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$(1, 2)$-rainbow connection number at most 3 in connected dense graphs

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Abstract

Let G be an edge-colored connected graph G. A path P in the graph G is called l-rainbow path if each subpath of length at most $l+1$ is rainbow. The graph G is called (k, l)-rainbow connected if any two vertices in G are connected by at least k pairwise internally vertex-disjoint l-rainbow paths. The smallest number of colors needed in order to make G (k, l)-rainbow connected is called the (k, l)-rainbow connection number of G and denoted by $rc_{k,l}(G)$. In this paper, we consider the $(1, 2)$-rainbow connection number at most 3 in some connected dense graphs. Our main results are as follows: (1) Let $n \geq 7$ be an integer and G be a connected graph of order n. If $\omega(G) \geq n-3$, then $rc_{1,2}(G) \leq 3$. Moreover, the bound of the clique number is sharpness. (2) Let $n \geq 7$ be an integer and G be a connected graph of order n. If $|E(G)| \geq \binom{n-3}{2} + 7$, then $rc_{1,2}(G) \leq 3$.

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1. Introduction

We use [18] for terminology and notation not defined here and consider simple, finite, and undirected graphs only. Let G be a graph. We denote by $V(G), E(G), n, m$ the vertex set, the edge set, the number of vertices, the number of edges, respectively. Let $v \in V(G)$ be a vertex. The degree of vertex v in G is denoted by $d_G(v)$ (simply $d(u)$ if G is known). A clique in a graph is a set of pairwise adjacent vertices. The clique number of G, written $\omega(G)$, is the maximum size of a clique in G. Let $K_{\omega(G)}$ be a clique of order $\omega(G)$ in G. Let uv be an edge of G and $c(uv)$ be its color. Let $p(G)$ denote the order of a longest path in G and $c(G)$ be the circumference of G. We abbreviate the set $\{1, 2, \dots, k\}$ by $[k]$.

Let G be a graph of order n with a vertex set $V(G) = \{v_1, \dots, v_n\}$ and an edge set $E(G)$, $u \notin V(G)$ be an arbitrary vertex, $k \in [n]$ be an arbitrary integer. $G \cup u$ is a new graph obtained from G and u with the vertex set $V(G \cup u) = V(G) \cup \{u\}$ and the edge set $E(G \cup u) = E(G) \cup \{uv_i \,|\forall i \in [k]\}$.

In the last years, the connection concepts of connected graphs appeared in graph theory and received many attentions. They have many applications in the transmission of information in networks. Let G be a connected and edge-colored graph.

The first connection concept introduced by Chartrand et al. [5] is *rainbow connection*. A *rainbow path* in an edge-colored graph G is a path P whose edges are assigned distinct colors. An edge-colored graph G is *rainbow connected* if every two vertices are connected by at least one rainbow path in G. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the smallest number of colors required to make it rainbow connected. After that, many researchers have studied problems on rainbow connection [10, 16, 17]. Moreover, it has been shown in [7] that computing $rc(G)$ for a given connected graph G is an NP-hard problem. Readers who are interested in this topic are referred to [14, 15].

Motivated by proper coloring and rainbow connection, Borozan et al. [2] and Andrews et al. [1], independently introduced the concept of *proper connection*. A path P in an edge-colored graph G is a *proper path* if any two consecutive edges receive distinct colors. An edge-colored graph G is *properly connected* if every two vertices are connected by at least one proper path in G. For a connected graph G, the *proper connection number* of G, denoted by $pc(G)$, is defined as the smallest number of colors required to make it properly connected. Some results on this topic can be found in [3, 4]. Very recently, it has been shown in [11] that computing $pc(G)$ for a given graph G is an NP-hard problem. For more details we refer to the survey [12].

Recently, the new concept of connection that is (k,l)-*rainbow connection* was defined in [13] as a generalization of rainbow connection and proper connection. The concept of l-rainbow coloring was also independently introduced and studied in [6, 8, 9, 20]. A path P in an edge-colored graph G is called an l-*rainbow path* if each subpath of length at most $l+1$ of P is rainbow. An edge-colored graph G is called (k,l)-*rainbow connected* if every two vertices are connected by at least k pairwise internally vertex-disjoint l-rainbow paths in G. For a connected graph G, the (k,l)-*rainbow connection number* of G, denoted by $rc_{k,l}(G)$, is defined as the smallest number of colors required to make it (k,l)-rainbow connected. From this definition, it can be readily seen that the $(1,1)$-rainbow connection number of a connected graph G is actually its proper connection number, i.e $rc_{1,1}(G) = pc(G)$. Meanwhile, the $(1,l)$-rainbow connection number of a connected graph G can be its rainbow connection number as long as l is large enough. Recently, there is a few results on this topic. In this paper, we consider the $(1,2)$-rainbow connection number of connected dense graphs with some additional properties. Clearly, $1 \leq rc_{1,2}(G) \leq m$. Moreover, $rc_{1,2}(G) = 1$ if and only if G is complete.

2. Auxiliary results

In this section, we introduce some definitions and basic results that will be essential tools in the proof of our results.

Definition 2.1. *Let $P = v_1v_2 \dots v_n$ be a path of order n. We color all edges of P alternately with colors 1, 2 and 3 that means every subpath of length at most 3 is rainbow.*

Similar to the proper connection number and the rainbow connection number, the following proposition is easily obtained in [20].

Proposition 2.1. *(Zhu et al. [20]) Let G be a nontrivial connected graph. If H is a connected spanning subgraph of G, then $rc_{1,2}(G) \leq rc_{1,2}(H)$. Particularly, $rc_{1,2}(G) \leq rc_{1,2}(T)$ for every spanning tree T of G.*

By using Proposition 2.1, the authors in [20] gave the $(1,2)$-rainbow connection number of the traceable graph, i.e. graphs containing a Hamiltonian path.

Proposition 2.2. *(Zhu et al. [20]) If G be a traceable graph, then $rc_{1,2}(G) \leq 3$.*

We present now the following proposition.

Proposition 2.3. *Let G be a traceable graph and $u \notin V(G)$ be an arbitrary vertex. If $H = G \cup u$ and $d_H(u) \geq 2$, then $rc_{1,2}(H) \leq 3$.*

Proof. Since G is a traceable graph of order n, by Proposition 2.2, $rc_{1,2}(G) \leq 3$. Let $P = v_1 \dots v_n$ be a path containing all vertices of G and v_i, v_j be two neighbours of u in G, where $i < j$. We consider that there are some vertices v_k between v_i and v_j in P. Otherwise, $v_1 \dots v_i u v_j \dots v_n$ is a path. By Proposition 2.1 and Proposition 2.2, $rc_{1,2}(H) \leq 3$.

All edges of P now are alternately assigned with colors 1, 2 and 3. Next, color the edge uv_i so that $c(uv_i) \notin \{c(v_iv_{i+1}), c(v_{i+1}v_{i+2})\}$ and color the edge uv_j so that $c(uv_j) \notin \{c(v_jv_{j-1}), c(v_{j-1}v_{j-2})\}$. It can be readily seen that every two vertices of $P \cup u$ is connected by at least one 2-rainbow path.

Thereby completing the proof. □

3. Main results

In this section, we study the $(1,2)$-rainbow connection number of connected dense graphs with some additional properties. The first result is investigated in a connected graph G with the condition of the clique number $\omega(G)$.

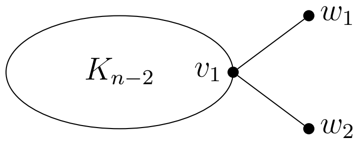

Figure 1. Graph H_{11}.

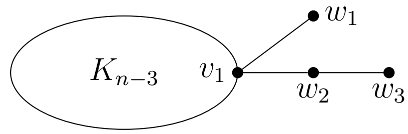

Figure 2. Graph H_{21}.

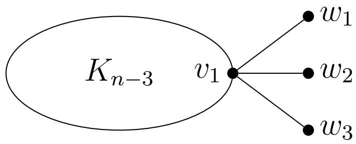

Figure 3. Graph H_{22}.

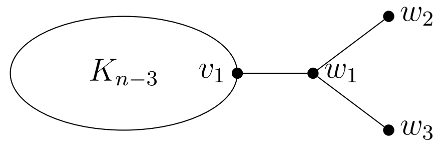

Figure 4. Graph H_{23}.

Theorem 3.1. *Let $n \geq 7$ be an integer. If G is a connected graph of order n with $\omega(G) \geq n-3$, then $rc_{1,2}(G) \leq 3$. Moreover, the bound of the clique number is sharpness.*

Proof. Let H be a minimally connected spanning subgraph of G such that $\omega(H) = \omega(G)$ and if the removal of any edges that are not in $K_{\omega(H)}$, then H is not connected. By Proposition 2.1, $rc_{1,2}(G) \leq rc_{1,2}(H)$. We only consider that H is nontraceable. Otherwise, by Proposition 2.2, $rc_{1,2}(G) \leq rc_{1,2}(H) \leq 3$. Note that H is connected. If $\omega(H) = n$ or $\omega(H) = n-1$, then H is traceable. Hence, we consider that $\omega(H) \in \{n-2, n-3\}$. Moreover, H is nontraceable. Let $V(K_{\omega(H)}) = \{v_1, v_2, \ldots, v_{\omega(H)}\}$ and $S = \{w_1, \ldots, w_{n-\omega(H)}\}$ be a vertex set of $K_{\omega(H)}$ and a vertex set not in $K_{\omega(H)}$, respectively.

Case 1. If $\omega(H) = n-2$ and H is nontraceable, then we have only one case that is $H \cong H_{11}$, see Figure 1. We color all edges of H_{11} as follows: $c(w_1v_1) = 1$, $c(w_2v_1) = 2$ and $c(v_iv_j) = 3$, where $v_iv_j \in K_{\omega(H_{11})}$. Since $H_{11} \setminus \{w_1, w_2\}$ is a clique, two vertices v_i, v_j are connected by at least one 2-rainbow path, say an edge. On the other hand, a 2-rainbow path between v_j and w_i, where $j \in [n-2]$, and $i \in [2]$ is $v_jv_1w_i$, and a 2-rainbow path between w_1, w_2 is $w_1v_1w_2$. Every two vertices of H_{11} is connected by at least one 2-rainbow path. Hence, $rc_{12}(H_{11}) \leq 3$. We obtain the result.

Case 2. Let $i \in [5]$. If $\omega(H) = n-3$ and H is nontraceable, then we have some cases that are $H \cong H_{2i}$, relabeling vertices of $K_{\omega(H_{2i})}$ if necessary, see Figures [2–6]. Since $n \geq 7$, there always exists a cycle of order 3 in $K_{\omega(H_{2i})}$, say $C_3 = v_1v_2v_3$.

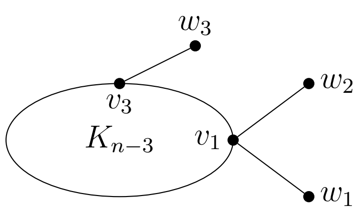

Figure 5. Graph H_{24}.

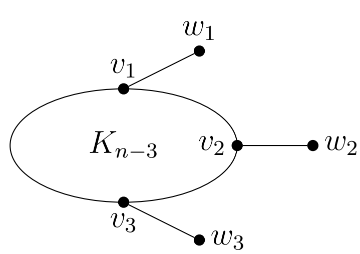

Figure 6. Graph H_{25}.

Now, we color all edges of H_{21}, H_{22}, H_{24}, H_{25} by 3 colors as follows:

(a) For graph H_{21}: $c(v_1w_1) = 1$, $c(v_1w_2) = 2$, $c(w_2w_3) = 3$, $c(v_2v_1) = 1$, $c(v_3v_1) = 2$.

(b) For graph H_{22}: $c(v_1w_k) = k$, where $k \in [3]$, $c(v_2v_3) = 1$, $c(v_3v_1) = 2$.

(c) For graph H_{24}: $c(v_1w_1) = 1$, $c(v_1w_2) = 2$, $c(v_3w_3) = 2$, $c(v_2v_3) = 1$.

(d) For graph H_{25}: $c(v_kw_k) = k$, where $k \in [3]$, $c(v_2v_3) = 1$, $c(v_tv_3) = 2$, where $t \in [n-3] \setminus \{2\}$.

All remaining edges of H_{21}, H_{22}, H_{24}, H_{25} are assigned to color 3.

For graph H_{23}, some edges are assigned as follows: $c(v_1w_1) = 1$, $c(w_2w_1) = 2$, $c(w_3w_1) = 3$, $c(v_tv_1) = 3$, where $t \in \{3, \ldots, n-3\}$, $c(v_{n-3}v_2) = 2$. Next, we color all edges of path $w_3w_1v_1v_2 \ldots v_{n-3}$ by alternating 3-colors. All remaining edges of H_{23} can be assigned by any color from $[3]$.

It can be readily seen that every two vertices of H_{2i} is connected by at least one 2-rainbow path. It follows that H_{2i} is $(1,2)$-rainbow connected with respect to this 3-coloring. Hence, $rc_{1,2}(H_{2i}) \leq 3$. We obtain the result.

Our proof is finished.

□

Remark 3.1. *The following example points out that Theorem 3.1 is best possible in sense of the clique number of graph G. For $n \geq 7$, let K_{n-4} be a complete graph and $K_{1,4}$ be a star. Next, identify the center of the star with an arbitrary vertex of K_{n-4}. Hence, the resulting graph G_4 has order n and clique number $\omega(G) = n-4$. It can be readily seen that $rc_{1,2}(G_4) \geq 4$.*

Next, we consider the $(1,2)$-rainbow connection number in connected graph with respect to their size.

Theorem 3.2. *Let $n \geq 7$ be an integer and G be a connected graph of order n. If $|E(G)| \geq \binom{n-3}{2} + 7$, then $rc_{1,2}(G) \leq 3$.*

For the proof of Theorem 3.2 we will make use the following result.

Theorem 3.3. *(Woodall et al. [19]) Let G be a graph of order $n = tm + r$, where $m \geq 1$, $t \geq 0$ and $1 \leq r \leq m$. If*

$$|E(G)| > t\binom{m+1}{2} + \binom{r}{2}$$

then $c(G) \geq m+2$

Proof. Since $|E(G)| \geq \binom{n-3}{2} + 7$, we observe that $|E(\bar{G})| \leq 3n - 13$, where $\bar{G}$ is the complement of G.

By Woodall's Theorem we conclude that $c(G) \geq n-2$. Now suppose, to the contrary, that $rc_{1,2}(G) \geq 4$. By using Proposition 2.2, G is not a traceable graph. Hence, we only consider that $c(G) = n-2$. Since G is connected, $p(G) = n-1$. Let $C = v_1 \ldots v_{n-2}v_1$ be a cycle of order $n-2$, which is clockwise oriented, and u, w be two vertices not belong to C. Clearly, $uw \notin E(G)$. Otherwise, G is traceable. Moreover, by using Proposition 2.3, we deduce that $d(u) = d(w) = 1$. We consider two cases as follows.

Case 1. $N(u) \cap N(w) = \{v_1\}$, renaming vertices if necessary. We construct 3-coloring of C as follows. Let $c(uv_1) = 2$ and $c(wv_1) = 3$. We color all edges of C alternately with colors 1, 2 and 3 so that $c(v_1v_2) = 1$, $c(v_2v_3) = 2$, $c(v_3v_4) = 3$ if $n = 3k$ or $c(v_1v_2) = 3$, $c(v_2v_3) = 1$, $c(v_3v_4) = 2$ if $n = 3k + 2$.

If $n = 3k + 1$, then $v_iv_{i+2} \in E(G)$ for some $i \in [n-2]$ (indices taken modulo 2). Otherwise, $|E(\bar{G})| \geq n - 3 + n - 3 + n > 3n - 13$ (note that $n - 2 \geq 5$), a contradiction. Choose i so that $i \neq 2$. Now, let C' be a new cycle obtained from C by replacing the path $v_iv_{i+1}v_{i+2}$ with the edge v_iv_{i+2}. color all the edges of C' as the same as the case $n = 3k$. Next assign the color of v_iv_{i+2} to both v_iv_{i+1} and $v_{i+1}v_{i+2}$.

It can be readily seen that G is $(1,2)$-rainbow connected with respect to this 3-coloring.

Case 2. $N(u) \cap N(w) = \emptyset$. Renaming vertices if necessary, we may assume that $N(u) = \{v_1\}$ and $N(w) = \{v_l\}$. Hence, $3 \leq l \leq n - 3$. If $n = 3k + 2$, then we color all edges of C alternately with colors 1, 2 and 3 so that $c(v_1v_2) = 1$, $c(v_2v_3) = 2$ and $c(v_3v_4) = 3$. Next, let $c(uv_1) = 3$ and $c(wv_l) = c(v_lv_{l+1})$.

By using a similar argument as in Case 1, there exist some edges $v_iv_{i+2} \in E(G)$ for $i \in [n-2]$ and $i \notin \{2, l-1\}$. If $n = 3k$, then let C' be a new cycle obtained from C by replacing the path $v_iv_{i+1}v_{i+2}$ with the edge v_iv_{i+2}. color all edge of C' and uv_1, wv_l as the same as the case $n = 3k + 2$. Next assign the color of v_iv_{i+2} to both v_iv_{i+1} and $v_{i+1}v_{i+2}$. If $n = 3k + 1$, then there are two edges v_iv_{i+2}, v_jv_{j+2} in G so that $j \neq i + 1$. Let C' a new cycle obtained from C by replacing the following paths: $v_iv_{i+1}v_{i+2}$ with the edge v_iv_{i+2} and $v_jv_{j+1}v_{j+2}$ with the edge v_jv_{j+2}. color all edge of C' and uv_1, wv_l as the same as the case $n = 3k + 2$. Next assign the color of v_iv_{i+2} to both v_iv_{i+2}, $v_{i+2}v_{i+2}$ and the color of v_jv_{j+2} to both v_jv_{j+1}, $v_{j+1}v_{i+2}$.

Clearly, G is $(1,2)$-rainbow connected with respect to this 3-coloring.

We complete our proof. □

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