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On energy, Laplacian energy and p-fold graphs

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Abstract

For a graph G having adjacency spectrum (A-spectrum) $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ and Laplacian spectrum (L-spectrum) $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$, the energy is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ and the Laplacian energy is defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. In this paper, we give upper and lower bounds for the energy of KK_n^j, $1 \leq j \leq n$ and as a consequence we generalize a result of Stevanovic et al. [22]. We also consider strong double graph and strong p-fold graph to construct some new families of graphs G for which $E(G) > LE(G)$.

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1. Introduction

Let G be a finite, simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A = (a_{ij})$ of G is a $(0, 1)$-square matrix of order n whose (i, j)-entry is equal to 1 if v_i is adjacent to v_j and equal to 0, otherwise. The spectrum of the adjacency matrix is called the A-spectrum of G. If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the adjacency spectrum of G, the energy [11] of G is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

This quantity introduced by I. Gutman has noteworthy chemical applications (see [13, 16, 21]).

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Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to G, where d_i is the degree of vertex v_i. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively called Laplacian and signless Laplacian matrices and their spectrum are respectively called Laplacian spectrum (L-spectrum) and signless Laplacian spectrum (Q-spectrum) of G. Being real symmetric, positive semi-definite matrices, we let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ to be respectively the L-spectrum and Q-spectrum of G. It is well known [8] that μ_n=0 with multiplicity equal to the number of connected components of G. Fiedler [8] showed that a graph G is connected if and only if its second smallest Laplacian eigenvalue is positive and called it as the algebraic connectivity of the graph G. Also it is well known that for a bipartite graph the L-spectra and Q-spectra are same [6]. For the sake of simplicity, we denote $a_i^{[t_j]}$ if the A-eigenvalue (L-eigenvalue) a_i occurs t_j times in the A-spectrum (L-spectrum).

The Laplacian energy of a graph G as put forward by Gutman and Zhou [14] is defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. This quantity, which is an extension of graph-energy concept has found remarkable chemical applications beyond the molecular orbital theory of conjugated molecules [20]. Both energy and Laplacian energy have been extensively studied in the literature (see [1, 2, 10, 7, 16, 23, 24, 25] and the references therein). It is easy to see that $tr(L(G)) = \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n-1} \mu_i = 2m$ and $tr(Q(G)) = \sum_{i=1}^{n} q_i = 2m$.

The strong double graph of a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the graph $SD(G)$ obtained by taking two copies of the graph G and joining each vertex v_i in one copy with the closed neighbourhood $N[v_i] = N(v_i) \cup \{v_i\}$ of the corresponding vertex in the other copy. For various properties of $SD(G)$ see [4]. The strong p-fold graph $SPF(G)$ of the graph G is a graph obtained by taking p-copies of the graph G and joining each vertex v_i in one copy with the closed neighbourhood $N[v_i] = N(v_i) \cup \{v_i\}$ of corresponding vertex in every other copy (e.g., see Figure 1). It is easy to see that the graphs $SD(G)$ and $SPF(G)$ are connected if and only if G is connected; and a vertex v_i is of degree d_i in G if and only if it is of degree $2d_i + 1$ and $pd_i + p - 1$ in $SD(G)$ and $SPF(G)$, respectively. Also the graphs $SD(G)$ and $SPF(G)$ always contain a perfect matching (1-factor). If K_p is the complete graph on p-vertices, it is easy to see that $SD(G) = G \circ K_2$ and $SPF(G) = G \circ K_p$, where $\circ$ represents the composition of the graphs.

Let $KK_n^j,\ 1 \leq j \leq n$ be the graph obtained by taking two copies of the graph K_n and joining a vertex in one copy with the $j,\ 1 \leq j \leq n$, vertices in another copy.

Gutman et al. [12] conjectured that the inequality $E(G) \leq LE(G)$ holds for all graphs. It was Stevanović et al. [22] who disproved the conjecture by furnishing an infinite family of graphs $G = KK_n^2$, for which the reverse inequality holds for all $n \geq 8$. As can be seen in [15], for $n = 7$, there is only one graph (see graph H in Figure 2) for which $E(G) > LE(G)$ holds. Using this graph Liu and Liu [15] constructed an infinite family of disconnected graphs for which $E(G) > LE(G)$ holds. Recently two of the authors [18] defined strong double graph $SD(G)$ of a graph G and showed that $E(G) > LE(G)$ holds for $SD(KK_n^2)$, for all $n \geq 9$. In this paper, we give upper and lower bounds for the energy of $KK_n^j,\ 1 \leq j \leq n$, and as a consequence we generalize a result of Stevanovic et al. [22]. We also consider strong double graph and strong

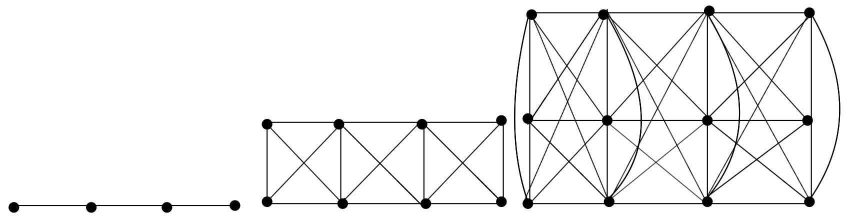

Figure 1. The strong double graph and strong 3-fold graph of P_4.

p-fold graph to construct some new families of graphs G for which

$$E(G) \ > \ LE(G). \tag{1}$$

Using singular value inequality it can be seen that for bipartite graphs the inequality $E(G) \leq LE(G)$ always holds [21]. So for the reverse inequality we will search for non-bipartite graphs. For other undefined notations and terminology from graph theory and spectral graph theory, the readers are referred to [5, 17].

Let $KK_n^j,\ 1 \leq j \leq n$ be the graph defined above. The A-spectrum and L-spectrum of KK_n^j were found in [9] and are given by the following results.

Lemma 1.1. *If $1 \leq j \leq n$, $n \geq 3$, the A-characteristic polynomial of KK_n^j is $(x+1)^{2n-4}h(x)$, where $h(x) = x^4 + (4-2n)x^3 + (n^2 - 6n + 6 - j)x^2 + (2n^2 - 6n + 2nj - j^2 - 3j + 4)x + (1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3jn - jn^2)$.*

Lemma 1.2. *If $1 \leq j \leq n$, $n \geq 3$, the L-characteristic polynomial of KK_n^j is $x(x-n)^{2n-j-2}(x-n-1)^{j-1}g(x)$, where $g(x) = x^2 - (n+1+j)x + 2j$.*

By Lemma 1.2, the L-spectrum of the graph KK_n^j is

$$\{n^{[2n-j-2]}, n+1^{[j-1]}, \tfrac{(n+j+1)+\sqrt{(n+j+1)^2-8j}}{2}, \tfrac{(n+j+1)-\sqrt{(n+j+1)^2-8j}}{2}, 0\},$$

with average vertex degree $n - 1 + \frac{j}{n}$. Therefore,

$$\begin{aligned} LE(KK_n^j) &= (2n-j-2)|n-n+1-\frac{j}{n}| + (j-1)|n+1-n+1-\frac{j}{n}| \\ &\quad + |\frac{(n+j+1)+\sqrt{(n+j+1)^2-8j}}{2} - n + 1 - \frac{j}{n}| + |0 - n + 1 - \frac{j}{n}| \\ &\quad + |\frac{(n+j+1)-\sqrt{(n+j+1)^2-8j}}{2} - n + 1 - \frac{3}{n}| \\ &= 3n - j + \frac{4j}{n} - 5 + \sqrt{(n+j+1)^2 - 8j}. \end{aligned}$$

So for any $j,\ 1 \leq j \leq n$, the Laplacian energy of the graph KK_n^j is

$$LE(KK_n^j) = 3n - j + \frac{4j}{n} - 5 + \sqrt{(n+j+1)^2 - 8j}. \tag{2}$$

It is easy to see that $LE(KK_n^j)$ is an increasing function of $j,\ 1 \leq j \leq n$. Therefore it follows that $\{KK_n^j,\ 1 \leq j \leq n\}$ gives a family of graphs where adding an edge one by one, increases the Laplacian energy monotonically. So we have the following observation.

Theorem 1.1. *Among the family $\{KK_n^j,\ 1 \leq j \leq n\}$, the graph KK_n^1 has the minimal Laplacian energy and the graph KK_n^n has the maximal Laplacian energy.*

Two graphs G_1 and G_2 of same order are said to be equienergetic if $E(G_1) = E(G_2)$ see [2]. In analogy to this two graphs G_1 and G_2 of same order are said to L-equienergetic if $LE(G_1) = LE(G_2)$ see [10, 18, 19]. Since cospectral (Laplacian cospectral) graphs are always equienergetic (L-equienergetic) the problem of constructing equienergetic (L-equienergetic) graphs is only considered for non-cospecral (non-Laplacian-cospectral) graphs.

For $j = n$, we have $LE(KK_n^n) = 3n - n + \frac{4n}{n} - 5 + \sqrt{(n+n+1)^2 - 8n} = 4n - 2 = LE(K_{2n})$. Since the L-spectrum of the graph K_{2n} is $\{2n^{[2n-1]}, 0\}$, it follows by Lemma 1.2, these graphs are non-Laplacian cospectral. Therefore we have the following.

Theorem 1.2. *For $j \in \mathbb{N},\ 1 \leq j \leq n$, the graphs KK_n^n and K_{2n} are non-Laplacian cospectral, Laplacian equienergetic graphs.*

Let G and H be two graphs with disjoint vertex sets. Let $u \in V(G)$ and $v \in V(H)$. Construct the graph $G \star H$ from copies of G and H, by identifying the vertices u and v. Thus $|V(G \star H)| = |V(G)| + |V(H)| - 1$. The graph $G \star H$ is known as the coalescence of G and H with respect to u and v. For $G = K_n$, $H = K_{n+1}$ and u(respectively v) any vertex of G(respectively H), we have $G \star H = K_n \star K_{n+1} = KK_n^n$. So we have the following consequence.

Corollary 1.1. *If $G = K_n$ and $H = K_{n+1}$, then*

$$\begin{aligned} LE(G \star H) &= LE(KK_n^n) = LE(K_{2n}) \\ &= 4n - 2 = 2n - 2 + 2(n+1) - 2 = LE(K_n) + LE(K_{n+1}). \end{aligned}$$

From this, it follows that the Laplacian energy of the coalescence of a complete graph on n vertices with a complete graph on $n+1$ vertices is the sum of their Laplacian energies, which in turn is same as the Laplacian energy of the complete graph on $2n$ vertices.

In [22], it is shown that inequality (1) holds for the graph KK_n^2. Here we first show that the inequality (1) also holds for the graphs KK_n^3 and KK_n^4, and using this argument, we prove a general result (Theorem 1.4), which generalizes Proposition 1 (of [22]).

Theorem 1.3. *For $n \geq 8$ and $j = 3, 4$, we have* $E(KK_n^j) > LE(KK_n^j)$.

Proof. For $j = 3$, it follows from Lemma 1.1, that the A-characteristic polynomial $P(KK_n^3, x)$ of the graph KK_n^3 is $P(KK_n^3, x) = (x+1)^{2n-4}h(x)$, where $h(x) = x^4 - 2(n-2)x^3 + (n^2 - 6n + 3)x^2 + (2n^2 - 14)x + (16n - 23 - 2n^2)$.

For $n \geq 8$, we have $h(n) = n^2 + 2n - 23 > 0$, $h(n-1) = -9 < 0$, $h(n-2) = (n-1)^2 > 0$, $h(1) = n^2 + 8n - 29 > 0$, $h(0) = -2n^2 + 16n - 23 < 0$, $h(-2.3) = -1.31n^2 + 8.594n + 4.3861 < 0$, $h(-3) = n^2 + 16n + 19 > 0$.

Therefore, $h(x)$ has three positive roots, one in each of the intervals $(0, 1), (n-2, n-1)$ and $(n-1, n)$, and a single negative root in the interval $(-3, -2.3)$. Assume that x_1, x_2, x_3, x_4 are the roots of $h(x)$ with $x_1, x_2, x_3 > 0$ and $x_4 < 0$. Therefore the A-spectrum of the graph KK_n^3 is $\{-1^{[2n-4]}, x_1, x_2, x_3, x_4\}$, with $x_1 + x_2 + x_3 + x_4 = 2(n-2)$. We have

$$\begin{aligned}
E(KK_n^3) &= (2n-4)|-1| + |x_1| + |x_2| + |x_3| + |x_4| \\
&= 2n - 4 + x_1 + x_2 + x_3 - x_4 \\
&= 2n - 4 + 2n - 4 - 2x_4 \\
&> 4n - 3.4.
\end{aligned}$$

By (2), the Laplacian energy of KK_n^3 is

$$LE(KK_n^3) = 3n - 8 + \frac{12}{n} + \sqrt{n^2 + 8n - 8}.$$

So $E(KK_n^3) - LE(KK_n^3) = n + 4.6 - \frac{12}{n} - \sqrt{n^2 + 8n - 8} = g(n)$. It is easy to see that $g(n) > 0$ for all $n \geq 8$. That is, $E(KK_n^3) > LE(KK_n^3)$, for all $n \geq 8$.

Using the same argument as above, it can be seen that for $j = 4$, the polynomial $h(x)$ has three positive roots, one in each of the intervals $(0, 1), (n-2, n-1)$ and $(n-1, n)$, and a single negative root in the interval $(-3, -2.4)$. So proceeding similarly the result follows. □

Now we obtain the lower and upper bounds for the energy of KK_n^j.

Theorem 1.4. *For $k \in \mathbb{N} - \{1\}, (k-1)^2 < j \leq k^2$ and $n \geq ((k-1)^2 + 2)^2 - (k-1)^2$, we have*

$$4n - 8 + 2k \quad < \quad E(KK_n^j) \quad < \quad 4n - 8 + 2(k+1).$$

Proof. By Lemma 1.1, the A-characteristic polynomial $P(KK_n^j, x)$ of the graph KK_n^j is

$$P(KK_n^j, x) = (x+1)^{2n-4}h(x),$$

where

$$\begin{aligned}
h(x) = x^4 &+ (4 - 2n)x^3 + (n^2 - 6n + 6 - j)x^2 \\
&+ (2n^2 - 6n + 2nj - j^2 - 3j + 4)x \\
&+ (1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3jn - jn^2).
\end{aligned}$$

Let x_1, x_2, x_3, x_4 be the zeros of the polynomial $h(x)$. Then the spectrum of the graph KK_n^j is $\{-1^{[2n-4]}, x_1, x_2, x_3, x_4\}$.

For $(k-1)^2 < j \le k^2$ and $n \ge ((k-1)^2+2)^2 - (k-1)^2$, we have the following.

$$\begin{aligned}
h(n) &= n^2 + 2n + 1 - 2j^2 - 2j \ > \ 0,\\
h(n-1) &= -j^2 \ < \ 0,\\
h(n-2) &= (n-1)^2 \ > \ 0,\\
h(0) &= 1 - 2j - 2j^2 - 2n + 3nj + nj^2 + n^2 - jn^2 \ < \ 0,\\
h(-k) &= k^4 + (2n-4)k^3 + (n^2 - 6n + 6 - j)k^2 - (2n^2 - 6n + 2nj - j^2 - 3j + 4)k\\
&\quad + (1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3jn - jn^2) \ < \ 0,\\
h(-(k+1)) &= k^4 + 2nk^3 + (n^2 - j)k^2 + (j^2 + j - 2nj)k + (jn + nj^2 - jn^2 - j^2) \ > \ 0.
\end{aligned}$$

Therefore, by Intermediate Value Theorem, it follows that $h(x)$ has three positive roots, one in each of the intervals $(0, n-2), (n-2, n-1)$ and $(n-1, n)$, and a single negative root in the interval $(-(k+1), -k)$. Assume that $x_1, x_2, x_3 \ > \ 0$ and $x_4 \ < \ 0$. Since $x_1 + x_2 + x_3 + x_4 = 2(n-2)$. We have

$$\begin{aligned}
E(KK_n^j) &= (2n-4)|-1| + |x_1| + |x_2| + |x_3| + |x_4|\\
&= 2n - 4 + x_1 + x_2 + x_3 - x_4\\
&= 2n - 4 + 2n - 4 - 2x_4\\
&= 4n - 8 - 2x_4.
\end{aligned}$$

The result follows from the fact that $x_4 \in (-(k+1), -k)$ implies $-(k+1) < x_4 < -k$, which implies $k < -x_4 < k+1$. □

Since $(k-1)^2 < j \le k^2$ implies $k - 1 < \sqrt{j} < k$, we have the following consequence of Theorem 1.4.

Corollary 1.2. *For $k \in \mathbb{N} - \{1\}, (k-1)^2 < j \le k^2$ and $n \ge ((k-1)^2+2)^2 - (k-1)^2$, we have*

$$E(KK_n^j) \ > \ 4n - 8 + 2\sqrt{j}.$$

A graph G on n vertices is said to be hyperenergetic if its energy exceeds the energy of the complete graph K_n, that is $E(G) \ > \ E(K_n) = 2(n-1)$. Since KK_n^j is a graph on $2n$ vertices, we have the following.

Corollary 1.3. *For $k \in \mathbb{N} - \{1, 2\}, (k-1)^2 < j \le k^2$ and $n \ge ((k-1)^2+2)^2 - (k-1)^2$, the graph KK_n^j is hyperenergetic.*

Proof. Since $k \ge 3$, we have by Theorem 1.4, $E(KK_n^j) \ > \ 4n-8+2k \ \ge \ 4n-2 \ = \ E(K_{2n})$. □

Corollary 1.4. *For $k \in \mathbb{N} - \{1\}$, $(k-1)^2 < j \le k^2$ and $n \ge ((k-1)^2+2)^2 - (k-1)^2$, we have*

$$E(KK_n^j) \; > \; LE(KK_n^j).$$

Proof. For $k = 2$, we have $j = 2, 3, 4$ and $n \ge 8$, the result follows by Proposition 1 (of [22]) and Theorem 1.3. So assume that $k \ge 3$. By equation (2) and Corollary 1.2, we have

$$\begin{aligned} E(KK_n^j) - LE(KK_n^j) &= 4n - 8 + 2\sqrt{j} - 3n + j - \frac{4j}{n} + 5 - \sqrt{(n+j+1)^2 - 8j} \\ &= n + 2\sqrt{j} + j - 3 - \frac{4j}{n} - \sqrt{(n+j+1)^2 - 8j} = g(n). \end{aligned}$$

It is easy to see that $g(n) > 0$, for $n \ge ((k-1)^2+2)^2 - (k-1)^2,\ k \ge 3$. Therefore the result follows. □

By a suitable labelling of vertices, the adjacency matrix $A = A(KK_n^j)$ of the graph KK_n^j, $1 \le j \le n$, can be put in the form

$$A = \begin{pmatrix} 0 & x_{2n-1} \\ x_{2n-1}^t & B \end{pmatrix},$$

where x_{2n-1} is a $(2n-1)$-vector having first $(n-1+j)$-entries equal to 1 and rest 0 and B is the adjacency matrix of the graph $K_{n-1} \cup K_n$.

Let the eigenvalues of A be $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{2n-1} \ge \lambda_{2n}$. Since the spectrum of B is $\{n-1, n-2, -1^{[2n-3]}\}$, by interlacing inequalities for principal submatrix, we have

$$\lambda_1 \ge n-1 \ge \lambda_2 \ge n-2 \ge \lambda_3 \ge -1 \ge \lambda_4 \ge -1 \ge \cdots \ge -1 \ge \lambda_{2n-1} \ge -1 \ge \lambda_{2n}.$$

From this it follows that $\lambda_1 \in (2n-1, n-1)$, $\lambda_2 \in (n-2, n-1)$, $\lambda_3 \in (-1, n-2)$, $\lambda_{2n} \in (-1, -2n+1)$ and $\lambda_4 = \lambda_5 = \cdots = \lambda_{2n-1} = -1$. This shows that the eigenvalue $\lambda_1,\ \lambda_2$ are always positive and λ_{2n} always negative, while as λ_3 may be positive or negative. Also it is clear from this and Lemma 1, that $\lambda_1, \lambda_2, \lambda_3, \lambda_{2n}$ are the zeros of the polynomial $h(x) = h(x) = x^4 + (4-2n)x^3 + (n^2-6n+6-j)x^2 + (2n^2-6n+2nj-j^2-3j+4)x + (1+nj^2-2j^2+n^2-2n-2j+3jn-jn^2)$. So $\lambda_1+\lambda_2+\lambda_3+\lambda_{2n} = 2n-4$ and $\lambda_1\lambda_2\lambda_3\lambda_{2n} = 1+nj^2-2j^2+n^2-2n-2j+3jn-jn^2$. Since $\lambda_1, \lambda_2 > 0$ and $\lambda_{2n} < 0$, it follows that $\lambda_3 > 0$ if and only if $1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3jn - jn^2 < 0$, which is so if and only if $2 \le j \le n-3$. Therefore we have the following result.

Theorem 1.5. *For $5 \le j \le n-3$ and $n \ge 9$, we have $E(KK_n^j) \; > \; LE(KK_n^j)$ if and only if*

$$n > \frac{j^2 - 3j + 16 + \sqrt{(j^2-3j+16)^2 + 4(j-4)(j^2-2j+16)}}{2(j-4)}.$$

Proof. Since, for $5 \le j \le n-3$, the eigenvalue $\lambda_3 > 0$, therefore we have

$$\begin{aligned} E(KK_n^j) &= (2n-4)|-1| + |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_{2n}| \\ &= 2n - 4 + \lambda_1 + \lambda_2 + \lambda_3 - \lambda_{2n} \\ &= 2n - 4 + 2n - 4 - 2\lambda_{2n} \\ &= 4n - 8 - 2\lambda_{2n}. \end{aligned}$$

Also by Theorem 1.1, we have $4n-4 = LE(KK_n^0) < LE(KK_n^1) < LE(KK_n^j) < LE(KK_n^n) = 4n-2$, for all $5 \leq j \leq n-3$. So instead of showing $E(KK_n^j) > LE(KK_n^j)$, we will show $E(KK_n^j) > LE(KK_n^n)$. We have

$$\begin{aligned} E(KK_n^j) - LE(KK_n^n) &= 4n - 8 - 2\lambda_{2n} - 4n + 2 \\ &= -6 - 2\lambda_{2n} > 0 \end{aligned}$$

if and only if $\lambda_{2n} < -3$ which, by the Intermediate Value Theorem, is equivalent to $h(-3) < 0$, that is $(j-4)n^2 - (j^2-3j+16)n - (j^2-2j+16) > 0$, that is $n > \frac{j^2-3j+16+\sqrt{(j^2-3j+16)^2+4(j-4)(j^2-2j+16)}}{2(j-4)}$.

□

The conditions of Theorem 1.5 are also sufficient for the graph KK_n^j to be hyperenergetic.

If u (respectively v) is a vertex in G (respectively H) and $G \star H$ is their coalescence, then it is shown in [21] that

$$E(G \star H) \leq E(G) + E(H), \tag{3}$$

with equality if and only if either u is an isolated vertex of G or v is an isolated vertex of H or both are isolated vertices.

For $j = n$, we have $KK_n^n = K_n \star K_{n+1}$. So for $G = K_n$ and $H = K_{n+1}$, we have by (3)

$$\begin{aligned} E(KK_n^n) &= E(K_n \star K_{n+1}) < E(K_n) + E(K_{n+1}) \\ &= 2n - 2 + 2(n+1) - 2 = 4n - 2 = LE(KK_n^n). \end{aligned}$$

From this it follows that the graph KK_n^n is not hyperenergetic.

2. On strong graphs and strong *p*-fold graphs

For a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the strong double graph $SD(G)$ is a graph obtained by taking two copies of G and joining each vertex v_i in one copy with the closed neighbourhood $N[v_i] = N(v_i) \cup \{v_i\}$ of corresponding vertex in another copy. In other words, strong double graph of the graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the graph $SD(G)$ with vertex set $V(SD(G)) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$, where the adjacency is defined as follows. $x_i(y_i)$ is adjacent to $x_j(y_j)$ if v_i adjacent to v_j; and x_i adjacent to y_j if $i = j$ or v_i adjacent to v_j (see Figure 1).

The following observations can be found in [18].

Lemma 2.1. *If λ_i, $i = 1, 2, \ldots, n$, is the A-spectrum of the graph G, then the A-spectrum of the graph $SD(G)$ is $2\lambda_i + 1$, $-1^{[n]}$, $i = 1, 2, \ldots, n$.*

Lemma 2.2. *If μ_i and d_i, $i = 1, 2, \ldots, n$, are respectively the L-spectrum and degree sequence of the graph G, then the L-spectrum of the graph $SD(G)$ is $2\mu_i, 2d_i + 2$, $i = 1, 2, \ldots, n$.*

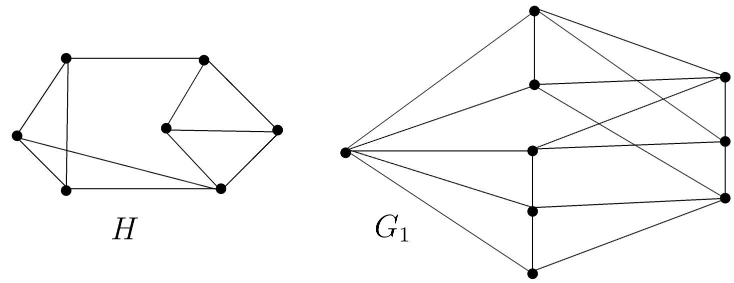

Figure 2. Graph H is the only graph on 7 vertices with $E(H) > LE(H)$. Graph G_1 is one of the graphs with $E(G_1) > LE(G_1)$, but $E(SPF(G_1)) \leq LE(SPF(G))$.

For the graph H (see Figure 2) it is shown in [15] that $E(H) > LE(H)$ and using this, an infinite families of graphs (disconnected) were constructed for which the inequality (1) holds. Here we show inequality (1) also holds for $SD(H)$. By direct calculation it can seen that the A-spectrum of H is

$$\{3.17741, 1.73205, 0.67836, 1^{[2]}, 1.73205, 1.85577\}$$

and its L-spectrum is

$$\{4+\sqrt{2}, 3+\sqrt{3}, 4^{[2]}, 4-\sqrt{2}, 3-\sqrt{3}, 0\}.$$

Using Lemmas 2.1 and 2.2, and the fact that the degree sequence of H is $[4,3,3,3,3,3,3]$, it follows that the A-spectrum and L-spectrum of the graph $SD(H)$ are respectively as

$$\{7.35482, 4.4641, 2.35672, -1^{[9]}, -2.4641, -2.71154\}$$

and

$$\{10, 8+2\sqrt{2}, 6+2\sqrt{3}, 8^{[8]}, 8-2\sqrt{2}, 6-2\sqrt{3}, 0\}.$$

Therefore $LE(SD(H)) = 28.299377 < 28.3512 = E(SD(H))$. That proves the assertion.

For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, let $SPF(G)$ be the graph obtained by taking p-copies of the graph G and joining each vertex v_i in one copy with the closed neighbourhood $N[v_i] = N(v_i) \cup \{v_i\}$ of the corresponding vertex in every other copy. By a suitable labelling of vertices, it can be seen that the adjacency matrix $\widehat{A}$ of the graph $SPF(G)$ is

$$\widehat{A} = \begin{pmatrix} A & A+I & \cdots & A+I \\ A+I & A & \cdots & A+I \\ \vdots & \vdots & \cdots & \vdots \\ A+I & A+I & \cdots & A \end{pmatrix},$$

where A is the adjacency matrix of G and I is the identity matrix of order equal to the order of A.

Therefore the characteristic polynomial

$$|\lambda I_{pn} - \widehat{A}| = \begin{vmatrix} \lambda I_n - A & -(A+I) & \cdots & -(A+I) \\ -(A+I) & \lambda I_n - A & \cdots & -(A+I) \\ \vdots & \vdots & \cdots & \vdots \\ -(A+I) & -(A+I) & \cdots & \lambda I_n - A \end{vmatrix},$$

Using elementary transformations $C_1 \to C_1 + C_2 + \cdots + C_p$ and then $R_i \to R_i - R_1$, for $i = 2, 3, \ldots, p$, it can be seen that the spectrum of the matrix $\widehat{A}$ and so the A-spectrum of the graph $SPF(G)$ is

$$\{-1^{[n(p-1)]}, px_1 + p - 1, px_2 + p - 1, \ldots, px_n + p - 1\}, \tag{4}$$

where $x_1, x_2, \ldots, x_n$ are the adjacency eigenvalues of the graph G.

Also the degree matrix $\widehat{D}$ of the graph $SPF(G)$ is

$$\widehat{D} = \begin{pmatrix} pD + (p-1)I & 0 & \cdots & 0 \\ 0 & pD + (p-1)I & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & pD + (p-1)I \end{pmatrix}.$$

So the Laplacian matrix $\widehat{L}$ of the graph $SPF(G)$ is

$$\widehat{L} = \begin{pmatrix} pD + (p-1)I - A & -(A+I) & \cdots & -(A+I) \\ -(A+I) & pD + (p-1)I - A & \cdots & -(A+I) \\ \vdots & \vdots & \cdots & \vdots \\ -(A+I) & -(A+I) & \cdots & pD + (p-1)I - A \end{pmatrix}.$$

Proceeding similarly as above, it can be seen that the L-spectrum of the graph $SPF(G)$ is

$$\{p\mu_1, p\mu_2, \ldots, p\mu_n, pd_1 + p^{[p-1]}, pd_2 + p^{[p-1]}, \ldots, pd_n + p^{[p-1]}\}, \tag{5}$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues of G and $d_1, d_2, \ldots, d_n$ are the degrees of the vertices in G.

The next result gives a two way infinite families of graphs G for which the inequality (1) holds.

Theorem 2.1. *For $j = 2, 3, 4,\ p = 2, 3$ and $n \geq 9$ and for $j = 2, 3, 4,\ p \geq 4$ and $n > pj$, we have*

$$E(SPF(KK_n^j)) \;>\; LE(SPF(KK_n^j)).$$

Proof. For $p = 2$ and $j = 2, 3, 4$, we have $SPF(KK_n^j) \cong SD(KK_n^2)$ or $SD(KK_n^3)$ or $SD(KK_n^4)$, respectively. If $SPF(KK_n^j) \cong SD(KK_n^2)$, then the result follows by Theorem 4.4 in [18]. If $SPF(KK_n^j) \cong SD(KK_n^3)$ or $SD(KK_n^3)$, then the result follows by proceeding similarly as in Theorem 4.4 in [18]. Also for $p = 3$ and $j = 2, 3, 4$, we have $SPF(KK_n^j) \cong KK_n^2 \circ K_3$ or $KK_n^3 \circ K_3$ or $KK_n^4 \circ K_3$, respectively. Here we will show the result holds for $SPF(KK_n^j) \cong KK_n^2 \circ K_3$, and the result for the other two cases follows similarly.

The A-spectrum of the graph $KK_n^2 \circ K_3$ is $\{-1^{[6n-4]}, 3x_1 + 2, 3x_2 + 2, 3x_3 + 2, 3x_4 + 2\}$, where x_1, x_2, x_2, x_4 are the zeros of $h(x) = x^4 - 2(n-2)x^3 + (n^2 - 6n + 4)x^2 + 2(n^2 - n - 3)x -$

$(n^2-8n+11)$. Proceeding similarly as in Theorem 1.3, it can be seen that $x_1, x_2, x_3 > 0$ and $x_4 \in (-3, -2.2)$ for $n \geq 9$. So, we have

$$E(KK_n^2 \circ K_3) = 12n - 6x_4 - 12 > 12n + 1.2.$$

Also the L-spectrum of the graph $KK_n^2 \circ K_3$ is $\{3n^{[6n-10]}, 3n+3^{[5]}, 3n+6^{[2]}, \frac{3(n+3)\pm\sqrt{n^2+6n-7}}{2}, 0\}$, with average vertex degree $3n-1+\frac{6}{n}$. We have

$$LE(KK_n^2 \circ K_3) = 9n - 13 + \frac{24}{n} + 3\sqrt{n^2+6n-7}.$$

Therefore $E(KK_n^2 \circ K_3) - LE(KK_n^2 \circ K_3) = 3n + 14.2 - \frac{24}{n} - 3\sqrt{n^2+6n-7} = g(n)$. It is easy to see that $g(n) > 0$, for all $n \geq 9$.

So assume that $p \geq 4$ and $j = 2, 3, 4$. Using (4) and Lemma 1.1, it follows that the A-spectrum of the graph $SPF(KK_n^j)$ is

$$\{-1^{[2pn-4]}, px_1 + (p-1), px_2 + (p-1), px_3 + (p-1), px_4 + (p-1)\},$$

where x_1, x_2, x_3, x_4 are the zeros of the polynomial $h(x) = x^4 - 2(n-2)x^3 + (n^2 - 6n + 6 - j)x^2 + (2n^2 - 6n + 2nj - j^2 - 3j + 4)x + (1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3nj - jn^2)$.

For $n > pj$, $p \geq 4$ and $j = 2, 3, 4$, we have $h(n) = n^2 + 2n - 2j^2 - 2j + 1 > 0$, $h(n-1) = -j^2 < 0$, $h(n-2) = (n-1)^2 > 0$, $h(1) = 16 - 6j - 3j^2 - 16n + 5jn + nj^2 + 4n^2 - jn^2 > 0$, $h(0) = 1 + nj^2 - 2j^2 + n^2 - 2n - 2j + 3nj - jn^2 < 0$, $h(-3) = 16 - 2j + j^2 + 16n - 3jn + nj^2 + 4n^2 - jn^2 > 0$,

$$h(-2.j) = \begin{cases} -0.56n^2 + 4.656n + 2.3936 < 0, & \text{if } j = 2 \\ -1.31n^2 + 8.594n + 4.3861 < 0, & \text{if } j = 3 \\ -2.04n^2 + 14.288n + 8.0016 < 0 & \text{if } j = 4. \end{cases}$$

Therefore, $h(x)$ has three positive roots, one in each of the intervals $(0,1), (n-2, n-1)$ and $(n-1, n)$, and a single negative root in the interval $(-3, -2.j)$. Assume that $x_1, x_2, x_3 > 0$ and $x_4 < 0$. We have

$$\begin{aligned} E(SPF(KK_n^j)) &= (2pn-4)|-1| + |px_1 + p - 1| + |px_2 + p - 1| \\ &+ |px_3 + p - 1| + |px_4 + p - 1| \\ &= 2pn - 4 + p(x_1 + x_2 + x_3) - px_4 + 2p - 2 \\ &= 2pn - 4 + p(2n - 4 - x_4) - px_4 + 2p - 2 \\ &= 4pn - 2p - 2px_4 - 6 \\ &> 4pn + 2p(1.j) - 6. \end{aligned}$$

Also by Lemma 1.2, equation (5) and the fact that the degree sequence of the graph KK_n^j is $[n+j-1, n^{[j]}, (n-1)^{[2n-j-1]}]$, it follows that the L-spectrum of the graph $SPF(KK_n^j)$ is

$\{pn^{[2pn-p(j+1)-1]}, p(n+1)^{[pj-1]}, p(n+j)^{[p-1]}, \frac{p((n+j+1)\pm\sqrt{(n+j+1)^2-8j)}}{2}, 0\}$

with average vertex degree $pn-1+\frac{pj}{n}$. Therefore, we have

$$\begin{aligned}
&LE(SPF(KK_n^j)\\
&=(2pn-p(j+1)-1)|pn-pn+1-\frac{pj}{n}|+(pj-1)|pn+p-pn+1-\frac{pj}{n}|\\
&+(p-1)|pn+pj-pn+1-\frac{pj}{n}|+|0-pn+1-\frac{pj}{n}|+\\
&+|\frac{p((n+j+1)-\sqrt{(n+j+1)^2-8j})}{2}-pn+1-\frac{pj}{n}|\\
&+|\frac{p((n+j+1)+\sqrt{(n+j+1)^2-8j})}{2}-pn+1-\frac{pj}{n}|\\
&=3pn-p(j+1)-4+\frac{4pj}{n}+p\sqrt{(n+j+1)^2-8j}.
\end{aligned}$$

For $n>pj$, $p\geq 4$ and $j=2,3,4$, we have $E(SPF(KK_n^j))-LE(SPF(KK_n^j))=pn+p(2(1.j)+j+1)-2-\frac{4pj}{n}-p\sqrt{(n+j+1)^2-8j}>0$. That is, $E(SPF(KK_n^j))>LE(SPF(KK_n^j))$, for all $n>pj$, $p\geq 4$ and $j=2,3,4$.

Remark 2.1. From Theorems 1.3 and 2.1, one may get an insight that the inequality $E(SPF(G))>LE(SPF(G))$ holds whenever the inequality $E(G)>LE(G)$ holds. This is not always true, in fact there are graphs G for which $E(G)>LE(G)$ hold, but $E(SPF(G))>LE(SPF(G))$ does not hold. For example, consider the graph G_1 as shown in Figure 2. For this graph A-spectrum is

$$\{-2.5616,-2.3444,-1.2837,-0.8643,-0.4633,0.6766,0.8543,1.9383,4.0482\}$$

and L-spectrum is

$$\{0,1.7888,3.1355,3.5858,4.1973,4.6874,5.3643,6.4142,6.8267\}.$$

So $E(G_1)=15.0347>14.9798=LE(G_1)$. By Lemma 1.1, the A-spectrum of the graph $SD(G_1)=S2F(G_1)$ is

$$\{-1^{[9]},-4.1232,-3.6888,-1.5676,-0.7286,0.0734,2.3532,2.7086,4.8766,9.0964\}.$$

Also by Lemma 1.2 and the fact the degree sequence of the graph G_1 is $[3,4,4,4,4,4,4,4,5]$, it follows that the L-spectrum of $SD(G_1)$ is

$$\{0,3.5776,6.271,7.1716,8.3946,9.3748,10.7286,12.8284,13.6534,8,10^{[7]},12\}.$$

Therefore $E(SD(G_1))=38.2162 < 41.1704=LE(SD(G_1))$.

Although the conjecture that "the inequality $LE(G)\geq E(G)$ holds for all G" has been disproved. This inequality holds for most of the graphs as shown in [12] and [22]. Therefore the following problem will be of great interest.

Problem 1. *Characterize all non-bipartite graphs G for which the inequality $LE(G)\geq E(G)$ holds.*

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