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A note on the Ramsey number for a cycle with respect to a disjoint union of wheels

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Abstract

Let K_n be a complete graph with n vertices. For graphs G and H, *the Ramsey number* $R(G, H)$ is the smallest positive integer n such that in every red-blue coloring on the edges of K_n, there is a red copy of graph G or a blue copy of graph H in K_n. Determining the Ramsey number $R(C_n, tW_m)$ for any integers $t \geq 1$, $n \geq 3$ and $m \geq 4$ in general is a challenging problem, but we conjecture that for any integers $t \geq 1$ and $m \geq 4$, there exists $n_0 = f(t, m)$ such that cycle C_n is tW_m–good for any $n \geq n_0$. In this paper, we provide some evidence for the conjecture in the case of $m = 4$ that if $n \geq n_0$ then the Ramsey number $R(C_n, tW_4) = 2n + t - 2$ with $n_0 = 15t^2 - 4t + 2$ and $t \geq 1$. Furthermore, if G is a disjoint union of cycles then the Ramsey number $R(G, tW_4)$ is also derived.

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1. Introduction

A notation $G(V, E)$, in short denoted as G, is a graph with the vertex set V and the edge set E. In this paper, we mention that all graphs are simple, undirected and finite. For subgraph H of G, a subgraph $G - H$ of G is constructed from G by deleting the vertex set and the edge set of H including all edges incident to the vertex set of H. Let A be any subset of the vertex set V of G,

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then *the induced subgraph* $G[A]$ of G, is the graph with the vertex set A and the edge set consists of all of the edges in E having both endpoints in A. We follow that *the complement* of a graph G is a graph $\overline{G}$ having the same vertex set V such that two any distinct vertices u, v in V of $\overline{G}$ are adjacent if and only if u and v are not adjacent in G. On the disjoint union of graphs, we use the definition as follow: for positive integers t and i with $i = 1, 2, \ldots, t$; and note that V_i and E_i are respectively the vertex set and the edge set of connected graph G_i. *The disjoint union of graphs*, $\bigcup_{i=1}^{t} G_i$, is a graph with the vertex set $\bigcup_{i=1}^{t} V_i$ and the edge set $\bigcup_{i=1}^{t} E_i$; and if $G_i \simeq G$ for each i then $\bigcup_{i=1}^{t} G_i = tG$.

For the definition of Ramsey number, we cite the results proposed by Sudarsana [13], [11], [12]; that is, for graphs G and H, *the Ramsey number* $R(G, H)$ is the smallest natural number n such that in every red and blue colorings of the edges of the complete graph K_n, there is a graph G in K_n which is all edges are red or a graph H in K_n which is all edges are blue. In the other terminology, a graph F is called (G, H)*–free*, if F contains no G and $\overline{F}$ contains no H. Furthermore, we have an equivalent definition of Ramsey number $R(G, H)$, that is, the smallest positive integer n such that there is no (G, H)–free graph on n vertices exists.

For graphs G and H with $|G| = n$, the chromatic number of H is χ and σ is *the chromatic surplus* of H, that is, the minimum cardinality of a color class taken over all proper colorings of H with χ colors. By using this basic terminology, in 1981 Burr [4] have been proposed the general lower bound of the Ramsey number $R(G, H)$, that is

$$R(G, H) \geq (n-1)(\chi-1) + \sigma \tag{1}$$

and the graph G is called H*–good* when the inequality in equation (1) is equals. The colletion known values of classical Ramsey number $r(n, m)$ and Ramsey number of graph $R(G, H)$ can be found in the dynamic survey of Radziszowski [8], in the other hand its applications colleted by Rosta [9]. On this paper, we denote by W_m for a wheel on $m + 1$ vertices. The notation tW_m discribe a graph with t copies of wheels of order $m + 1$. Surahmat et al. [14] proved that cycle C_n is W_m–good; Chen et al. [6] showed that P_n is W_m–good for even m and $n \geq m - 1 \geq 3$; P_n is tW_4–good for $n \geq 15t^2 - 4t + 2$, $t \geq 1$ [13], and Sudarsana [12] recently proved that C_n is tK_m–good. Meanwhile, S_n is not W_6–good for $n \geq 3$ [5].

In the case of $H \simeq tW_m$, the chromatic surplus $\sigma(H)$ equals t. In general, determining Ramsey number $R(C_n, tW_m)$ for any $m \geq 4$ is a notoriously hard problem. The following theorem shows that if $n \geq n_0$ then the cycle C_n is tW_4–good with $n_0 = 15t^2 - 4t + 2$ and $t \geq 1$.

Theorem 1.1. *For $t \geq 1$ and $f(t) = 15t^2 - 4t + 2$. If $n \geq f(t)$ then $R(C_n, tW_4) = 2n + t - 2$.*

In proving of Theorem 1.1, we use the result of Bondy [3] stated below and the above mentioned result of Surahmat et al. [14].

Lemma 1.1. *[3] Let G be a graph of order n. If the minimum degree of G satisfies $\delta(G) \geq \frac{n}{2}$ then either G is pancyclic or n is even and $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.*

Theorem 1.2. *[14] Let n and m be positive integers. If m is even and $n \geq \frac{5m}{2} - 1$ then $R(C_n, W_m) = 2n - 1$.*

Let G be a graph containing all H–good components, the general formula for finding the exact value of Ramsey number $R(G, H)$ have been found by Bielak [2] and Sudarsana et al. [11]. In some particular graphs, showed by Stahl [10] and Baskoro et al. [1]. Hence, these results give a motivation to study the families of graphs which have H–good properties.

In particular, when G is a disjoint union of cycles then by using the results of Sudarsana et al. [11] and Theorem 1.1 we obtain the Corollary 1.1 below for finding the exact value of Ramsey number $R(G, tW_4)$.

Corollary 1.1. *Let t and k be positive integers and $f(t) = 15t^2 - 4t + 2$. Let $G \simeq \bigcup_{i=1}^{k} l_i C_{n_i}$, where $l_i \geq 1$ and each C_{n_i} is a cycle of order n_i. If $n_1 \geq n_2 \geq \cdots \geq n_k \geq f(t)$ then*

$$R(G, tW_4) = \max_{1 \leq i \leq k} \left\{ n_1 + \sum_{j=i}^{k} l_j n_j \right\} + t - 2. \tag{2}$$

2. The Proof of Theorem

We first show the following lemma which is used to prove Theorem 1.1.

Lemma 2.1. *For positive integers n and t with $t \leq \frac{1}{2}(\lfloor \frac{n}{2} \rfloor - 1)$. Then, $R(C_n, tP_3) = n + t - 1$.*

Proof. The graph $K_{n-1} \cup K_{t-1}$ is (C_n, tP_3)-free with $n + t - 2$ vertices which is concludes the lower bound $R(C_n, tP_3) \geq n + t - 1$.

To obtain the upper bound $R(C_n, tP_3) \leq n + t - 1$, we use induction technique on t. The Result of Faudree et al. [7], $R(P_l, P_3) = l$ for $l \geq 4$, implies that the assertion holds for $t = 1$. Let F be a graph with $|F| = n + t - 1$ containing no C_n and then by induction on t, we have $(t-1)P_3$ in $\overline{F}$. Let $B = \{x_1, y_1, z_1, \ldots, x_{t-1}, y_{t-1}, z_{t-1}\}$ be the vertex set of $(t-1)$ copies of P_3 in $\overline{F}$ and each of path P_3^i has edges $x_i y_i$ and $y_i z_i$ for $i = 1, 2, \ldots, (t-1)$. Let us now consider the subgraph $F[A]$ of F induced by $A = V(F) \backslash B$, where $V(F)$ is the vertex set of graph F. It is clear that the induced subgraph $F[A]$ has order $n - 2t + 2$. Suppose on the contrary that $\overline{F}$ contains no tP_3 and hence the subgraph $F[A]$ has minimum degree at least $n - 2t$ since otherwise the subgraph $\overline{F}[A]$ contains P_3 which together with B produce a copy of tP_3 in $\overline{F}$.

We next consider the relation between the vertices in A and in B. Note that $\overline{F}$ does not contain tP_3. Thus there are at least two vertices of each $\{x_i, y_i, z_i\}$ adjacent to all but at most three vertices in A since otherwise we have two copies of P_3 between the vertices in $\{x_i, y_i, z_i\}$ and in A which together with $B \setminus \{x_i, y_i, z_i\}$ forms a copy of tP_3 in $\overline{F}$. Now without loss of generality, we may assume that each x_i and z_i are adjacent to all but at most three vertices in A. Let $F[D]$ be the subgraph of F induced by the set $D = A \cup \{x_1, z_1, x_2, z_2, \ldots, x_{t-1}, z_{t-1}\}$. Immediatelly, we obtain that $F[D]$ has order n with minimum degree $\delta(F[D]) \geq n - 2t - 1$. Since $t \leq \frac{1}{2}(\lfloor \frac{n}{2} \rfloor - 1)$, it follows that $\delta(F[D]) \geq \frac{n}{2}$. By the result of Bondy's in Lemma 1.1 implies that $F[D]$ contains cycle of order n in F, contradicting with our assumption on F. Hence $\overline{F}$ contains a copy of tP_3 as claimed. This concludes the proof. □

In the following lemma is the weaker form of Theorem 1.1.

Lemma 2.2. *Let $t \geq 1$ be an integer and $f(t) = 15t^2 - 4t + 2$. If F is a graph with order $2n + t - 2$ containing C_{n-1} and $n \geq f(t)$ then F contains C_n or $\overline{F}$ contains tW_4.*

Proof. Let F be a graph with $|F| = 2n + t - 2$ containing C_{n-1}. We will show that F contains C_n or $\overline{F}$ contains tW_4.

Since F contains C_{n-1}, we have that the order of $F - C_{n-1}$ is $n + t - 1$. Note that if $t \geq 1$ then $n \geq f(t) > 4t+2$ (this fact is equivalent with condition in Lemma 2.1), and hence Lemma 2.1 give an implication that there is C_n in $F - C_{n-1}$ or tP_3 is in the complement of $F - C_{n-1}$. If $F - C_{n-1}$ contains C_n then we are done. Therefore, we have cycle C_{n-1} in F and $\overline{F}$ contains t disjoint copies $P_3^1, P_3^2, \ldots, P_3^t$ of path with three vertices. It can be verified that there is no common vertex between C_{n-1} and tP_3.

Assume that there is no C_n in F, we will find tW_4 in $\overline{F}$. Let us consider the relation between the vertices in $A = \{x_1, x_2, \ldots, x_{n-1}\}$ and in $B = V(P_3^1) \cup V(P_3^2) \cup \cdots \cup V(P_3^t)$. Suppose that the neighborhood $N_A(u)$ in A of a vertex $u \in B$ satisfies $|N_A(u) \cap V(C_{n-1})| \geq 5t - 1$. Let $x_i, x_j \in N_A(u) \cap V(C_{n-1})$ with $i < j$. Note that $j - i > 1$ since otherwise we can see that the cycle C_{n-1} including the vertex u extend to a cycle of order n. Meanwhile, if x_{i+1} and x_{j+1} are adjacent in F then we also obtain the cycle $C_n' = x_i u x_j x_{j-1} \ldots x_{i+1} x_{j+1} x_{j+2} \ldots x_{n-1} x_1 x_2 \ldots x_i$ of length n in F. If $x_{i+1}x_{j+1}$ is not an edge in F for every pair $x_i, x_j \in N_A(u) \cap V(C_{n-1})$ then the set $\{x_{i+1}, x_{j+1} \in N_A(u) \cap V(C_{n-1})\} \cup \{u\}$ is $5t$ independent vertices in F forming tW_4 in $\overline{F}$. Therefore, we obtain $|N_A(u) \cap V(C_{n-1})| \leq 5t - 2$ for each u in B. Thus,

$$\left| A \setminus \bigcup_{u \in B} N_A(u) \right| \geq n - 1 - 3t(5t - 2). \tag{3}$$

Note that $n \geq f(t)$, and so the equation (3) implies that the set A contains at least $2t + 1$ vertices which are not adjacent to all vertices in B and then will form tW_4 in $\overline{F}$. This completes the proof of Lemma 2.2. □

In the rest, we are now ready to give the proof of Theorem 1.1 as the main result.

Proof of Theorem 1.1. Immediately we can verified that the graph $2K_{n-1} \cup K_{t-1}$ is (C_n, tW_4)–free on $2n + t - 3$ vertices to give the lower bound $R(C_n, tW_4) \geq 2n + t - 2$.

By using induction on t, we will show the upper bound $R(C_n, tW_4) \leq 2n + t - 2$. From Theorem 1.2, we have $R(C_n, W_4) = 2n - 1$ for $n \geq 3$. Note that if $t = 1$ then $n \geq f(1) > 3$. Therefore, Theorem 1.1 is true for $n \geq f(1) \geq 3$. In what follows we assume $t \geq 2$ and Theorem 1.1 is also holds for $n \geq f(t-1)$, that is $R(C_n, (t-1)W_4) \leq 2n + t - 3$.

We are now show that Theorem 1.1 is also valid for $n \geq f(t)$. Let F be a graph with $|F| = 2n + t - 2$. We shall show that there is a C_n in F or tW_4 is in $\overline{F}$. Note that if $t \geq 1$ then $n \geq f(t) > 3$ and then by Theorem 1.2 in Surahmat et al. [14] we have that there is a C_n in F or $\overline{F}$ contains W_4. Then the proof is end, if we have C_n in F. Thus we have W_4 in $\overline{F}$. Hence, $|F - W_4| = 2(n-2) + (t-1) - 2$ and by induction on t note that $n - 1 \geq f(t) - 1 > f(t-1)$, we obtain C_{n-2} in $F - W_4$ or $(t-1)W_4$ in the complement of $F - W_4$. Meanwhile, if there is a graph $(t-1)W_4$ in the complement of $F - W_4$ then together with the first one will forms tW_4 in $\overline{F}$ and we are done. Thus F has a cycle C_{n-2} and $|F - C_{n-2}| = n + t$. Note that if $t \geq 1$ then

$n \geq f(t) > 4t + 2$, and by Lemma 2.1 we obtain C_n in $F - C_{n-2}$ or tP_3 is in the complement of $F - C_{n-2}$. If $F - C_{n-2}$ contains C_n then we are done.

Thus, we have C_{n-2} in F and $\overline{F}$ contains t disjoint copies $P_3^1, P_3^2, \ldots, P_3^t$ of path with three vertices. It is clear that the subgraphs C_{n-2} and tP_3 has no vertices in common.

Assume that there is no C_{n-1} in F. We will show that $\overline{F}$ contains tW_4. Let us now focus to see the relation between the vertices in A and in B; where $A = \{x_1, x_2, \ldots, x_{n-2}\}$ is the vertex set of cycle C_{n-2} and $B = V(P_3^1) \cup V(P_3^2) \cup \cdots \cup V(P_3^t)$. Suppose that the neighborhood $N_A(u)$ in A of a vertex $u \in B$ satisfies $|N_A(u) \cap V(C_{n-2})| \geq 5t - 1$. Let $x_i, x_j \in N_A(u) \cap V(C_{n-2})$ with $i < j$. Note that $j - i > 1$ since otherwise C_{n-2} together with the vertex u forms a cycle of order $n - 1$. Therefore, Lemma 2.2 implies that $\overline{F}$ contains tW_4. If x_{i+1} and x_{j+1} are adjacent in F then we get the cycle $C'_{n-1} = x_i u x_j x_{j-1} \ldots x_{i+1} x_{j+1} x_{j+2} \ldots x_{n-2} x_1 x_2 \ldots x_i$ of length $n - 1$ in F, and then by Lemma 2.2 we obtain a copy of tW_4 in $\overline{F}$. If $x_{i+1}x_{j+1}$ is not an edge in F for every pair $x_i, x_j \in N_A(u) \cap V(C_{n-2})$ then the set $\{x_{i+1}, x_{j+1} \in N_A(u) \cap V(C_{n-2})\} \cup \{u\}$ contains $5t$ vertices which is independent in F and so forms tW_4 in $\overline{F}$. Hence, we find $|N_A(u) \cap V(C_{n-2})| \leq 5t - 2$ for each u in B . Therefore,

$$\left| A \setminus \bigcup_{u \in B} N_A(u) \right| \geq n - 2 - 3t(5t - 2). \tag{4}$$

By subtitution of condition $n \geq f(t)$ to the equation (4) give an implication that the set A contains at least $2t$ vertices which are not adjacent to all vertices in B and hence $\overline{F}$ contains tW_4. This concludes that F has C_{n-1} or $\overline{F}$ has tW_4. If we have tW_4 in $\overline{F}$ then the proof is done. Therefore, we have a copy of cycle C_{n-1} in F. Now, by Lemma 2.2 we obtain a copy of C_n in F or a copy of tW_4 in $\overline{F}$. The proof of Theorem 1.1 is now complete.

To the rest of this paper, let us present the conjecture as an open problem to further work on to the readers. By regarding Theorem 1.1, we believe that the following conjecture is true.

Conjecture 1. *For $t \geq 1$ and $m \geq 4$, there exists $n_0 = f(t, m)$ such that if $n \geq n_0$ then the cycle C_n is tW_m–good.*

3. Remark and Conclusion

In the rest of paper, we state the new result corresponding to the goodness of cycle C_n with respect to tW_4. In addition, if G is a disjoint union of cycles then we also obtain the exact value of Ramsey number $R(G, tW_4)$. This work is an effort to have a result of the Ramsey numbers $R(C_n, tW_m)$ and $R(G, tW_m)$ for any positive integers t, n and m. In future, it is not only possible obtaining the new technic to prove the clasical Ramsey number in two colors but also a wide multi colors Ramsey number in general as well.

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