \{v_1, \ldots, v_{n_1}, u_1, \ldots, u_{n_2}\}$ and $E(K_{n_1,n_2}) = \{v_iu_j \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$. Then we have $V(M(K_{n_1,n_2})) = V(K_{n_1,n_2}) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{ij} \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$.

Assume first $n_1 = n_2$. If $n_1 \equiv 0 \mod 3$, then consider

$$S = \{m_{11}, m_{12}, m_{23}, m_{33}, \ldots, m_{(n_1-1)n_1}, m_{n_1n_1}\}.$$

By construction, S is a total dominating set of $M(K_{n_1,n_2})$ and $|S| = n_1 + \frac{n_1}{3} = n_2 + \frac{n_1}{3} = n_2 + \lceil \frac{2n_1-n_2}{3} \rceil$. If $n_1 \equiv 1 \mod 3$, then consider

$$S = \{m_{11}, m_{12}, m_{23}, m_{33}, \ldots, m_{(n_1-2)(n_1-1)}, m_{(n_1-1)(n_1-1)}\} \cup \{m_{n_1(n_1-1)}, m_{n_1n_1}\}.$$

By construction, S is a total dominating set of $M(K_{n_1,n_2})$ and $|S| = n_1 + \lceil\frac{n_1}{3}\rceil = n_2 + \lceil\frac{n_1}{3}\rceil = n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$. If $n_1 \equiv 2 \mod 3$, then consider

$$S = \{m_{11}, m_{12}, m_{23}, m_{33}, \ldots, m_{(n_1-1)(n_1-1)}, m_{(n_1-1)n_1}\} \cup \{m_{n_1n_1}\}.$$

By construction, S is a total dominating set of $M(K_{n_1,n_2})$ and $|S| = n_1 + \lceil\frac{n_1}{3}\rceil = n_2 + \lceil\frac{n_1}{3}\rceil = n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$.

Assume that $n_1 + 1 \leq n_2 \leq 2n_1 - 1$. Consider

$$S' = \{m_{11}, m_{1n_1+1}, \ldots, m_{(n_2-n_1)(n_2-n_1)}, m_{(n_2-n_1)n_2}\}.$$

Let $G = K_{n_1,n_2}[u_{n_2-n_1+1}, \ldots, u_{n_1}, v_{n_2-n_1+1}, \ldots, v_{n_1}]$. Then G is isomorphic to a graph of the form $K_{n,n}$, where $n = 2n_1 - n_2$. This implies that by the first part of the proof, we can construct S'' a total dominating set of $M(G)$ with $|S''| = 2n_1 - n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$. Consider $S = S' \cup S''$. Then S is a total dominating set of $M(K_{n_1,n_2})$ and $|S| = 2(n_2-n_1) + 2n_1 - n_2 + \lceil\frac{2n_1-n_2}{3}\rceil = n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$.

This implies that if $n_1 \leq n_2 \leq 2n_1 - 1$, then $\gamma_t(M(K_{n_1,n_2})) \leq n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$.

Assume now that $n_2 \geq 2n_1$. Consider

$$S = \{m_{11}, m_{1n_1+1}, \ldots, m_{n_1n_1}, m_{n_12n_1}\} \cup \{m_{n_12n_1+1}, \ldots, m_{n_1n_2}\},$$

then S is a total dominating set of $M(K_{n_1,n_2})$ with $|S| = n_2$, and hence, $\gamma_t(M(K_{n_1,n_2})) \leq n_2$.

On the other hand, assume first $n_1 = n_2$. By Theorem 2.8, we have $\gamma_t(M(K_{n_1,n_2})) \geq \lceil\frac{2(n_1+n_2)}{3}\rceil = n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$.

Assume that $n_1 + 1 \leq n_2 \leq 2n_1 - 1$. Let S be a total dominating set of $M(K_{n_1,n_2})$. By Lemma 2.1, we can assume that $S \subseteq \mathcal{M}$. The construction of the first part of the proof is optimal since S' and S'' have the smallest possible size by the argument discussed when $n_1 = n_2$ and $n_2 \geq 2n_1$.

This implies that if $n_1 \leq n_2 \leq 2n_1 - 1$, then $\gamma_t(M(K_{n_1,n_2})) = n_2 + \lceil\frac{2n_1-n_2}{3}\rceil$.

Assume that $n_2 \geq 2n_1$, then by [9, Proposition 3.13], we have $n_2 = \gamma(M(K_{n_1,n_2})) \leq \gamma_t(M(K_{n_1,n_2})) \leq n_2$. This implies that $\gamma_t(M(K_{n_1,n_2})) = n_2$.

□

3. The middle graph of a tree

Similarly to [9, Proposition 2.4], if we consider T a tree and we denote by $\mathrm{leaf}(T) = \{v \in V(T) \mid d_T(v) = 1\}$ the set of leaves of T, then we have the following result.

Proposition 3.1. *Let T be a tree with $n \geq 2$ vertices. Then*

$$\gamma_t(M(T)) \geq |\,\mathrm{leaf}(T)|.$$

Proof. Fix $\mathrm{leaf}(T) = \{v_1, \ldots, v_k\}$, for some $k \leq n$. If $n = 2$, then T is isomorphic to P_2 and hence $\gamma_t(M(T)) = 2 = |\,\mathrm{leaf}(T)|$. Assume that $n \geq 3$ and let S be a total dominating set of $M(T)$. Then, for each $i = 1, \ldots, k$, $S \cap N_{M(T)}[v_i] \neq \emptyset$. Since, if $i \neq j$, then $N_{M(T)}[v_j] \cap N_{M(T)}[v_i] = \emptyset$, we have that $|S| \geq k$. As a consequence, $\gamma_t(M(T)) \geq k = |\,\mathrm{leaf}(T)|$.

□

Remark 3.2. *Notice that by Proposition 2.3, the inequality described in Proposition 3.1 is sharp.*

It is sufficient to add some assumptions on the diameter of a tree T, to compute $\gamma_t(M(T))$ explicitly.

Theorem 3.3. *Let T be a tree of order $n \geq 4$ with $\operatorname{diam}(T) = 3$. Then*

$$\gamma_t(M(T)) = \begin{cases} n-2, & \textit{if there are two vertices with } d_T(v) \geq 3, \\ n-1, & \textit{otherwise.} \end{cases}$$

Proof. The assumption that $\operatorname{diam}(T) = 3$ implies that T is a tree which is obtained by joining central vertex v of $K_{1,p}$ and the central vertex w of $K_{1,q}$ where $p + q = n - 2$. Let $\operatorname{leaf}(T) = \{v_i \mid 1 \leq i \leq n-2\}$ be the set of leaves of T. Obviously $V(T) = \operatorname{leaf}(T) \cup \{v, w\}$ and $|\operatorname{leaf}(T)| = n - 2$. Define $v_{n-1} = v$ and $v_n = w$.

Assume first that $p, q \geq 2$, i.e. there are two vertices with $d_T(u) \geq 3$. Since $S = \{m_{i(n-1)} \mid 1 \leq i \leq p\} \cup \{m_{in} \mid p+1 \leq i \leq n-2\}$ is a total dominating set of $M(T)$ with $|S| = n-2$, then $\gamma_t(M(T)) \leq n-2$. On the other hand, by Proposition 3.1, we have $\gamma_t(M(T)) \geq n-2$.

Assume that $p \geq 2$ and $q = 1$, i.e. there is only one vertex with $d_T(u) \geq 3$. Let S be a total dominating set of $M(T)$. By Lemma 2.1, we can assume that $S \subseteq E(T)$. Since $N_{M(T)}(v_i) = \{m_{i(n-1)}\}$ for all $1 \leq i \leq p = n-3$ and $N_{M(T)}(v_{n-2}) = \{m_{(n-2)n}\}$, then $\{m_{i(n-1)} \mid 1 \leq i \leq p\} \cup \{m_{(n-2)n}\} \subseteq S$. Moreover, $N_{M(T)}(m_{(n-2)n}) = \{m_{(n-1)n}, v_n, v_{n-2}\}$ implies that $m_{(n-1)n} \in S$. This implies that $|S| \geq n-1$, and hence $\gamma_t(M(T)) \geq n-1$. On the other hand, by Theorem 2.8, $\gamma_t(M(T)) \leq n-1$.

Assume that $p, q = 1$, i.e. there are no vertices with $d_T(u) \geq 3$. This implies that T is isomorphic to P_4 and $n = 4$, and hence by Proposition 2.5, $\gamma_t(M(T)) = 3 = n-1$. $\square$

In general, the opposite implication of Theorem 3.3 does not hold as the next example shows.

Example 3.4. *Let T be the tree in Figure 2. Then a direct computation shows that $\operatorname{diam}(T) = 4$ and $\gamma_t(M(T)) = 5 = n - 2$.*

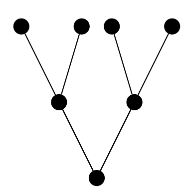

Figure 2. A tree on 7 vertices.

Proposition 3.5. *Let T be a tree of order $n \geq 3$ with $\operatorname{diam}(T) = 2$. Then*

$$\gamma_t(M(T)) = n - 1.$$

Proof. The assumption that $\operatorname{diam}(T) = 2$ implies that T is isomorphic to $K_{1,n-1}$. As a consequence, by Proposition 2.3, that $\gamma_t(M(T)) = n - 1$. $\square$

Remark 3.6. *By the proof of Theorem 3.3, differently from the case of domination (see [9, Theorem 3.2]), $\gamma_t(M(G)) = n - 1$ does not implies that G is isomorphic to $K_{1,n-1}$.*

4. Operations on graphs

In this section, similarly to [9], we study the total domination number of the middle graph of the corona, 2-corona and join with K_p of a graph.

Definition 4.1. *The* corona $G \circ K_1$ *of a graph* G *is the graph of order* $2|V(G)|$ *obtained from* G *by adding a pendant edge to each vertex of* G.

Example 4.2. *Consider the graph* P_3*, then the graph* $P_3 \circ K_1$ *is the one in Figure 3.*

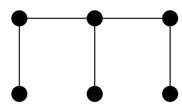

Figure 3. The graph $P_3 \circ K_1$.

Theorem 4.3. *For any connected graph* G *of order* $n \geq 2$*,*

$$\gamma_t(M(G \circ K_1)) = n + \gamma(M(G)).$$

Proof. Fix $V(G) = \{v_1, \dots, v_n\}$. Then $V(G \circ K_1) = \{v_1, \dots, v_{2n}\}$ and $E(G \circ K_1) = \{v_1 v_{n+1}, \dots, v_n v_{2n}\} \cup E(G)$. Then $V(M(G \circ K_1)) = V(G \circ K_1) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{i(n+i)} \mid 1 \leq i \leq n\} \cup \{m_{ij} \mid v_i v_j \in E(G)\}$.

Let S' be a minimal dominating set of $M(G)$. By construction, $S = S' \cup \{m_{i(n+i)} \mid 1 \leq i \leq n\}$ is a total dominating set of $M(G \circ K_1)$ with $|S| = n + \gamma(M(G))$. This implies that $\gamma_t(M(G \circ K_1)) \leq n + \gamma(M(G))$.

On the other hand, let S be a total dominating set of $M(G \circ K_1)$. By Lemma 2.1, we can assume that $S \subseteq \mathcal{M}$. Since $N_{M(G \circ K_1)}(v_{n+i}) = \{m_{i(n+i)}\}$, for all $1 \leq i \leq n$, then $m_{i(n+i)} \in S$, for all $1 \leq i \leq n$. In addition, $N_{M(G \circ K_1)}(m_{i(n+i)}) = \{v_i, v_{n+i}\} \cup N_{M(G)}(v_i)$, for all $1 \leq i \leq n$, then $N_{M(G)}(v_i) \cap S \neq \emptyset$, for all $1 \leq i \leq n$. As a consequence, $S \cap E(G)$ is a dominating set of $M(G)$ and hence $|S| \geq n + \gamma(M(G))$. As a consequence, $\gamma_t(M(G \circ K_1)) \geq n + \gamma(M(G))$. $\square$

Definition 4.4. *The* 2-corona $G \circ P_2$ *of a graph* G *is the graph of order* $3|V(G)|$ *obtained from* G *by attaching a path of length* 2 *to each vertex of* G *so that the resulting paths are vertex-disjoint.*

Example 4.5. *Consider the graph* P_3*, then the graph* $P_3 \circ P_2$ *is the one in Figure 4.*

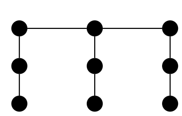

Figure 4. The graph $P_3 \circ P_2$

Theorem 4.6. *For any connected graph G of order $n \geq 2$,*

$$\gamma_t(M(G \circ P_2)) = 2n.$$

Proof. Fix $V(G) = \{v_1, \dots, v_n\}$. Then $V(G \circ P_2) = \{v_1, \dots, v_{3n}\}$ and $E(G \circ P_2) = \{v_i v_{n+i}, v_{n+i}v_{2n+i} \mid 1 \leq i \leq n\} \cup E(G)$. Then $V(M(G \circ P_2)) = V(G \circ P_2) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{i(n+i)}, m_{(n+i)(2n+i)} \mid 1 \leq i \leq n\} \cup \{m_{ij} \mid v_i v_j \in E(G)\}$.

Let S be a total dominating set of $M(G \circ P_2)$. By Lemma 2.1, we can assume that $S \subseteq \mathcal{M}$. Since $N_{M(G \circ P_2)}(v_{2n+i}) = \{m_{(n+i)(2n+i)}\}$, for every $1 \leq i \leq n$, we have $m_{(n+i)(2n+i)} \in S$ for every $1 \leq i \leq n$. In addition, $N_{M(G \circ P_2)}(m_{(n+i)(2n+i)}) = \{m_{i(n+i)}, v_{2n+i}, v_{n+i}\}$, for every $1 \leq i \leq n$, implies that $m_{i(n+i)} \in S$, for every $1 \leq i \leq n$. This implies that $|S| \geq 2n$, and hence $\gamma_t(M(G \circ P_2)) \geq 2n$.

On the other hand, $S = \{m_{i(n+i)}, m_{(n+i)(2n+i)} \mid 1 \leq i \leq n\}$ is a total dominating set of $M(G \circ P_2)$ with $|S| = 2n$. This implies that $\gamma_t(M(G \circ P_2)) \leq 2n$. □

Definition 4.7. *The* join $G + H$ *of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{vw \mid v \in V(G), w \in V(H)\}$.*

Example 4.8. *Consider the graphs $G = K_3$ and $H = P_2$, then graph $G + H$ is the one in Figure 5.*

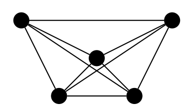

Figure 5. The graph $K_3 + P_2$.

Theorem 4.9. *For any connected graph G of order $n \geq 2$,*

$$\gamma_t(M(G + \overline{K_p})) = \begin{cases} p, & \text{if } p \geq 2n, \\ \lceil \frac{2(n+p)}{3} \rceil, & \text{if } \frac{n}{2} \leq p \leq 2n - 1. \end{cases}$$

Proof. Fix $V(G) = \{v_1, \dots, v_n\}$ and $V(\overline{K_p}) = \{v_{n+1}, \dots, v_{n+p}\}$. Then $V(M(G + \overline{K_p})) = V(G + \overline{K_p}) \cup \mathcal{M}_1 \cup \mathcal{M}_2$ where $\mathcal{M}_1 = \{m_{ij} \mid v_i v_j \in E(G)\}$ and $\mathcal{M}_2 = \{m_{i(n+j)} \mid 1 \leq i \leq n, 1 \leq j \leq p\}$.

Case $p \geq 2n$. Let S be a total dominating set of $M(G + \overline{K_p})$. By Lemma 2.1, we can assume $S \subseteq \mathcal{M}_1 \cup \mathcal{M}_2$. Since, if $j \neq k$, $N_{M(G+\overline{K_p})}(v_{n+j}) \cap N_{M(G+\overline{K_p})}(v_{n+k}) = \emptyset$, then for every $1 \leq j \leq p$ there exists $1 \leq i \leq n$ such that $m_{i(n+j)} \in S$, and hence $|S| \geq p$. As a consequence, $\gamma_t(M(G + \overline{K_p})) \geq p$. On the other hand, $S = \{m_{i(n+i)}, m_{i(2n+i)} \mid 1 \leq i \leq n\} \cup \{m_{1(3n+i)} \mid 1 \leq i \leq p - 2n\}$ is a total dominating set of $M(G + \overline{K_p})$ with $|S| = p$. This implies that $\gamma_t(M(G + \overline{K_p})) \leq p$.

Case $p = 2n - 1$. Since $S = \{m_{i(n+i)}, m_{i(2n+i)} \mid 1 \leq i \leq n - 1\} \cup \{m_{n(2n)}, m_{n(2n-1)}\}$ is a total dominating set of $M(G + \overline{K_{2n-1}})$ with $|S| = 2n$, then $\gamma_t(M(G + \overline{K_{2n-1}})) \leq 2n$. On the other hand, by Theorem 2.8, $\gamma_t(M(G + \overline{K_{2n-1}})) \geq \lceil \frac{2(3n-1)}{3} \rceil = 2n$, and hence $\gamma_t(M(G + \overline{K_{2n-1}})) = 2n = \lceil \frac{2(n+p)}{3} \rceil$.

Case $n + 3 \leq p \leq 2n - 2$. Assume that $p = n + k$ with $3 \leq k \leq n - 2$. The graph $G + \overline{K_p}$ has k subgraphs isomorphic to P_3 and one subgraph isomorphic to $P_{2(n-k)}$ that are all disjoint. In fact, the k subgraphs $(G+\overline{K_p})[v_1, v_{n+1}, v_{2n+1}], \ldots, (G+\overline{K_p})[v_k, v_{n+k}, v_{2n+k}]$ are all isomorphic to P_3 and the subgraph $(G+\overline{K_p})[v_{k+1}, \ldots, v_n, v_{n+k+1}, \ldots, v_{2n}]$ has a subgraph isomorphic to $P_{2(n-k)}$. By Proposition 2.5, $\gamma_t(M(G+\overline{K_p})) \leq 2k + \lceil \frac{2(2(n-k))}{3} \rceil = \lceil \frac{2(n+p)}{3} \rceil$. By Theorem 2.8, we obtain the desired equality.

Case $p = n+2$. If $n \equiv 0 \mod 3$, consider

$$S = \{m_{1(n+1)}, m_{1(n+2)}, m_{2(n+3)}, m_{3(n+3)}, \ldots, m_{(n-1)(2n)}, m_{n(2n)}, m_{n(2n+1)}, m_{n(2n+2)}\}.$$

Then S is a total dominating set of $M(G+\overline{K_p})$ with $|S| = \lceil \frac{2(n+p)}{3} \rceil$. If $n \equiv 1 \mod 3$, consider

$$S = \{m_{1(n+1)}, m_{1(n+2)}, m_{2(n+3)}, m_{3(n+3)}, \ldots, m_{n(2n)}, m_{n(2n+1)}, m_{n(2n+2)}\}.$$

Then S is a total dominating set of $M(G+\overline{K_p})$ with $|S| = \lceil \frac{2(n+p)}{3} \rceil$. If $n \equiv 2 \mod 3$, consider

$$S = \{m_{1(n+1)}, m_{1(n+2)}, m_{2(n+3)}, m_{3(n+3)}, \ldots, m_{n(2n+1)}, m_{n(2n+2)}\}.$$

Then S is a total dominating set of $M(G+\overline{K_p})$ with $|S| = \lceil \frac{2(n+p)}{3} \rceil$. This implies that $\gamma_t(M(G+\overline{K_p})) \leq \lceil \frac{2(n+p)}{3} \rceil$. By Theorem 2.8, we then obtain that $\gamma_t(M(G+\overline{K_p})) = \lceil \frac{2(n+p)}{3} \rceil$.

Case $n-1 \leq p \leq n+1$. If $p = n-1$, then the graph $G+\overline{K_p}$ contains the path $P : v_1 v_{n+1} v_2 v_{n+2} \cdots v_{n+p} v_n$. If $p = n$, then $G+\overline{K_p}$ contains the path $P' : v_1 v_{n+1} v_2 v_{n+2} \cdots v_{n+p-1} v_n v_{n+p}$. If $p = n+1$, then $G+\overline{K_p}$ contains the path $P'' : v_{n+1} v_1 v_{n+2} v_2 v_{n+3} \cdots v_{n+p-1} v_n v_{n+p}$. Since the paths P, P' and P'' are all isomorphic to P_{n+p}, we can apply Theorem 2.10, and obtain that $\gamma_t(M(G+\overline{K_p})) = \lceil \frac{2(n+p)}{3} \rceil$.

Case $\frac{n}{2} \leq p \leq n-2$. Assume that $p = n-k$ with $2 \leq k \leq \frac{n}{2}$. If n is even and $p = \frac{n}{2}$ (or equivalently $k = \frac{n}{2}$), then the set $S = \{m_{i(n+i)}, m_{(i+\frac{n}{2})(n+i)} \mid 1 \leq i \leq \frac{n}{2}\}$ is a total dominating set of $M(G+\overline{K_p})$ with $|S| = n = \lceil \frac{2(n+p)}{3} \rceil$. As a consequence, $\gamma_t(M(G+\overline{K_p})) \leq \lceil \frac{2(n+p)}{3} \rceil$, and by Theorem 2.8, we obtain the desired equality.

Assume that $2 \leq k \leq \frac{n}{2} - 1$. The graph $G+\overline{K_p}$ has k subgraphs isomorphic to P_3 and one subgraph isomorphic to $P_{2(n-2k)}$ that are all disjoint. In fact, the k induced subgraphs $(G+\overline{K_p})[v_1, v_{n+1}, v_{k+1}], \ldots, (G+\overline{K_p})[v_k, v_{n+k}, v_{2k}]$ are all isomorphic to P_3 and the induced subgraph $(G+\overline{K_p})[v_{2k+1}, \ldots, v_n, v_{n+k+1}, \ldots, v_{2n-k}]$ has a subgraph isomorphic to $P_{2(n-2k)}$. By Proposition 2.5, this implies that $\gamma_t(M(G+\overline{K_p})) \leq 2k + \lceil \frac{2(2(n-2k))}{3} \rceil = \lceil \frac{2(n+p)}{3} \rceil$. By Theorem 2.8, we obtain the desired equality. □

Similarly to [9], when p is small relatively to n, $\gamma_t(M(G+\overline{K_p}))$ is strongly related to $\gamma_t(M(G))$.

Theorem 4.10. *For any connected graph G of order $n \geq 2$ and any integer $1 \leq p \leq \frac{n}{2} - 1$,*

$$\lceil \frac{2(n+p)}{3} \rceil \leq \gamma_t(M(G+\overline{K_p})) \leq$$
$$2p + \min\{\gamma_t(M(G[A])) \mid A \subseteq V(G),$$
$$|A| = n - 2p, G[A] \textit{ has no isolated vertices}\}.$$

Proof. Fix $V(G) = \{v_1, \dots, v_n\}$ and $V(\overline{K_p}) = \{v_{n+1}, \dots, v_{n+p}\}$. Then $V(M(G+\overline{K_p})) = V(G+\overline{K_p}) \cup \mathcal{M}_1 \cup \mathcal{M}_2$ where $\mathcal{M}_1 = \{m_{ij} \mid v_iv_j \in E(G)\}$ and $\mathcal{M}_2 = \{m_{i(n+j)} \mid 1 \leq i \leq n, 1 \leq j \leq p\}$.

By Theorem 2.8, we obtain the first inequality. Let now $A \subseteq V(G)$ be a subset with $|A| = n - 2p$ and suppose $G[A]$ has no isolated vertices. Without loss of generalities, we can suppose that $A = \{v_{2p+1}, \dots, v_n\}$. Consider S' be a minimal total dominating set of $M(G[A])$, then $S = S' \cup \{m_{i(n+i)}, m_{(p+i)(n+i)} \mid 1 \leq i \leq p\}$ is a total dominating set of $M(G + \overline{K_p})$. Since this arguments works for every $A \subseteq V(G)$ such that $|A| = n - 2p$ and $G[A]$ has no isolated vertices, we obtain the second inequality. □

If we apply Lemma 2.2 to the graph $G + \overline{K_1}$, we obtain the following result.

Lemma 4.11. *Let G be a graph of order $n \geq 2$ with no isolated vertices. Then*

$$\gamma_t(M(G)) \leq \gamma_t(M(G + \overline{K_1})) \leq \gamma_t(M(G)) + 1.$$

Notice that both inequalities described in Lemma 4.11 are sharp as we can see form the following examples.

Example 4.12. *Consider the graph $G = C_5$. Then $G + \overline{K_1}$ is isomorphic to W_6. This implies that by Corollary 2.11, $\gamma_t(M(G)) = 4 = \gamma_t(M(G + \overline{K_1}))$.*

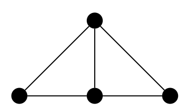

Figure 6. The graph $P_3 + \overline{K_1}$.

Example 4.13. *Consider the graph $G = P_3$. Then $G + \overline{K_1}$ is the graph in Figure 6. By Proposition 2.5 and Theorem 2.10, we have that $\gamma_t(M(P_3)) = 2$ and $\gamma_t(M(P_3 + \overline{K_1})) = 3$.*

Proposition 4.14. *For any star graph $K_{1,n}$ on $n + 1$ vertices, with $n \geq 4$, we have*

$$\gamma_t(M(K_{1,n} + \overline{K_1})) = n.$$

Proof. Fix $V(K_{1,n}) = \{v_0, v_1, \dots, v_n\}$, $V(\overline{K_1}) = \{v_{n+1}\}$ and $E(K_{1,n}) = \{v_0v_1, v_0v_2, \dots, v_0v_n\}$. Then $V(M(K_{1,n}+\overline{K_1})) = V(K_{1,n}) \cup \mathcal{M}$, where $\mathcal{M} = \{m_i \mid 1 \leq i \leq n\} \cup \{m_{i(n+1)} \mid 0 \leq i \leq n\}$.

By Proposition 2.3 and Lemma 4.11, $\gamma_t(M(K_{1,n} + \overline{K_1})) \geq n$. On the other hand, since $S = \{m_i \mid 1 \leq i \leq n-2\} \cup \{m_{(n-1)(n+1)}, m_{n(n+1)}\}$ is a total dominating set of $M(K_{1,n} + \overline{K_1})$ with $|S| = n$, then $\gamma_t(M(K_{1,n} + \overline{K_1})) \leq n$. □

Remark 4.15. *Proposition 4.14 shows that the upper bound of Theorem 4.10 is sharp. In fact, if $A \subseteq V(K_{1,n})$ with $|A| = n - 2$ and $G[A]$ has no isolated vertices, then $G[A]$ is isomorphic to $K_{1,n-2}$, and hence by Proposition 2.3, $\gamma_t(M(G[A])) = n - 2$.*

Proposition 4.16. *Let G be a graph of order $n \geq 2$ and $1 \leq p \leq \frac{n}{2} - 1$. If G has a subgraph isomorphic to a path graph P_n, then*

$$\gamma_t(M(G + \overline{K_p})) = \lceil \frac{2(n+p)}{3} \rceil.$$

Proof. By hypothesis, the graph $G + \overline{K_p}$ contains a subgraph isomorphic to P_{n+p}. By Theorem 2.10 we obtain the desired equality. □

As a direct consequence of Proposition 4.16, we obtain the following result.

Corollary 4.17. *Let G be a graph of order $n \geq 2$ and $1 \leq p \leq \frac{n}{2} - 1$. If G is isomorphic to a path graph P_n, or a cycle graph C_n, or a wheel graph W_n, or a complete graph K_n, then*

$$\gamma_t(M(G + \overline{K_p})) = \lceil \frac{2(n+p)}{3} \rceil.$$

5. Nordhaus-Gaddum relations

Since the work [11] appeared, several other authors studied Nordhaus-Gaddum type relations for many graph invariants. We refer to [1] for a survey on the subject.

Theorem 5.1. *Consider a graph G on $n \geq 2$ vertices such that G and $\overline{G}$ have no isolated vertices and no components isomorphic to K_2. Then*

$$2(n-1) \geq \gamma_t(M(G)) + \gamma_t(M(\overline{G})) \geq 2\lceil \frac{2n}{3} \rceil$$

and

$$(n-1)^2 \geq \gamma_t(M(G)) \cdot \gamma_t(M(\overline{G})) \geq (\lceil \frac{2n}{3} \rceil)^2.$$

Proof. By applying Theorem 2.8 to each component of G and $\overline{G}$, we obtain that $n-1 \geq \gamma_t(M(G)) \geq \lceil \frac{2n}{3} \rceil$ and $n - 1 \geq \gamma_t(M(\overline{G})) \geq \lceil \frac{2n}{3} \rceil$. □

Remark 5.2. *If in Theorem 5.1 we allow G or $\overline{G}$ to have components isomorphic to K_2, then the described upper bounds might not work. To see this it is enough to consider the graph C_4. In fact, $\overline{C_4}$ consists of two copies of K_2, and then $\gamma_t(M(C_4)) = 3$ and $\gamma_t(M(\overline{C_4})) = 4$.*

As the next example shows, all the inequalities of Theorem 5.1 are sharp.

Example 5.3. *Consider the graph P_4, then by Proposition 2.5, we have $\gamma_t(M(P_4)) = 3$. On the other hand, $\overline{P_4}$ is isomorphic to P_4, and hence $\gamma_t(M(\overline{P_4})) = 3$. Since $n = 4$, then $6 = \gamma_t(M(P_4)) + \gamma_t(M(\overline{P_4})) = 2(n-1) = 2\lceil \frac{2n}{3} \rceil$, and $9 = \gamma_t(M(P_4)) \cdot \gamma_t(M(\overline{P_4})) = (n-1)^2 = (\lceil \frac{2n}{3} \rceil)^2$.*

Acknowledgement

During the preparation of this article the fourth author was supported by JSPS Grant-in-Aid for Early-Career Scientists (19K14493).

References

[1] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Applied Mathematics*, **161** (2013), 466–546.

[2] J.A. Bondy and U.S.R. Murty, *Graph theory*, Graduate texts in mathematics, vol. 244, Springer Science and Media, 2008.

[3] E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi, Total domination in graphs, *Networks*, **10** (1980), 211–219.

[4] T. Hamada and I. Yoshimura, Traversability and connectivity of the middle graph of a graph, *Discrete Mathematics*, **14** (1976) 247–255.

[5] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[7] M.A. Henning and A. Yeo, *Total domination in graphs*, Springer Monographs in Mathematics, 2013.

[8] F. Kazemnejad and S. Moradi, Total domination number of central graphs, *Bulletin of the Korean Mathematical Society*, **56** (4) (2019), 1059–1075.

[9] F. Kazemnejad, B. Pahlavsay, E. Palezzato and M. Torielli, Domination number of middle graphs. ArXiv:2008.02975.

[10] F. Kazemnejad, B. Pahlavsay, E. Palezzato, and M. Torielli, Total dominator coloring number of middle graphs. *To appear in Discrete Mathematics, Algorithms and Applications*, 2022.

[11] E.A. Nordhaus and J.W. Gaddum, On complementary graphs, *Amer. Math. Monthly*, **63** (1956), 175–177.

[12] L. Ouldrabah, M. Blidia, and A. Bouchou, Roman domination in oriented trees, *Electronic Journal of Graph Theory and Applications*, **9** (1) (2021), 95–103.

[13] B. Pahlavsay, E. Palezzato, and M. Torielli, 3-tuple total domination number of rook's graphs. *Discussiones Mathematicae Graph Theory*, **42** (2022), 15–37

[14] B. Pahlavsay, E. Palezzato, and M. Torielli, Domination in latin square graphs. *Graphs and Combinatorics*, **37** (3) (2021), 971–985.