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Weak edge triangle free detour number of a graph

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Abstract

For any two vertices u and v in a connected graph $G = (V, E)$, a $u - v$ path P is called a $u - v$ *triangle free path* if no three vertices of P induce a triangle. The *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\triangle f}(u, v)$ is called a $u - v$ *triangle free detour*. A set $S \subseteq V$ is called a *weak edge triangle free detour set* of G if every edge of G has both ends in S or it lies on a triangle free detour joining a pair of vertices of S. The *weak edge triangle free detour number* $wdn_{\triangle f}(G)$ of G is the minimum order of its weak edge triangle free detour sets and any weak edge triangle free detour set of order $wdn_{\triangle f}(G)$ is a *weak edge triangle free detour basis* of G. Certain properties of these concepts are studied. The weak edge triangle free detour numbers of certain classes of graphs are determined. Its relationship with the triangle free detour diameter is discussed and it is proved that for any three positive integers a, b and n of integers with $3 \leq b \leq n - a + 1$ and $a \geq 4$, there exists a connected graph G of order n with triangle free detour diameter $D_{\triangle f} = a$ and $wdn_{\triangle f}(G) = b$. It is also proved that for any three positive integers a, b and c with $3 \leq a \leq b$ and $c \geq b + 2$, there exists a connected graph G such that $R_{\triangle f} = a$, $D_{\triangle f} = b$ and $wdn_{\triangle f}(G) = c$.

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1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand et al. [6]. The *neighbourhood of a vertex* v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph $\langle N(v)\rangle$ induced by its neighbourhood $N(v)$ is complete.

The concept of geodetic number was introduced by Harary et al. [4, 5, 9]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. A set $S \subseteq V$ is called *geodetic set* of G if every vertex of G lies on a geodesic joining a pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a *geodetic basis* of G.

The concept of detour number was introduced by Chartrand et al. [3]. The *detour distance* $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ *detour*. A set $S \subseteq V$ is called *detour set* of G if every vertex of G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is called a *detour basis* of G.

The concept of edge detour number was introduced by Santhakumaran an Athisayanathan [11, 12]. A set $S \subseteq V$ is called an *edge detour set* of G if every edge of G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is called an *edge detour basis* of G. A graph G is called an *edge detour graph* if it has an edge detour set.

The concept of weak edge detour number was introduced by Santhakumaran and Athisayanathan [13]. A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge of G has both ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a *weak edge detour basis* of G.

The concept of triangle free detour distance was introduced by Keerthi Asir and Athisayanathan [10]. The *triangle free detour eccentricity* $e_{\triangle f}(v)$ of a vertex v in G is the maximum triangle free detour distance from v to a vertex of G. The *triangle free detour radius*, $R_{\triangle f}$ of G is the minimum triangle free detour eccentricity among the vertices of G, while the *triangle free detour diameter*, $D_{\triangle f}$ of G is the maximum triangle free detour eccentricity among the vertices of G.

The concept of triangle free detour number was introduced by Sethu Ramalingam and Athisayan athan [14]. A set $S \subseteq V$ is called a *triangle free detour set* of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The *triangle free detour number* $dn_{\triangle f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\triangle f}(G)$ is called a *triangle free detour basis* of G.

In general, there are graphs G for which there exist edges which do not lie on a triangle free detour joining any pair of vertices of V. For the graph G given in Figure 1, the edge u_1u_2 does not lie on a triangle free detour joining any pair of vertices of V. This motivates us to introduce the concept of weak edge triangle free detour set of a graph.

The following theorems will be used in the sequel.

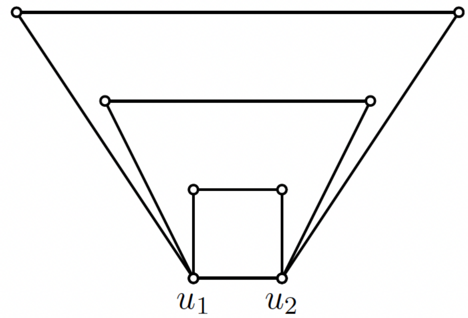

Figure 1. $G : dn_{\triangle f}(G) \neq 2$.

Theorem 1.1. [14] *Each extreme vertex of a graph G belongs to every triangle free detour set of G.*

Theorem 1.2. [14] *If G is a connected graph of order n and triangle free detour diameter $D_{\triangle f}$, then $dn_{\triangle f}(G) \leq n - D_{\triangle f} + 1$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Weak edge triangle free detour number of a graph

Definition 2.1. *Let G be a connected graph. A set $S \subseteq V$ is called a* ***weak edge triangle free detour set*** *of G if every edge of G has both ends in S or it lies on a triangle free detour joining a pair of vertices of S. The* ***weak edge triangle free detour number*** *$wdn_{\triangle f}(G)$ of G is the minimum order of its weak edge triangle free detour sets and any weak edge triangle free detour set of order $wdn_{\triangle f}(G)$ is a* ***weak edge triangle free detour basis*** *of G.*

Example 2.1. For the graph G given in Figure 2, it is clear that no two element subset of V is a weak edge triangle free detour set of G. It is easily seen that the set $S_1 = \{u, v, x\}$ is a weak edge triangle free detour basis of G so that $wdn_{\triangle f}(G) = 3$.

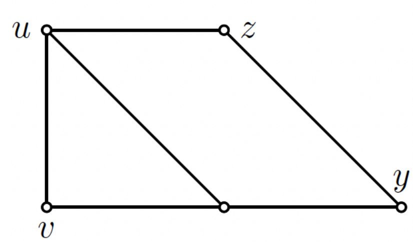

Figure 2. G : The Set S_1 with $wdn_{\triangle f}(G) = 3$.

Also the set $S_2 = \{u, v, z\}$ is another weak edge triangle free detour basis of G. Thus there can be more than one weak edge triangle free detour basis for a graph G. For the graph G given in

Figure 3, $S = \{u, v\}$ is a weak edge detour basis of G so that $dn_w(G) = 2$ and $S = \{u, v, x, y, z\}$ is a weak edge triangle free detour basis so that $wdn_{\triangle f}(G) = 5$. Hence the weak edge detour number and the weak edge triangle free detour number of a graph G are different.

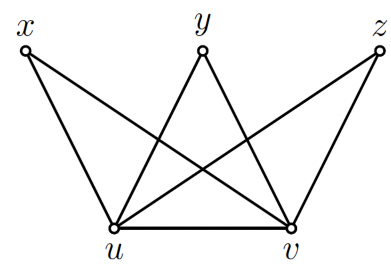

Figure 3. G : The Set S with $wdn_{\triangle f}(G) = 5$.

Theorem 2.1. *For a connected graph G of order* n, $2 \leq dn_w(G) \leq wdn_{\triangle f}(G) \leq n$.

Proof. A weak edge detour set needs at least two vertices so that $dn_w(G) \geq 2$. Since every weak edge triangle free detour set is also a weak edge detour set so that $dn_w(G) \leq wdn_{\triangle f}(G)$. Also the set of all vertices of G is a weak edge triangle free detour set of G so that $wdn_{\triangle f}(G) \leq n$. Thus $2 \leq dn_w(G) \leq wdn_{\triangle f}(G) \leq n$. □

Remark 2.1. The bounds in Theorem 2.1 are sharp. The set of two end-vertices of a path P_n is its unique weak edge triangle free detour set so that $wdn_{\triangle f}(G) = 2$. For the complete graph K_n, $wdn_{\triangle f}(K_n) = n$. Thus the path P_n has the smallest weak edge triangle free detour number 2 and the complete graph K_n has the largest possible weak edge triangle free detour number n.

Definition 2.2. *A vertex v in a graph G is a* ***weak edge triangle free detour vertex*** *if v belongs to every weak edge triangle free detour basis of G. If G has a unique weak edge triangle free detour basis S, then every vertex of S is a weak edge triangle free detour vertex of G.*

Example 2.2. For the graph G given in Figure 2, $S_1 = \{u, v, x\}$, $S_2 = \{u, v, z\}$, $S_3 = \{z, v, y\}$ and $S_4 = \{x, y, v\}$ are the only weak edge triangle free detour bases of G so that v is the weak edge triangle free detour vertex of G.

Remark 2.2. A cut-vertex may or may not belong to a weak edge triangle free detour basis of a graph G. For the graph G given in Figure 4, $S_1 = \{u, v, w\}$, $S_2 = \{u, w, y\}$, $S_3 = \{v, w, x\}$ and $S_4 = \{w, x, y\}$ are the only weak edge triangle free detour bases of G. The cut-vertex w belongs to every weak edge triangle free detour basis so that the cut-vertex w is the unique weak edge triangle free detour vertex of G.

For the graph G in Figure 5, $S = \{u, v, x, y\}$ is a unique weak edge triangle free detour basis and the cut-vertex w is not a weak edge triangle free detour vertex of G.

In the following theorem we show that there are certain vertices in a connected graph G that are weak edge triangle free detour vertices of G.

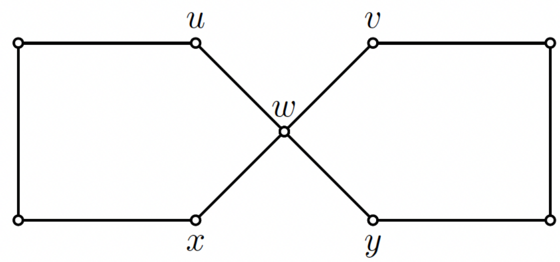

Figure 4. G : The Sets S_1, S_2, S_3 and S_4 with $wdn_{\triangle f}(G) = 3$.

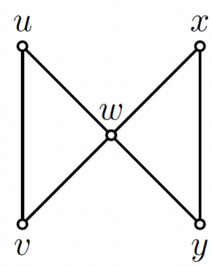

Figure 5. G : The Set S with $wdn_{\triangle f}(G) = 4$.

Theorem 2.2. *Every extreme-vertex of a connected graph G belongs to every weak edge triangle free detour set of G. Also, if the set S of all extreme-vertices of G is a weak edge triangle free detour set, then S is the unique weak edge triangle free detour basis for G.*

Proof. Let u be an extreme vertex of G and let S be a weak edge triangle free detour set of G. Suppose that $u \notin S$.

Case 1. $N[u] = K_2$. Clearly u is an end-vertex and u does not lie on any triangle free detour joining a pair of vertices $x, y \in S$. So that S is not a triangle free detour set which is a contradiction.

Case 2. $N[u] = K_n (n \geq 3)$. Since $u \notin S$, then u is an internal vertex of a $x - y$ triangle free detour path say P, for some $x, y \in S$. Let v and w be the neighbours of u on *P*. Then v and w are not adjacent and so u is not an extreme vertex, which is a contradiction. If S is the set of all extreme vertices of G, then by the first part of this theorem, $dn_{\triangle f}(G) \geq |S|$. If S is a triangle free detour set of G, then $wdn_{\triangle f}(G) \leq |S|$. Hence $wdn_{\triangle f}(G) = |S|$ and S is the unique weak edge triangle free detour basis for G. □

Corollary 2.1. *If T is a tree with k end-vertices, then $wdn_{\triangle f}(T) = k$.*

Proof. This follows from Theorem 2.2. □

In the following theorems we give the weak edge triangle free detour basis of certain graphs.

Theorem 2.3. *Let G be the complete graph K_n of order n. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if $S = V(G)$.*

Proof. This follows from Theorem 2.2. □

Theorem 2.4. *Let G be an even cycle of order $n \geq 4$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two adjacent vertices or two antipodal vertices of G.*

Proof. Let $S = \{u, v\}$ be any set of two vertices of G. If u and v are adjacent, then $D_{\triangle f}(u, v) = n - 1$ and every edge $e \neq uv$ of G lies on the $u - v$ triangle free detour and the both ends of the edge uv belong to S. If u and v are antipodal, then $D_{\triangle f}(u, v) = \frac{n}{2}$ and every edge e of G lies on a $u - v$ triangle free detour in G. Thus S is a weak edge triangle free detour set of G. Since $|S| = 2$, S is a weak edge triangle free detour basis of G.

Conversely, assume that S is a weak edge triangle free detour basis of G. Let S' be any set of two adjacent vertices or two antipodal vertices of G. Then as in the first part of this theorem, S' is a weak edge triangle free detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\} \subseteq V$. If u and v are not adjacent and u and v are not antipodal, then the edges of the $u - v$ geodesic that do not lie on the $u - v$ triangle free detour in G so that S is not a weak edge triangle free detour set of G, which is a contradiction. □

Theorem 2.5. *Let G be an odd cycle of order $n \geq 5$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two adjacent vertices of G.*

Proof. Let G be an odd cycle $C_n(n \geq 5)$. If $\{u, v\}$ is any set of two adjacent vertices of G. It is clear that $D_{\triangle f}(u, v) = n - 1$. Then every edge $e \neq uv$ of G lies on the $u - v$ triangle free detour and the both ends of the edge uv belong to S so that S is a weak edge triangle free detour set of G. Since $|S| = 2$, S is a weak edge triangle free detour basis of G.

Conversely, assume that S is a weak edge triangle free detour basis of G. Let $S^{'}$ be any set of two adjacent vertices of G. Then as in the first part of this theorem S' is a weak edge triangle free detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\} \subseteq V$. If u and v are not adjacent, then the edges of $u - v$ geodesic do not lie on the $u - v$ triangle free detour in G so that S is not a weak edge triangle free detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices of G. □

Theorem 2.6. *Let G be a complete bipartite graph $K_{n,m}(2 \leq n \leq m)$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two vertices of G.*

Proof. Let G be the complete bipartite graph $K_{n,m}$. Let X and Y be bipartite sets of G with $|X| = n$ and $|Y| = m$. Let $S = \{u, v\}$ be any set of two vertices of G.

Case 1. Let $u, v \in X$. It is clear that $D_{\triangle f}(u, v) = 2n - 2$. Let $xy \in E$ such that $x \in X$ and $y \in Y$. If $x \neq u$, then the edge xy lies on the $u - v$ triangle free detour $P : u, y, x, \ldots, v$ of length $2n - 2$. If $x = u$, then the edge xy lies on the $u - v$ triangle free detour $P : u = x, y, \ldots, v$ of length $2n - 2$. Hence S is a weak edge triangle free detour basis of G.

Case 2. Let $u, v \in Y$. It is clear that $D_{\triangle f}(u, v) = 2n$. Let $xy \in E$ such that $x \in X$ and $y \in Y$. If $y \neq v$, then the edge xy lies on the $u - v$ triangle free detour $P : u, x, y, \ldots, v$ of length $2n$. If $y = v$, then the edge xy lies on the $v - u$ triangle free detour $P : v = y, x, \ldots, u$ of length $2n$. Hence S is a weak edge triangle free detour set of G. In all cases, since $|S| = 2$, S is a weak edge triangle free detour basis of G.

Case 3. $u \in X$ and $v \in Y$. It is clear that $D_{\triangle f}(u,v) = 2n-1$. Let $xy \in E$. If $xy = uv$, then both of its ends are in S. Let $xy \neq uv$ be such that $x \in X$ and $y \in Y$. If $x \neq u$ and $y \neq v$, then the edge xy lies on the $u-v$ triangle free detour $P: u, y, x, \ldots, v$ of length $2n-1$. If $x = u$ and $y \neq v$, then the edge xy lies on the $u-v$ triangle free detour $P: u = x, y, \ldots, v$ of length $2n-1$. Hence S is a weak edge triangle free detour set of G.

Conversely, let S be a weak edge triangle free detour basis of G. Let S' be any set consisting of two vertices of G. Then as in the first part of this theorem, S' is a weak edge triangle free detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two vertices of G. □

Theorem 2.7. *Let G be the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 6$). Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of every vertex of G.*

Proof. Let $K_1 = \{w\}$ and $C_{n-1} : v_1, v_2, \ldots, v_{n-1}, v_1$ be the cycle of length $n-1$. Let $W_n = K_1 + C_{n-1}$ $(n \geq 6)$ be a wheel and S be a weak edge triangle free detour basis consists of every vertex of G. Then all the edges of C_{n-1} lie on the triangle free detour joining any pair of two adjacent vertices of C_{n-1}. Also $D_{\triangle f}(w, v_i) = 1 (1 \leq i \leq n-1)$, then every edge of W_n lies on a triangle free detour joining a pair of vertices of S. Hence S is a weak edge triangle free detour set of G. Since $|S| = n$, S is a weak edge triangle free detour basis of G.

Conversely, let S be a weak edge triangle free detour basis of G. Let S_1 be any set consisting of every vertex of G. Then as in the first part of this Theorem, S_1 is a weak edge triangle free detour basis of G. Hence $|S| = |S_1| = n$ and it follows that S consists of every vertex of G. □

Corollary 2.2. *Let G be a connected graph of order n.*

(a) If G is the path P_n, then $wdn_{\triangle f}(G) = 2$.
(b) If G is the complete graph K_n, then $wdn_{\triangle f}(G) = n$.
(c) If G is the cycle $C_n (n \geq 4)$, then $wdn_{\triangle f}(G) = 2$.
(d) If G is the complete bipartite graph $K_{n,m} (2 \leq n \leq m)$, then $wdn_{\triangle f}(G) = 2$.
(e) If G is the wheel $W_n (n \geq 4)$, then $wdn_{\triangle f}(W_n) = n - 1$ for $n = 5$ and $wdn_{\triangle f}(W_n) = n$ for $n = 4$ and $n \geq 6$.

Proof. (a) This follows from Corollary 2.1.
(b) This follows from Theorem 2.3.
(c) This follows from Theorems 2.4 and 2.5.
(d) This follows from Theorem 2.6.
(e) This follows from Theorem 2.7. □

The following theorems give realization results.

Theorem 2.8. *For any two positive integers k and n with $2 \leq k \leq n$, there exists a connected graph G of order n with $wdn_{\triangle f}(G) = k$.*

Proof. **Case 1.** $2 \leq k = n$. Any complete graph G has the desired property.

Case 2. $2 \leq k < n$, let P be a path of order $n-k+2$. Then the graph G obtained from P by adding $k-2$ new vertices to P and joining them to any cut-vertex of P is a tree of order n and so by Corollary 2.1, $wdn_{\triangle f}(G) = k$. □

Theorem 2.9. *For each positive integer $k \geq 3$, there exists a connected graph G with a vertex v of degree k in G such that v does not belongs to a weak edge triangle free detour basis of G and $wdn_{\triangle f}(G) = k$.*

Proof. For $k \geq 2$, let G be the graph obtained from the complete graph K_3, where $V(K_3) = \{v_1, v_2, v_3\}$, by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ and joining each $u_i (1 \leq i \leq k-2)$ to v_1. Then $deg_G\ v_1 = k$. Let $S = \{u_1, u_2, \ldots, u_{k-2}\}$. Then neither S nor $S \cup \{v_i\} (1 \leq i \leq 3)$ is a weak edge triangle free detour set of G. However, $S \cup \{v_2, v_3\}$ is a weak edge triangle free detour set of G and hence by Theorem 2.2, $S \cup \{v_2, v_3\}$ is a weak edge triangle free detour basis of G so that $wdn_{\triangle f}(G) = k$. □

Theorem 2.10. *For any two positive integers a and b with $3 \leq a \leq b$, there exists a connected graph G such that $dn_{\triangle f}(G) = a$ and $wdn_{\triangle f}(G) = b$.*

Proof. **Case 1.** $a = b$. Any tree with a end vertices has the desired property.

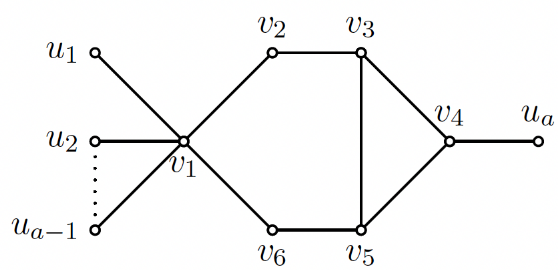

Figure 6. G : The Set S with $a = b-1$, $dn_{\triangle f}(G) = a$ and $wdn_{\triangle f}(G) = b$.

Case 2. $a = b - 1$. Consider the graph G given in Figure 6. Let $S = \{u_1, u_2, \ldots, u_{a-1}, u_a\}$ be the set of all end vertices of G. By Theorems 1.1 and 2.2, S is contained in every triangle free detour set and every weak edge triangle free detour set of G. It is easily seen that S is a triangle free detour set of G and so $dn_{\triangle f}(G) = a$, but S is not a weak edge triangle free detour set of G. Let $T = S \cup \{v_3\}$. Then T is a weak edge triangle free detour set of G and so $wdn_{\triangle f}(G) = b = a+1$.

Case 3. $a \leq b-2$. Let $P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2}$ be a path of order $b-a+2$ and $P_2 : x, y$ be a path of order 2. Let H be the graph obtained by joining the vertex x with u_{b-a+2} and also joining each vertex $u_i (1 \leq i \leq b-a+1)$ with y. Let G be the graph obtained by adding $a-2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ to H and joining each $v_i (1 \leq i \leq a-2)$ to x in H. The graph G is shown in Figure 7. Let $S = \{v_1, v_2, \ldots, v_{a-2}, u_1\}$ be the set of all extreme vertices of G. By Theorems 1.1 and 2.2, S is contained in every triangle free detour set and every weak edge triangle free detour set of G. Let $S^{'} = S \cup \{y\}$. It is easily verified that $S^{'}$ is a triangle free detour basis of G and so $dn_{\triangle f}(G) = a$. Let $S^{''} = S \cup \{x, y, u_2, u_3, \ldots, u_{b-a}\}$. It is easily seen that $S^{''}$ is a weak edge triangle free detour basis of G and so $wdn_{\triangle f}(G) = b$. □

Theorem 2.11. *For any two positive integers a and b with $4 \leq a \leq b$, there exists a connected graph G with $dn_w(G) = a$ and $wdn_{\triangle f}(G) = b$.*

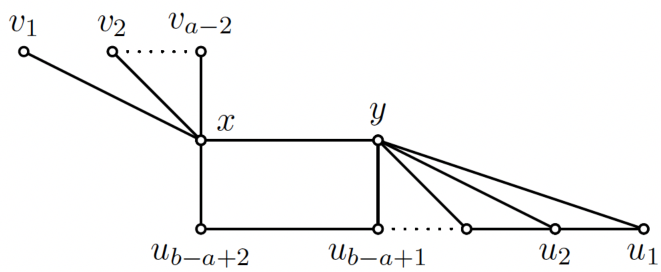

Figure 7. G : The Set S with $a \leq b-2$, $dn_{\triangle f}(G) = a$ and $wdn_{\triangle f}(G) = b$.

Proof. **Case 1.** For $4 \leq a = b$, any tree with a end vertices has the desired properties, By Corollary 2.1.

Case 2. For $4 \leq a < b$. Let $P_4 : x, u_1, u_2, u_3$ be path of order 4. Let G_1 be the graph obtained from the path P_4 by adding $a-3$ new vertices $w_1, w_2, \ldots, w_{a-3}$ and joining each vertex $w_i(1 \leq i \leq a-3)$ to u_3 in P_4. Let G_2 be the graph obtained from the graph G_1 by adding $b-a+2$ new vertices $v_1, v_2, \ldots, v_{b-a+2}$ and joining each vertex $v_j(1 \leq j \leq b-a+2)$ to u_1 and u_2 in P_4. Let $G = G_2$ be the required graph of order $b+3$ is shown in Figure 8. Since $S = \{x, w_1, w_2, \ldots, w_{a-3}\}$ is the set of all end vertices of G, then every weak edge detour set of G contains S and S is not a weak edge detour set of G. Let $S_1 = S \cup \{u_1, u_2\}$. It is easily verified that S_1 is a weak edge detour set of G and so that $dn_w(G) = |S_1| = a$.

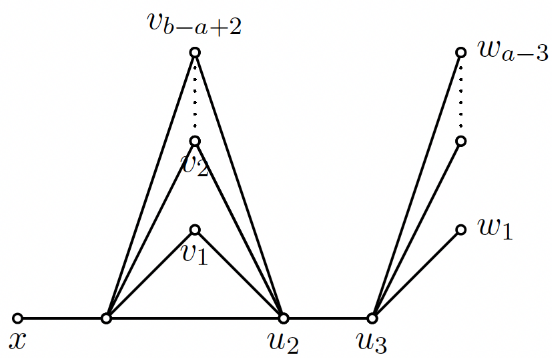

Figure 8. G : The Set S with $4 \leq a < b$, $dn_w(G) = a$ and $wdn_{\triangle f}(G) = b$.

Next, we show that $wdn_{\triangle f}(G) = b$. By Theorem 2.2, every weak edge triangle free detour set of G contains S. Clearly, S is not a weak edge triangle free detour set of G. It is easily verified that each $v_i(1 \leq i \leq b-a+2)$ must belong to every weak edge triangle free detour set of G. Thus $T = S \cup \{v_1, v_2, \ldots, v_{b-a+2}\}$ is a weak edge triangle free detour set of G, it follows from Theorem 2.2 that T is a weak edge triangle free detour basis of G and so $wdn_{\triangle f}(G) = b$. □

3. Weak edge triangle free detour number and triangle free detour diameter of a graph

We have seen that by Theorem 1.2, $dn_{\triangle f}(G) \leq n - D_{\triangle f} + 1$. However, in the case of weak edge triangle free detour number of a graph, this is not true.

Remark 3.1. In the case of weak edge triangle free detour number $wdn_{\triangle f}(G)$ of a graph G, there are graphs for which $wdn_{\triangle f}(G) = n - D_{\triangle f} + 1$, $wdn_{\triangle f}(G) > n - D_{\triangle f} + 1$ and $wdn_{\triangle f}(G) < n - D_{\triangle f} + 1$. For any cycle C_n of order $n \geq 4$, $D_{\triangle f} = n - 1$ and $wdn_{\triangle f}(C_n) = 2$ so that $wdn_{\triangle f}(G) = n - D_{\triangle f} + 1$. For any wheel W_n of order $n \geq 6$, $D_{\triangle f} = n - 2$ and $wdn_{\triangle f}(W_n) = n$ so that $wdn_{\triangle f}(W_n) > n - D_{\triangle f} + 1$. For the graph G in Figure 9, $n = 6$, $D_{\triangle f} = 4$ and $wdn_{\triangle f}(G) = 2$ so that $wdn_{\triangle f}(W_n) < n - D_{\triangle f} + 1$.

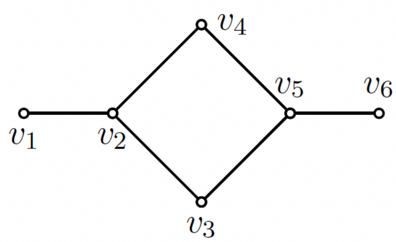

Figure 9. $G : n = 4$ with $D_{\triangle f}(G)$ = 4 and $wdn_{\triangle f}(G) = 2$.

Theorem 3.1. *For every tree T of order n and triangle free detour diameter $D_{\triangle f}$, c $n - D_{\triangle f} + 1$ if and only if T is a caterpillar.*

Proof. Let T be any tree. Let $P : u = v_0, v_1, v_2, \ldots, v_{D_{\triangle f}}$ be a triangle free detour diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_1, v_2, \ldots, v_{D_{\triangle f-1}}$. Then $D_{\triangle f} - 1 + l + k = n$. By Corollary 2.1, $wdn_{\triangle f}(T) = k$ and so $wdn_{\triangle f}(T) = n - D_{\triangle f} - l + 1$. Hence $wdn_{\triangle f}(T) = n - D_{\triangle f} + 1$ if and only if $l = 0$ if and only if all the internal vertices of T lie on the triangle free detour diametrical path P if and only if T is a caterpillar. □

Theorem 3.2. *For any three positive integers a, b and c with $3 \leq a < b$ and $c \geq b + 2$, there exists a connected graph G such that $R_{\triangle f} = a$, $D_{\triangle f} = b$ and $wdn_{\triangle f}(G) = c$.*

Proof. **Case 1.** Let $b \leq 2a$.

Subcase 1.1. Let a be an odd integer. Then $a \geq 3$. Let $C_{a+1} : v_0, v_1, \ldots, v_a, v_0$ be a cycle of order $a + 1$ and let $P_{b-a+1} : u_0, u_1, \ldots, u_{b-a}$ be a path of order $b - a + 1$. Let H be the graph obtained from C_{a+1} and P_{b-a+1} by identifying v_0 of C_{a+1} with u_0 of P_{b-a+1}. The required graph G is obtained from H by adding $c - 2$ new vertices $w_1, w_2, \ldots, w_{c-2}$ to H and joining each $w_i(1 \leq i \leq c - 2)$ to the vertex u_{b-a-1} is shown in Figure 10. It is easily verified that $a \leq e_{\triangle f}(x) \leq b$ for any vertex x in G and $e_{\triangle f}(v_0) = a$ and $e_{\triangle f}(v_1) = b$. Thus $R_{\triangle f} = a$ and $D_{\triangle f} = b$. Now, we show that $wdn_{\triangle f}(G) = c$. Let $S = \{w_1, w_2, \ldots, w_{c-2}, u_{b-a}\}$ be the set of all end-vertices of G. Let $T = S \cup \{v\}$, where $v = v_{\frac{a+1}{2}}$ is the antipodal vertex of v_0 in C_{a+1}. Then

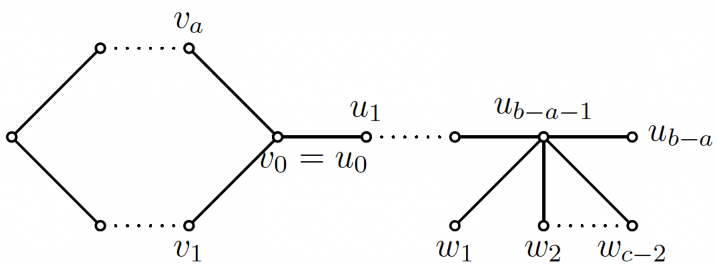

Figure 10. G : a is an odd integer with $R_{\triangle f}(G) = a$, $D_{\triangle f}(G) = b$ and $wdn_{\triangle f}(G) = c$.

T is a weak edge triangle free detour set of G and hence it follows from Theorem 2.2 that T is a weak edge triangle free detour basis of G so that $wdn_{\triangle f}(G) = c$.

Subcase 1.2. Let a be an even integer. Construct the graph H as above Subcase(1.1) in Figure 10. Then G is obtained from H by adding $c-3$ new vertices $w_1, w_2, w_3, \ldots, w_{c-3}$ to H and joining each $w_i (1 \leq i \leq c-3)$ to the vertex u_{b-a-1} and is shown in Figure 11. It is easily verified that $a \leq e_{\triangle f}(x) \leq b$ for any vertex x in G and $e_{\triangle f}(v_0) = a$ and $e_{\triangle f}(v_1) = b$. Thus $R_{\triangle f} = a$ and $D_{\triangle f} = b$. Now, we show that $wdn_{\triangle f}(G) = c$. Let $S = \{w_1, w_2, \ldots, w_{c-2}, u_{b-a}\}$ be the set of all end-vertices of G. As in Case 1, S is not a weak edge triangle free detour set of G. Let $S_1 = S \cup \{v\}$, where v is any vertex of G such that $v \notin S$. It is easy to see that S_1 is not a weak edge triangle free detour set of G. Clearly the set $T = S \cup \{v_1, v_a\}$ is a weak edge triangle free detour set of G. Hence it follows from Theorem 2.2 that T is a weak edge triangle free detour basis of G and $wdn_{\triangle f}(G) = c$.

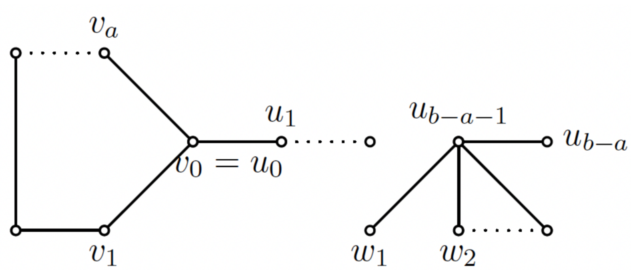

Figure 11. G : a is an even integer with $R_{\triangle f}(G) = a$, $D_{\triangle f}(G) = b$ and $wdn_{\triangle f}(G) = c$.

Case 2. Let $b > 2a$. Let $P_{2a} : v_1, v_2, \ldots, v_{2a}$ be a cycle of order $2a$ and $W_{b+2} = K_1 + C_{b+1}$ be the wheel with $V(C_{b+1}) = \{u_1, u_2, \ldots, u_{b+1}\}$,$K_1 = \{u_0\}$. Let F be the graph obtained from W_{b+2} and P_{2a} by identifying u_0 of W_{b+2} with v_1 of P_{2a} as shown in Figure 12. It is easily verified that $a \leq e_{\triangle f}(x) \leq b$ for any vertex x in G and $e_{\triangle f}(v_a) = a$ and $e_{\triangle f}(u_1) = b$. Thus $R_{\triangle f} = a$ and $D_{\triangle f} = b$. Let $S = \{u_1, u_2, \ldots, u_{b+1}, v_{2a}\}$ is a weak edge triangle free detour set of G so that $wdn_{\triangle f}(G) = b + 2 = c$. □

Theorem 3.3. *For any three positive integers a, b and n with $3 \leq b \leq n - a + 1$ and $a \geq 4$, there exists a connected graph G of order n with triangle free detour diameter $D_{\triangle f} = a$ and $wdn_{\triangle f}(G) = b$.*

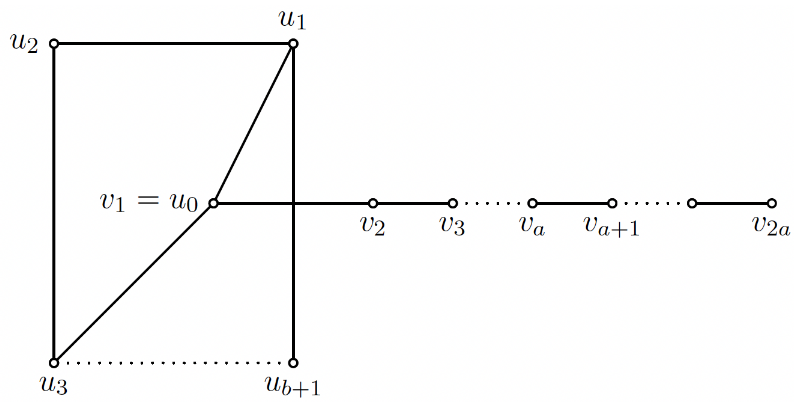

Figure 12. $G : b > 2a$ with $R_{\triangle f}(G) = a$, $D_{\triangle f}(G) = b$ and $wdn_{\triangle f}(G) = c$.

Proof. **Case 1.** When a is even, let G be the graph obtained from the cycle $C_a : u_1, u_2, \ldots, u_a, u_1$ of order a by adding $b-1$ new vertices $v_1, v_2, \ldots, v_{b-1}$ and joining each vertex $v_i (1 \leq i \leq b-1)$ to u_1 and adding $n-a-b+1$ new vertices $w_1, w_2, \ldots, w_{n-a-b+1}$ and joining each vertex $w_i (1 \leq i \leq n-a-b+1)$ to both u_1 and u_3 and is shown in Figure 13. It is easily verified that the order of the graph G is n and triangle free detour diameter $D_{\triangle f} = a$.

Now, we show that $wdn_{\triangle f}(G) = b$. Let $S = \{v_1, v_2, \ldots, v_{b-1}\}$ be the set of all end-vertices of G. Since no edge of G other than the edges $u_1 v_i (1 \leq i \leq b-1)$ lies on a triangle free detour joining a pair of vertices of S, S is not a weak edge triangle free detour set of G. Let $T = S \cup \{v\}$, where v is the antipodal vertex of u_1 in C_a. Then every edge of G lies on a triangle free detour joining a vertex $v_i (1 \leq i \leq b-1)$ and v so that T is a weak edge triangle free detour set of G. Now, it follows from Theorem 2.2 that T is a weak edge triangle free detour basis of G and so $wdn_{\triangle f}(G) = b$.

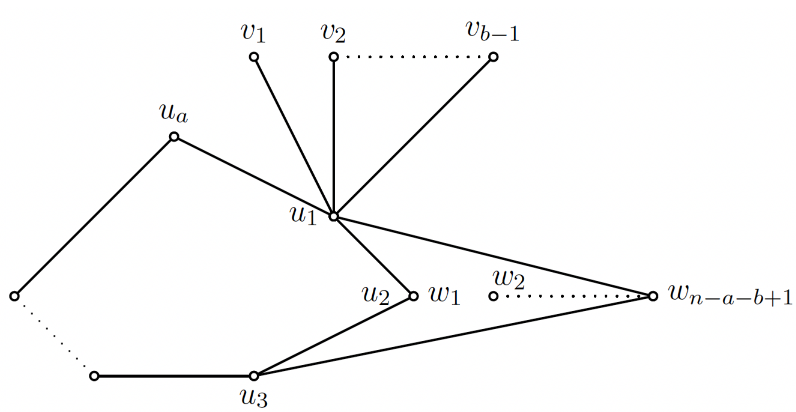

Figure 13. $G : a$ is even with $D_{\triangle f}(G) = a$ and $wdn_{\triangle f}(G) = b$.

Case 2. When a is odd, let G be the graph obtained from the cycle $C_a : u_1, u_2, \ldots, u_a, u_1$ of order a by adding $b-2$ new vertices $v_1, v_2, \ldots, v_{b-2}$ and joining each vertex $v_i (1 \leq i \leq b-2)$ to u_1 and adding $n-a-b+2$ new vertices $w_1, w_2, \ldots, w_{n-a-b+2}$ and joining each vertex $w_i (1 \leq$

$i \leq n-a-b+2$) to both u_1 and u_3. It is easily verified that the order of the graph G is n, triangle free detour diameter $D_{\triangle f} = a$ and is shown in Figure 14. Now, we show that $wdn_{\triangle f}(G) = b$. Let $S = \{v_1, v_2, \ldots, v_{b-2}\}$ be the set of all end-vertices of G. As in Case 1, S is not a weak edge triangle free detour set of G. Let $S_1 = S \cup \{v\}$, where v is the any vertex of G such that $v \neq v_i (1 \leq i \leq b-2)$. It is easy to see that S_1 is not a weak edge triangle free detour set of G. Now, the set $T = S \cup \{u_2, u_a\}$ is a weak edge triangle free detour set of G. Hence it follows from Theorem 2.2 that T is a weak edge triangle free detour basis of G and so $wdn_{\triangle f}(G) = b$. □

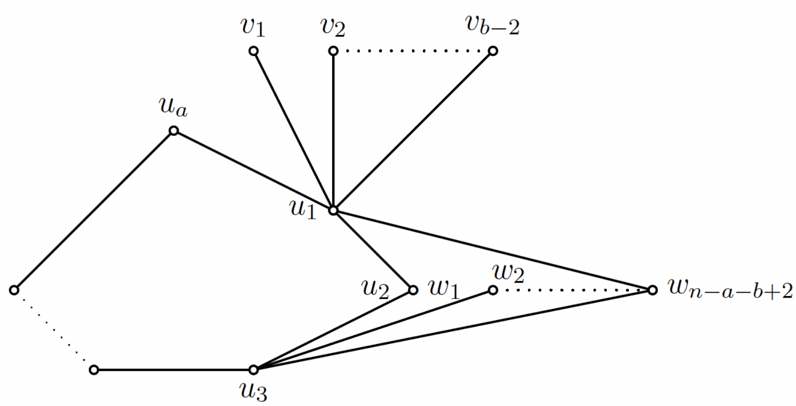

Figure 14. G : a is odd with $D_{\triangle f}(G) = a$ and $wdn_{\triangle f}(G) = b$.

Problem 3.1. *For any three positive integers* a, b *and* c *with* $3 \leq a \leq b$ *and* $c \leq b+1$, *does there exists a connected graph* G *such that* $R_{\triangle f} = a$, $D_{\triangle f} = b$ *and* $wdn_{\triangle f}(G) = c$.

Problem 3.2. *Characterize graphs* G *of order* n *for which* $wdn_{\triangle f}(G) = n-1$.

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