Electronic Journal of Graph Theory and Applications **9** (1) (2021), 125–135

Electronic Journal of
Graph Theory and Applications

On two Laplacian matrices for skew gain graphs

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Abstract

Gain graphs are graphs where the edges are given some orientation and labeled with the elements (called gains) from a group so that gains are inverted when we reverse the direction of the edges. Generalizing the notion of gain graphs, skew gain graphs have the property that the gain of a reversed edge is the image of edge gain under an anti-involution. In this paper, we study two different types, Laplacian and g-Laplacian matrices for a skew gain graph where the skew gains are taken from the multiplicative group $F^\times$ of a field F of characteristic zero. Defining incidence matrix, we also prove the matrix tree theorem for skew gain graphs in the case of the g-Laplacian matrix.

Keywords: graph, adjacency matrix, Laplacian matrix, incidence matrix, graph eigenvalue, skew gain graph
Mathematics Subject Classification : 05C22, 05C50, 05C76
DOI: 10.5614/ejgta.2021.9.1.12

1. Introduction

Throughout this article, unless otherwise mentioned, by a graph we mean a finite, connected, simple graph and any terms which are not mentioned here, the reader may refer to [8].

A gain graph is a graph with some orientation for the edges such that each edge has a gain, that is a label from a group so that reversing the direction of edge inverts the gain [12]. Generalizing the notion of gain graphs, the skew gain graphs are defined such that the gain of an edge (we call it as skew gain) is related to the skew gain of the reverse edge by an anti-involution [6]. Let Γ be an arbitrary group. A function $f : \Gamma \to \Gamma$ is an *involution* if $f(f(x)) = x$ for all $x \in \Gamma$. A

Received: 8 July 2020, Revised: 30 December 2020, Accepted: 20 January 2021.

function $f : \Gamma \to \Gamma$ is called an *anti-homomorphism* if $f(xy) = f(y)f(x)$ for all $x, y \in \Gamma$. For an abelian group an anti-homomorphism is always a homomorphism. An involution $f : \Gamma \to \Gamma$ which is an anti-homomorphism is called an *anti-involution*. We use $\mathrm{Inv}(\Gamma)$ to denote the set of all anti-involutions on Γ. We define $g : \Gamma \to \Gamma$ such that $g(x) = xf(x)$ for all $x \in \Gamma$.

Definition 1.1 ([7]). *Let $G = (V, \overrightarrow{E})$ be a graph, where V denotes the vertex set of G and $\overrightarrow{E}$ the edge set of G with some prescribed orientation for the edges, and let Γ be an arbitrary group. If $f \in \mathrm{Inv}(\Gamma)$ be an anti-involution then the* skew-gain graph $\Phi_f = (G, \Gamma, \varphi, f)$ *is such that the* skew gain function $\varphi : \overrightarrow{E} \to \Gamma$ *satisfies* $\varphi(\overrightarrow{vu}) = f(\varphi(\overrightarrow{uv}))$.

The adjacency matrix of a skew gain graph is defined when the skew gains are taken from the multiplicative group $F^{\times}$ where F is a field of characteristic zero. Here $f \in \mathrm{Inv}(\Gamma)$ is an involutive automorphism. We use the notation $u \sim v$ when the vertices u and v are adjacent and similar notation for the incidence of an edge on a vertex.

Definition 1.2 ([11]). *Given a skew gain graph $\Phi_f = (G, F^{\times}, \varphi, f)$ its adjacency matrix $A(\Phi_f) = (a_{ij})_n$ is defined as the square matrix of order $n = |V(G)|$ where*

$$a_{ij} = \begin{cases} \varphi(\overrightarrow{v_i v_j}), & \textit{if } v_i \sim v_j, \\ 0, & \textit{otherwise.} \end{cases}$$

such that whenever $a_{ij} \neq 0$, $a_{ji} = f(a_{ij})$, which is the anti-involution.

The general expression for computing the coefficients of the characteristic polynomial of the adjacency matrix of skew gain graphs are studied in [11]. The Laplacian matrix of a graph and matrix tree theorem are well studied by many which can be referred to for instance from [9]. The matrix tree theorem for signed graph can be seen in Zaslavsky [13] and on a more general setting in Chaiken [3]. For more recent investigations on the spectrum of a graph and energy of a graph one can refer to [4] and [5]. In this paper, we define Laplacian matrix and g-Laplacian matrix of skew gain graphs by defining the corresponding degree and g-degree matrices and prove the matrix tree theorem for skew gain graphs.

2. Laplacian matrix for skew gain graphs

Definition 2.1. *Given a skew gain graph $\Phi_f = (G, F^{\times}, \varphi, f)$, the degree of the vertex v_i in Φ_f is denoted by $d(v_i)$, and it is obtained by adding the multiplicative identity of the field $F^{\times}, d_i$ times, where d_i is the degree of the vertex v_i in the underlying graph G.*
Degree matrix $D(\Phi_f)$ can be defined as the diagonal matrix $D(\Phi_f) = \mathrm{diag}(d(v_i))$.

Definition 2.2. *Given a skew gain graph $\Phi_f = (G, F^{\times}, \varphi, f)$ its Laplacian matrix is defined as $L(\Phi_f) = D(\Phi_f) - A(\Phi_f)$. We define the Laplacian charactersitic polynomial of the skew gain graph Φ_f as $det(xI - L(\Phi_f))$. The Laplacian spectrum of a skew gain graph Φ_f refers to the eigenvalues of the Laplacian matrix $L(\Phi_f)$ and their multiplicities.*

Lemma 2.1 ([1]). *Let $P = (p_{ij})$ be an $n \times n$ matrix. Then the determinant of P has the expansion*

$$\det(P) = \sum sgn(\pi) p_{1\pi(1)} p_{2\pi(2)} \dots p_{n\pi(n)}$$

where the summation is over all permutations π on the set $\{1,2,3, \dots, n\}$ and $sgn(\pi)$ is the sign of the permutation π. If π is an even cycle, then $sgn(\pi) = -1$ and if π is an odd cycle, then $sgn(\pi) = +1$. Thus the sign of an arbitrary permutation π is $(-1)^{N_e}$, where N_e is the number of even cycles in cyclic representation of π.

Let $\mathfrak{L}(G)$ denotes the set of all elementary subgraphs L of G (of all orders) and $K_e(L)$ denotes the number of components in L having even order. Also let $\mathcal{M}(G)$ denotes the set of all matchings M in the graph G and $K(M)$ denotes the number of edges in M. We denote the Laplacian characteristic polynomial of skew gain graph by $\chi(\Phi_f, x) = \det(xI - L(\Phi_f))$.

Theorem 2.3. *If $\Phi_f = (G, F^\times, \varphi, f)$ is a skew gain graph where $G = (V, \overrightarrow{E})$ is a graph of order n, then*

$$\chi(\Phi_f, x) = \prod_{v \in V(G)} (x - d(v)) + \sum_{L \in \mathfrak{L}(G)} (-1)^{K_e(L)} \prod_{K_2 \in L} g(\varphi(\vec{e})) \prod_{C \in L} (\varphi(C) + f(\varphi(C))) \prod_{v \notin V(L)} (x - d(v)).$$

Proof. Let $d(v_i)$ denotes the degree of the vertex v_i in Φ_f and let the adjacency matrix of skew gain graph Φ_f be $A(\Phi_f) = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 0 & a_{23} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 0 \end{pmatrix}$.
Then the Laplacian characteristic polynomial of skew gain graph Φ_f is

$$\chi(\Phi_f, x) = \det(xI - L(\Phi_f)) = \det \begin{pmatrix} x - d(v_1) & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & x - d(v_2) & a_{23} & \dots & a_{2n} \\ \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & x - d(v_n) \end{pmatrix}.$$

Using Lemma 2.1, corresponding to the identity permutation, we get the term $\prod_{v \in V(G)} (x - d(v))$.
Now, for any non-identity permutation π, consider the term $sgn(\pi) a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)}$. Any permutation π can be expressed as a product of disjoint cycles. Thus if π fixes the i^{th} element, $a_{ii} = x - d(v_i)$. Now a cycle (ij) of length two in π corresponds to $a_{ij}.a_{ji}$ which corresponds to the edges $\overrightarrow{v_i v_j}$ and $\overrightarrow{v_j v_i}$ in G. Any cycle $(pqr \dots t)$ of length greater than 2 corresponds to $a_{pq} a_{qr} \dots a_{tp}$ which gives a cycle $v_p v_q v_r \dots v_t v_p$ in G. Thus, corresponding to the non-identity permutation π we get an elementay subgraph L of G and $a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)}$ becomes $\prod_{K_2 \in L} g(\varphi(\vec{e})) \prod_{C \in L} (\varphi(C) + f(\varphi(C)) \prod_{v \notin V(L)} (x - d(v))$.
Now, $sgn(\pi) = (-1)^{N_e}$, where N_e is the number of even cycles in π, which is same as the number of components in L having even order. □

When the underlying graph of $\Phi_f = (G, F^\times, \varphi, f)$ is a cycle or path, we call it as a skew gain cycle or skew gain path respectively.

Corollary 2.1. *If $\Phi_f = (P_n, F^{\times}, \varphi, f)$ is a skew gain path, then its Laplacian characteristic polynomial is*

$$\chi(\Phi_f, x) = (x-2)^{n-2}(x-1)^2 + \sum_{M \in \mathcal{M}(P_n)} (-1)^{K(M)} \prod_{\vec{e} \in M} g(\varphi(\vec{e})) \prod_{v \notin V(M)} (x - d(v)).$$

Corollary 2.2. *If $\Phi_f = (C_n, F^{\times}, \varphi, f)$ is a skew gain cycle, then its Laplacian characteristic polynomial is*

$$\chi(\Phi_f, x) = (x-2)^n + (-1)^{n-1}(\varphi(C_n) + f(\varphi(C_n))) + \sum_{M \in \mathcal{M}(C_n)} (-1)^{K(M)} \prod_{\vec{e} \in M} g(\varphi(\vec{e})) \prod_{v \notin V(M)} (x - d(v)).$$

Proof. The only elementary subgraph $L \in \mathfrak{L}(C_n)$ containing cycle as a component is C_n itself. If n is even then $(-1)^{K_e(L)} = -1 = (-1)^{n-1}$ and if n is odd $(-1)^{K_e(L)} = (-1)^0 = 1 = (-1)^{n-1}$. All other elementary subgraphs contains K_2 as components which can be considered as matchings in C_n and hence using theorem 2.3 we get

$$\chi(\Phi_f, x) = (x-2)^n + (-1)^{n-1}(\varphi(C_n) + f(\varphi(C_n))) + \sum_{M \in \mathcal{M}(C_n)} (-1)^{K(M)} \prod_{\vec{e} \in M} g(\varphi(\vec{e})) \prod_{v \notin V(M)} (x - d(v)).$$

□

Corollary 2.3. *If $\Phi_f = (K_{1,n}, F^{\times}, \varphi, f)$ is a skew gain graph with underlying graph as the star $K_{1,n}$, then its Laplacian characteristic polynomial is*

$$\chi(\Phi_f, x) = (x-1)^n (x-n) - (x-1)^{(n-1)} \sum_{\vec{e} \in E(K_{1,n})} g(\varphi(\vec{e})).$$

Now we move to the Laplacian spectrum of some particular classes of skew gain graphs. First of all, it is worthwhile to point out that if $\Phi_f = (G, F^{\times}, \varphi, f)$ is a skew gain graph where G is d-regular, then the Laplacian eigenvalues of $L(\Phi_f)$ are $d - \lambda$ where λ is an eigenvalue of its adjacency matrix $A(\Phi_f)$.

To find the spectrum of bipartite skew gain graphs, we define for a matrix $B = (a_{ij}) \in M_{m \times n}(F)$, $B^f = (b_{ij}) \in M_{m \times n}(F)$ where $f \in \text{Inv}(F^{\times})$ as $b_{ij} = \begin{cases} f(a_{ij}), & \text{if } a_{ij} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$

Theorem 2.4. *Let $\Phi_f = (G, F^{\times}, \varphi, f)$ be a skew gain graph where $G = K_{m,m}$ is a complete bipartite graph. Then the eigenvalues of $L(\Phi_f)$ are $m - \lambda$ such that λ^2 is an eigenvalue of $B(B^f)^T$.*

Proof. The adjacency eigenvalues of $\Phi_f = (G, F^{\times}, \varphi, f)$, where $G = K_{m,m}$ is a complete bipartite graph, are λ such that λ^2 is an eigenvalue of $B(B^f)^T$ [11]. Hence , as Φ_f is regular with degree m, we get the eigenvalues of $L(\Phi_f)$ are $m - \lambda$ such that λ^2 is an eigenvalue of $B(B^f)^T$. □

Theorem 2.5. *Let $\Phi_f = (G, F^\times, \varphi, f)$ be a skew gain graph where $G = K_{1,n}$ is a star of order $n+1$. Then* $\det(L(\Phi_f)) = n - \sum_{\vec{e}\in E(G)} g(\varphi(\vec{e}))$.

Proof. When we put $x = 0$ in the characteristic polynomial of $L(\Phi_f)$ in Corollary 2.3, we get the constant term in the polynomial as $(-1)^{n-1}\big(n - \sum_{\vec{e}\in E(G)} g(\varphi(\vec{e}))\big)$, which is equal to $(-1)^{n+1}\det(L(\Phi_f))$. Hence $\det(L(\Phi_f)) = n - \sum_{\vec{e}\in E(G)} g(\varphi(\vec{e}))$. □

Theorem 2.6. *If $\Phi_f = (G, F^\times, \varphi, f)$ is a skew gain graph where $G = K_{1,n}$ is a star of order $n+1$, then the Laplacian spectrum of Φ_f is* $\begin{pmatrix} \frac{n+1+\sqrt{(n+1)^2-4\det(L(\Phi_f))}}{2} & \frac{n+1-\sqrt{(n+1)^2-4\det(L(\Phi_f))}}{2} & 1 \\ 1 & 1 & n-1 \end{pmatrix}$.

Proof. By Corollary 2.3, Laplacian characteristic polynomial of Φ_f is

$$\begin{aligned}\chi(\Phi_f, x) &= (x-1)^{n-1}\big((x-n)(x-1) - \sum_{\vec{e}\in E(G)} g(\varphi(\vec{e}))\big)\\ &= (x-1)^{n-1}\big((x^2-(n+1)x+n - \sum_{\vec{e}\in E(G)} g(\varphi(\vec{e}))\big)\end{aligned}$$

From this the Laplacian spectrum of Φ_f becomes
$\begin{pmatrix} \frac{n+1+\sqrt{(n+1)^2-4\det(L(\Phi_f))}}{2} & \frac{n+1-\sqrt{(n+1)^2-4\det(L(\Phi_f))}}{2} & 1 \\ 1 & 1 & n-1 \end{pmatrix}$. □

3. g-Laplacian matrix for skew gain graphs

In this section, we take ordered fields F and for $a \in F^\times$, $\sqrt{a}$ is the principal square root of a which belongs to the algebraic closure of the field F. Now we define the g-Laplacian matrix of a skew gain graph as follows. For an oriented edge $\vec{e_j} = \overrightarrow{v_i v_k}$ we take v_i as the tail of that edge and v_k as its head and we write $t(\vec{e_j}) = v_i$ and $h(\vec{e_j}) = v_k$.

Definition 3.1. *Given a skew gain graph $\Phi_f = (G, F^\times, \varphi, f)$ its g-Laplacian matrix is defined as $L_g(\Phi_f) = D_g(\Phi_f) - A(\Phi_f)$ where the diagonal matrix $D_g(\Phi_f)$ is* $\operatorname{diag}\Big(\sum_{\vec{e}:v_i\sim\vec{e}} \sqrt{g(\varphi(\vec{e})}\Big)$ *where $\sqrt{a}$ for $a \in F$ belongs to the algebraic closure of the field F. The matrix $D_g(\Phi_f)$ is the g-degree matrix of Φ_f.*

The incidence matrix for a skew gain graph Φ_f can be defined as follows

Definition 3.2. *Given a skew gain graph $\Phi_f = (G, F^\times, \varphi, f)$ its (oriented) incidence matrix is defined as* $\mathrm{H}(\Phi_f) = (b_{ij})$ *where*

$$b_{ij} = \begin{cases} g(\varphi(\vec{e_j})), & \text{if } t(\vec{e_j}) = v_i,\\ -f(\varphi(\vec{e_j}))\sqrt{g(\varphi(\vec{e_j}))}, & \text{if } h(\vec{e_j}) = v_i,\\ 0, & \text{otherwise.}\end{cases}$$

The definitions of Laplacian, g-Laplacian and incidence matrix of a skew gain graph coincide with the corresponding definitions for ordinary graphs, signed graphs and gain graphs which are extensively studied in [1, 9, 10, 13]. Now we define a matrix operation for the incidence matrix $\mathrm{H}(\Phi_f)$ as follows:
$\mathrm{H}^{\#}$ is the transpose of the matrix obtained by replacing each column element as under:
(i) $g(\varphi(\vec{e_j}))$ replaced by $(\sqrt{g(\varphi(\vec{e_j}))})^{-1}$ and
(ii) $-f(\varphi(\vec{e_j}))\sqrt{g(\varphi(\vec{e_j}))}$ replaced by $-(f(\varphi(\vec{e_j})))^{-1}$

Theorem 3.3. *For a skew gain graph $\Phi_f = (G, F^{\times}, \varphi, f)$, $L_g(\Phi_f) = \mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ and $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_m}$ be the vertices and edges in G, respectively. Denoting $\mathrm{H}(\Phi_f)$ by $(\eta_{v_i\vec{e_j}})$ and $\mathrm{H}^{\#}(\Phi_f)$ by $(\eta'_{\vec{e_i}v_j})$, let the i^{th} row vector of $\mathrm{H}(\Phi_f)$ be $(\eta_{v_i\vec{e_1}}, \eta_{v_i\vec{e_2}}, \ldots, \eta_{v_i\vec{e_m}})$ and j^{th} column of $\mathrm{H}^{\#}(\Phi_f)$ be $(\eta'_{\vec{e_1}v_j}, \eta'_{\vec{e_2}v_j}, \ldots, \eta'_{\vec{e_m}v_j})^T$.

Now, the $(i,j)^{th}$ entry of $\mathrm{HH}^{\#}$ is $\sum_{k=1}^{m} \eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j}$.

For $i = j$, $\eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j} \neq 0$ if and only if $\vec{e_k}$ is incident to v_i. If $t(\vec{e_k}) = v_i$ then $\eta_{v_i\vec{e_k}} = g(\varphi(\vec{e_k}))$ in $\mathrm{H}(\Phi_f)$ and hence $\eta'_{\vec{e_k}v_j} = \sqrt{g(\varphi(\vec{e_k}))^{-1}}$ in $\mathrm{H}^{\#}(\Phi_f)$ so that $\eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j} = \sqrt{g(\varphi(\vec{e_k})}$ in $\mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)$. If $h(\vec{e_k}) = v_i$ then $\eta_{v_i\vec{e_k}} = -f(\varphi(\vec{e_k}))\sqrt{g(\varphi(\vec{e_k}))}$ and hence $\eta'_{\vec{e_k}v_j} = -f(\varphi(\vec{e_k}))^{-1}$ so that $\eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j} = \sqrt{g(\varphi(\vec{e_k})}$. Thus, the diagonal entries in $\mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)$ is $\sum_{\vec{e}: v_i \sim \vec{e}} \sqrt{(g(\varphi(\vec{e})}$.

For $i \neq j$, $\eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j} \neq 0$ if and only if $\vec{e_k}$ is an edge joining v_i and v_j. If $\vec{e_k} = \overrightarrow{v_iv_j}$ then $\eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j} = g(\varphi(\vec{e_k})).(-f(\varphi(\vec{e_k}))^{-1}) = -\varphi(\vec{e_k})$ and if $\vec{e_k} = \overrightarrow{v_jv_i}$ then

$$\eta_{v_i\vec{e_k}}\eta'_{\vec{e_k}v_j} = -f(\varphi(\vec{e_k}))\sqrt{g(\varphi(\vec{e_k}))}.\sqrt{g(\varphi(\vec{e_k}))^{-1}} = -f(\varphi(\vec{e_k}))$$

.

In both cases, the $(i,j)^{th}$ entry of $\mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)$ coincides with the $(i,j)^{th}$ entry of $L_g(\Phi_f)$ and hence the proof. □

From Definition 3.2, we will have the following deductions:
(i) In the case of real weighted graphs where $f(x) = x$ so that $g(x) = x^2$ (which is a particular skew gain graphs which we can be used to deal with weighted signed graphs also), the incidence matrix $\mathrm{H} = (b_{ij})$ has

$$b_{ij} = \begin{cases} w(\vec{e_j})^2, & \text{if } t(\vec{e_j}) = v_i, \\ -w(\vec{e_j})^2, & \text{if } h(\vec{e_j}) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) In the case of complex skew gain graph with $f(z) = \overline{z}$, so that $g(z) = |z|^2$, the incidence matrix $\mathrm{H} = (b_{ij})$ has

$$b_{ij} = \begin{cases} |w(\vec{e_j})|^2, & \text{if } t(\vec{e_j}) = v_i, \\ -\overline{w(\vec{e_j})}|w(\vec{e_j})|, & \text{if } h(\vec{e_j}) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.1 ([2]). *Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Then $rank(AB) \leq min\{rank(A), rank(B)\}$. Also $rank(A) \leq min\{m, n\}$.*

Lemma 3.2 ([2]). *If $A \in M_n(F)$ is a block triangular matrix of the form* $A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & A_{kk} \end{pmatrix}$*, where each A_{ii} is a square matrix and the 0's are zero matrices of appropriate size, then* $\det(A) = \prod_{i=1}^{k} \det(A_{ii})$.

Theorem 3.4. *If $\Phi_f = (G, F^\times, \varphi, f)$ is a skew gain graph, where G is a tree of order n, then* $\det L_g(\Phi_f) = 0$.

Proof. A connected tree on n vertices have $n-1$ edges. Thus the incidence matrix $H(\Phi_f)$ has order $n \times n-1$. Now, by Lemma 3.1, rank($\mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)$) is less than or equal to $n-1$ which implies $\det(\mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)) = 0$. Thus by Theorem 3.3,

$$\det(L_g(\Phi_f)) = \det(\mathrm{H}(\Phi_f)\mathrm{H}^{\#}(\Phi_f)) = 0.$$

□

Theorem 3.5. *If $\Phi_f = (C_n, F^\times, \varphi, f)$ is a skew gain cycle then*

$$\det L_g(\Phi_f) = 2\sqrt{\prod_{\vec{e} \in E(C_n)} g(\varphi(\vec{e}))} - [\varphi(C_n) + f(\varphi(C_n))].$$

Proof. Let the skew gain cycle be $C_n = v_1\vec{e_1}v_2\vec{e_2}v_3\vec{e_3}\dots v_{n-1}\vec{e_{n-1}}v_n\vec{e_n}v_1$. Its incidence matrix H is

$$\begin{pmatrix} g(\varphi(\vec{e_1})) & 0 & \dots & 0 & -f(\varphi(\vec{e_n}))\sqrt{g(\varphi(\vec{e_n}))} \\ -f(\varphi(\vec{e_1}))\sqrt{g(\varphi(\vec{e_1}))} & g(\varphi(\vec{e_2})) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g(\varphi(\vec{e_{n-1}})) & 0 \\ 0 & 0 & \dots & -f(\varphi(\vec{e_{n-1}}))\sqrt{g(\varphi(\vec{e_{n-1}}))} & g(\varphi(\vec{e_n})) \end{pmatrix}.$$

Expanding along the first row to find the determinant of H, we get
$\det(\mathrm{H}) = g(\varphi(\vec{e_1}))M_{1,1} + (-1)^n f(\varphi(\vec{e_n}))\sqrt{g(\varphi(\vec{e_n}))}M_{1,n}$, where

$$M_{1,1} = \det \begin{pmatrix} g(\varphi(\vec{e_2})) & \dots & 0 & 0 \\ \dots & & & \\ 0 & \dots & g(\varphi(\vec{e_{n-1}})) & 0 \\ 0 & \dots & -f(\varphi(\vec{e_{n-1}}))\sqrt{g(\varphi(\vec{e_{n-1}}))} & g(\varphi(\vec{e_n})) \end{pmatrix}$$

$$M_{1,n} = \det \begin{pmatrix} -f(\varphi(\vec{e_1}))\sqrt{g(\varphi(\vec{e_1}))} & g(\varphi(\vec{e_2})) & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & g(\varphi(\vec{e_{n-1}})) \\ 0 & 0 & \dots & -f(\varphi(\vec{e_{n-1}}))\sqrt{g(\varphi(\vec{e_{n-1}}))} \end{pmatrix}$$

Clearly, $M_{1,1}$ and $M_{1,n}$ are determinant of triangular matrices and hence it is the product of the diagonal entries. Thus
$M_{1,1} = g(\varphi(\vec{e_2}))g(\varphi(\vec{e_3}))\dots g(\varphi(\vec{e_n}))$ and
$M_{1,n} = (-1)^{n-1} f(\varphi(\vec{e_1}))\sqrt{g(\varphi(\vec{e_1}))} f(\varphi(\vec{e_2}))\sqrt{g(\varphi(\vec{e_2}))}\dots f(\varphi(\vec{e_{n-1}}))\sqrt{g(\varphi(\vec{e_{n-1}}))}$.

Hence $\det(\mathrm{H}) = \prod_{\vec{e}\in E(C_n)} g(\varphi(\vec{e})) - f(\varphi(C_n))\sqrt{\prod_{\vec{e}\in E(C_n)} g(\varphi(\vec{e}))}$.

Now, considering the matrix $\mathrm{H}^\#$,

$$\mathrm{H}^\# = \begin{pmatrix} \sqrt{g(\varphi(\vec{e_1}))}^{-1} & 0 & \dots & 0 & -f(\varphi(\vec{e_n}))^{-1} \\ -f(\varphi(\vec{e_1}))^{-1} & \sqrt{g(\varphi(\vec{e_2}))}^{-1} & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & \sqrt{g(\varphi(\vec{e_{n-1}}))}^{-1} & 0 \\ 0 & 0 & \dots & -f(\varphi(\vec{e_{n-1}}))^{-1} & \sqrt{g(\varphi(\vec{e_n}))}^{-1} \end{pmatrix}^T.$$

Finding its determinant in a similiar way, we get

$$\begin{aligned} \det(\mathrm{H}^\#) &= \prod_{\vec{e}\in E(C_n)} \sqrt{g(\varphi(\vec{e}))}^{-1} - f(\varphi(C_n))^{-1} \\ &= \sqrt{\prod_{\vec{e}\in E(C_n)} g(\varphi(\vec{e}))}^{-1} - f(\varphi(C_n))^{-1}. \end{aligned}$$

Now from Theorem 3.3, $L_g = \mathrm{H}\mathrm{H}^\#$ which gives $\det(L_g) = \det(\mathrm{H})\det \mathrm{H}^\#$.
Thus, $\det L_g(\Phi_f) = 2\sqrt{\prod_{\vec{e}\in E(C_n)} g(\varphi(\vec{e}))} - [\varphi(C_n) + f(\varphi(C_n))]$. □

Theorem 3.6. *If $\Phi_f = (G, F^\times, \varphi, f)$ is a skew gain graph of order n where G is a unicyclic graph with unique cycle C then*

$$\det L_g(\Phi_f) = \sqrt{\prod_{\vec{e}\notin E(C)} g(\varphi(\vec{e}))}\Big(2\sqrt{\prod_{\vec{e}\in E(C)} g(\varphi(\vec{e}))} - [\varphi(C) + f(\varphi(C))]\Big).$$

Proof. Let $C = v_1\vec{e_1}v_2\vec{e_2}\dots v_p\vec{e_p}v_1$ be the unique cycle and define the orientation of edges as for $i<j$ the edge $\overrightarrow{e_{i,j}}$ has tail $t(\overrightarrow{e_{i,j}}) = v_i$ and head $h(\overrightarrow{e_{i,j}}) = v_j$. We get the incidence matrix $\mathrm{H}(\Phi_f)$ as an upper triangular block martix with diagonal blocks $A_1, A_2, \dots, A_k, k = n-p+1$, where A_1 corresponds to the vertices and edges in the cycle C and $A_i, i = 2,3,\dots n-p+1$ are one element matrices $[-f(\varphi(\vec{e}))\sqrt{g(\varphi(\vec{e}))}]$ corresponding to the edges $\vec{e}$ not in C. Then, by Lemma

3.2, $\det(H(\Phi_f)) = \prod \det(A_i)$. Now using Theorem 3.5,

$$\det(\mathrm{H}) = \prod_{\vec{e}\notin E(C)} (-f(\varphi(\vec{e}))(\sqrt{g(\varphi(\vec{e}))})\big(\prod_{\vec{e}\in E(C)} g(\varphi(\vec{e})) - f(\varphi(C))\sqrt{\prod_{\vec{e}\in E(C)} g(\varphi(\vec{e}))}\big).$$

Similiarly we get

$$\det(\mathrm{H}^{\#}) = \prod_{\vec{e}\notin E(C)} (-f(\varphi(\vec{e}))^{-1})\big(\sqrt{\prod_{\vec{e}\in E(C)} g(\varphi(\vec{e}))}^{-1} - f(\varphi(C))^{-1}\big).$$

Since $\det L_g(\Phi_f) = \det \mathrm{HH}^{\#}$ we get,

$$\det L_g(\Phi_f) = \sqrt{\prod_{\vec{e}\notin E(C)} g(\varphi(\vec{e}))}\Big(2\sqrt{\prod_{\vec{e}\in E(C)} g(\varphi(\vec{e}))} - [\varphi(C) + f(\varphi(C))]\Big).$$ □

A 1-tree is a connected unicyclic graph and a 1-forest is a disjoint union of 1-trees. A spanning subgraph of G which is a 1-forest is called as an essential spanning subgraph of G. We denote the collection of all essential spanning subgraphs of G by $\mathfrak{E}(G)$

Theorem 3.7. *If $\Phi_f = (G, F^\times, \varphi, f)$ is a skew gain graph where G is a 1-forest, then*

$$\det L_g(\Phi_f) = \prod_{\Psi\in G} \sqrt{\prod_{\vec{e}\notin C_\Psi} g(\varphi(\vec{e}))}\Big(2\sqrt{\prod_{\vec{e}\in C_\Psi} g(\varphi(\vec{e}))} - [\varphi(C_\Psi) + f(\varphi(C_\Psi))]\Big)$$

where the product runs over all component 1-trees Ψ having unique cycle C_Ψ.

Proof. By suitable reordering of vertices and edges, if necessary, we can make the matrix $L_g(\Phi_f)$ as a block diagonal matrix where the blocks corresponds to the 1-tree components of the 1-forest. Then, by Lemma 3.2, determinant $\det(L_g(\Phi_f)) = \prod_{\Psi\in\mathfrak{E}(G)} \det(L_g(\Psi))$. Now by applying Theorem 3.6, we get the required expansion. □

Now we can prove the matrix - tree theorem for skew gain graphs.

Lemma 3.3. *Let $\Phi_f = (G, F^\times, \varphi, f)$ be a skew gain graph on n vertices and Ψ be a spanning subgraph of Φ_f having exactly n edges. Then $\det(L_g(\Psi)) \neq 0$ implies Ψ is an essential spanning subgraph of Φ_f.*

Proof. Let Ψ be a spanning subgraph of Φ_f having exactly n edges and let $\det(L_g(\Psi)) \neq 0$. We have to prove Ψ is an essential spanning subgraph of Φ_f. That is, we have to prove that the components of Ψ are 1-trees.
By suitable ordering of vertices and edges, we can make the matrix $L_g(\Psi)$ as a block diagonal matrix $\mathrm{diag}(A_i)$ where the blocks A_i corresponds to the components of Ψ. Thus, $\det(L_g(\Psi)) = \prod_{A_i\in\Psi} \det(L_g(A_i))$.
If Ψ contains an isolated vertex, then the matrix $L_g(\Psi)$ has a zero row which implies $\det(L_g(\Psi)) = 0$, a contradiction.

If A_i is a tree for some i, then by Theorem 3.4 we get $\det(L_g(A_i)) = 0$ which implies $\det(L_g(\Psi)) = 0$, again a contradiction.
Claim: If A_k is a component of Ψ then A_k have same number of edges and vertices.
Suppose A_k, for some k, has p vertices and $p+t$ edges where $t \geq 1$. Then the $n-p$ vertices and $n-p-t$ edges not in A_k forms either a tree or a disconnected graph having trees as components. Both cases leads to $\det(L_g(\Psi)) = 0$, a contradiction. Hence our claim.
Now all the components of Ψ have same number of edges and vertices implies the components of Ψ are 1-trees. Hence Ψ is a spanning 1-forest. That is Ψ is an essential spanning subgraph of Φ_f. □

Theorem 3.8. *If $\Phi_f = (G, F^\times, \varphi, f)$ is a skew gain graph on n vertices, then*

$$\det(L_g(\Phi_f) = \sum_{\Psi \in \mathfrak{E}(G)} \prod_{\psi \in \Psi} \sqrt{\prod_{\vec{e} \notin C_\psi} g(\varphi(\vec{e}))} \Big(2\sqrt{\prod_{\vec{e} \in C_\psi} g(\varphi(\vec{e}))} - [\varphi(C_\psi) + f(\varphi(C_\psi))] \Big)$$

where the summation runs over all essential spanning subgraphs Ψ of Φ_f and $\psi \in \Psi$ denotes the component 1-trees ψ in the spanning 1-forest Ψ.

Proof. Since $L_g(\Phi_f) = \mathrm{H}(\Phi_f)\mathrm{H}^\#(\Phi_f)$, by the Binet-Cauchy theorem [2] we get,

$$\det(L_g(\Phi_f)) = \sum_J \det(\mathrm{H}(J)) \det(\mathrm{H}^\#(J)) = \sum_J \det L_g(J)$$

where J is a spanning subgraph of G with exactly n edges. Then, by Lemma 3.3, we get $\det(L_g(\Phi_f)) = \sum_{\Psi \in \mathfrak{E}(G)} \det L_g(\Psi)$, where the summation runs over all essential spanning subgraphs of Φ_f. Hence by Theorem 3.7, we get required the expansion. □

Conclusion

In this paper, we dealt with two types of Laplacian matrices for skew gain graphs. An expression for finding the Laplacian characteristic polynomial is given in Theorem 2.3. The Laplacian spectrum of some classes of skew gain graphs are also studied. By defining the g-Laplacian matix and the incidence matrix for skew gain graphs, we have proved the matrix - tree theorem for skew gain graphs.

Acknowledgement

The first author would like to acknowledge her gratitude to the Department of Science and Technology, Govt. of India for the financial support under INSPIRE Fellowship scheme Reg No: IF180462.

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