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Degree sum adjacency polynomial of standard graphs and graph operations

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Abstract

In this paper, we explore the characteristic polynomials of degree sum adjacency matrix $DS_A(G)$ of a simple undirected graph G. We state a relation between the structure of a graph and the coefficients of its DS_A polynomial. A walk generating function is expressed in terms of DS_A polynomial. Then, we obtain the degree sum adjacency polynomial for some standard graphs, derived graphs and for graph operations.

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1. Introduction

Spectral graph theory focuses on the study of the eigenvalues and its relation to the structural properties of a graph. Thus, for a given graph many matrices were defined in this field which records the information about the vertices and the edges of a graph. To state a few, the most explored and widely studied matrices are the adjacency matrix, the laplacian matrix, the signless laplacian matrix, and many more.

In chemistry, many matrices are defined with respect to the distance, incidence and other factors. This motivated many researchers to explore different matrices [11, 13, 14] and study their properties and energy [1, 10]. Zagreb index defined as the sum of the degrees of adjacent vertices

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have been studied intensively [4, 5, 6, 7, 15], which relates to the degree sum adjacency (DS_A) matrix. This motivated us to explore the DS_A polynomial for a graph and its operations. In this paper, we consider the degree sum adjacency matrix defined by Zaferani [14] and we discuss relation between the structure of a graph and the coefficients of DS_A polynomial. Then we determine the generating function for the number of walks of each length with respect to the degree sum adjacency matrix. Later we study the DS_A polynomial of complementary graphs, some regular graphs, derived graphs and graph operations in terms of its adjacency polynomial. The proof techniques of the results in this paper are analogous to the results in [3].

Let G be a simple graph with n vertices and m edges. The adjacency matrix of a graph G is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$, if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. The adjacency eigenvalues are denoted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and they satisfy all the basic relations [3]. The adjacency polynomial of a graph G is denoted by,

$$\phi\,(G:\lambda) = det(\lambda I - A) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$

Let the vertices $v_1, v_2, \ldots, v_n$ of G have the degrees $d_1, d_2, \ldots, d_n$. Then $DS_A(G) = [ds_{ij}]$ is the *degree sum adjacency matrix* [14] of G whose elements are defined as,

$$ds_{ij} = \begin{cases} d_i + d_j, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

G:

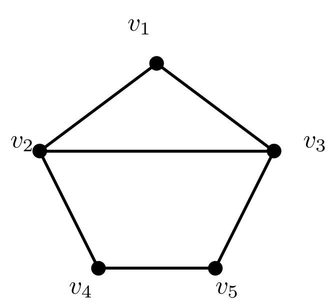

Figure 1.

$$DS_A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 5 & 5 & 0 & 0 \\ 5 & 0 & 6 & 5 & 0 \\ 5 & 6 & 0 & 0 & 5 \\ 0 & 5 & 0 & 0 & 4 \\ 0 & 0 & 5 & 4 & 0 \end{bmatrix} \end{array}$$

Graph and its DS_A matrix

The degree sum adjacency polynomial of a graph G is defined as

$$P_{DS_A(G)}(\beta) = det(\beta I - DS_A(G)) = \beta^n + a_1\beta^{n-1} + a_2\beta^{n-2} + \cdots + a_n. \tag{2}$$

As $DS_A(G)$ is a real symmetric matrix, its eigenvalues must be real and can be arranged as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

Lemma 1.1. *[2] The eigenvalues of matrix $xI + yJ$ of order $n \times n$ are $x + ny$ with multiplicity one and x with multiplicity $n - 1$.*

2. Characteristic polynomial of degree sum adjacency matrix

In this section, first we obtain the explicit values of some coefficients of polynomial as defined in Eq. (2). Then obtain the relation between the DS_A characteristic polynomial of a graph and that of its complement.

Some propositions relating the coefficients a_i of $P_{DS_A(G)}(\beta)$ to structural properties of G:
A degree sum adjacency matrix of any simple graph G is,

$$DS_A(G) = \begin{pmatrix} 0 & ds_{12} & \cdots & ds_{1n} \\ ds_{21} & 0 & \cdots & ds_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ds_{n1} & ds_{n2} & \cdots & 0 \end{pmatrix}. \tag{3}$$

Then the coefficients of DS_A polynomial of G can be expressed using Sach's theorem as follows. Let G be a graph having n vertices and i be any positive number. Then Sach's graphs S_i are the subgraphs of G with i vertices having disjoint union of K_2 and/or C_n. Let the number of components of $s \in S$ and number of cycles of $s \in S$ be $P(s)$ and $c(s)$ respectively. Then the coefficient a_i of β^{n-i} in Eq. (2) is given by

$$a_i = \sum_{s \in S_i} (-1)^{P(s)} \left(\substack{\text{square of degree sum} \\ \text{along the edge}}\right) \cdot 2^{c(s)} \left(\substack{\text{product of degrees} \\ \text{along the edges of c(s)}}\right).$$

Here we state first few coefficients of DS_A polynomial.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= -\sum_{j<k} ds_{jk}^2 \\
a_3 &= -2(\text{multiplying sum of the degrees along the edges of the triangle }) \\
a_4 &= \left(\sum_{i<j,k<l} ds_{ij}^2 \cdot ds_{kl}^2 \substack{\text{where } ds_{ij} \text{ and } ds_{kl} \\ \text{are the matching edges}} \right) - 2 \left(\substack{\text{multiplying sum of degrees} \\ \text{along the edges of } C_4}\right) \\
a_5 &= 2 \left(\substack{\text{multiplying sum of the degrees along} \\ \text{the edges of the triangle and disjoint edge}}\right) - 2 \left(\substack{\text{multiplying sum of degrees} \\ \text{along the edges of } C_5}\right) \\
a_6 &= -\left(\sum_{\substack{i<j,k<l, \\ m<n}} ds_{ij}^2 \cdot ds_{kl}^2 \cdot ds_{mn}^2 \substack{\text{where } ds_{ij}, ds_{kl} \text{ and } ds_{mn} \\ \text{are the matching edges}} \right) + 2 \left(\substack{\text{multiplying sum of degrees} \\ \text{along the edges of } C_4 \\ \text{and an disjoint edge}}\right) \\
&\quad +4 \left(\substack{\text{multiplying sum of degrees along} \\ \text{the edges of two disjoint triangles}}\right) - 2 \left(\substack{\text{multiplying sum of degrees} \\ \text{along the edges of } C_6}\right).
\end{aligned}$$

Relation between DS_A polynomial of a graph and its complement:

A walk of length k in a graph is any sequence of vertices $v_1, v_2, \ldots, v_{k+1}$ (not necessarily different) such that there is an edge from v_i to v_{i+1}, for each $i = 1, 2, \ldots, k$. To obtain the DS_A polynomial of a complement graph we first find the generating function to get the number of walks of length k in G with respect to its DS_A matrix.

Theorem 2.1. *Let $\overline{G}$ be the complement of G and let $H_{DS_A(G)}(t) = \sum_{k=0}^{\infty} N_k t^k$ be the function that generates the number N_k of walks of length k in G, $(k = 0, 1, 2, \ldots)$ with respect to its DS_A matrix. Then*

$$H_{DS_A(G)}(t) = \frac{1}{2rt}\left\{\frac{\left(\frac{r}{n-r-1}\right)^n (-1)^n P_{DS_A(\overline{G})}\left[-\left(\frac{1+2rt}{t}\right)\left(\frac{n-r-1}{r}\right)\right]}{P_{DS_A(G)}\left(\frac{1}{t}\right)}\right\}. \tag{4}$$

Proof. The proof of this theorem is analogous to the proof obtained for adjacency matrix of a graph G [3]. Let sum (A) denote the sum of all entries of matrix A.

$$|B + xJ| = |B| + x \text{ sum (adj B)} \tag{5}$$

$$adjB = B^{-1}|B| \tag{6}$$

$$\sum_{k=0}^{\infty} a^k t^k = 1 + at + a^2t^2 + \cdots = \frac{1}{1-at}. \tag{7}$$

$$N_k = \sum_{i.j} ds_{ij}^k = \text{sum}(DS_A)^k, \tag{8}$$

where B is any n ordered non singular matrix, J is a square matrix whose all entries are equal to one, x is any arbitrary number and N_k is the number of all walks of length k in G with respect to the DS_A matrix.

Let $H_{DS_A(G)}(t) = \sum_{k=0}^{\infty} N_k t^k$ denote the generating function that gives the number of walks N_k each of length k in G. Using Eq. (8), Eq. (7) and Eq. (6) we get.

$$\begin{aligned}\sum_{k=0}^{\infty} N_k t^k &= \sum_{k=0}^{\infty} \text{sum}(DS_A)^k t^k = \text{sum} \sum_{k=0}^{\infty}(DS_A)^k t^k \\ &= \text{sum} \frac{1}{(I-(DS_A)t)} = \text{sum } (I - DS_A t)^{-1} \\ &= \frac{\text{sum} adj(I - DS_A t)}{|I - DS_A t|}.\end{aligned} \tag{9}$$

From Eq. (5) we have

$$\text{sum } adjB = \frac{1}{x}\left\{|B + xJ| - |B|\right\}.$$

Substituting $B = I - DS_A t$, Eq. (9) reduces to

$$\sum_{k=0}^{\infty} N_k t^k = \frac{1}{x}\left\{\frac{|I - DS_A t + xJ| - |I - DS_A t|}{|I - DS_A t|}\right\}. \tag{10}$$

Substituting $x = 2rt$ in the Eq.(10) we get

$$\sum_{k=0}^{\infty} N_k t^k = \frac{1}{2rt}\left\{\frac{|I - DS_A t + 2rtJ| - |I - DS_A t|}{|I - DS_A t|}\right\}. \tag{11}$$

But

$$\overline{DS_A} = 2(n-r-1)\,(J - I - A)$$

$$\overline{DS_A} = 2(n-r-1)\left(J - I - \frac{DS_A}{2r}\right)$$

$$2r(\overline{DS_A}) = 2(n-r-1)\,(-2rI + 2rJ - DS_A)\,.$$

Multiplying both sides by t we get,

$$-DS_A t + 2rtJ = \frac{rt(\overline{DS_A})}{n-r-1} + 2rtI.$$

Using the above result in Eq.(11) we get

$$\begin{aligned}
\sum_{k=0}^{\infty} N_k t^k &= \frac{1}{2rt}\left\{\frac{\left|I + \left(\frac{r\overline{DS_A}}{n-r-1} + 2rI\right)t\right|}{|I - DS_A t|} - 1\right\} \\
&= \frac{1}{2rt}\left\{\frac{\left|\left(\frac{1+2rt}{t}\right)I + \frac{r\overline{DS_A}}{n-r-1}\right|}{\left|\frac{I}{t} - DS_A\right|} - 1\right\} \\
&= \frac{1}{2rt}\left\{\frac{(-1)^n\left(\frac{r}{n-r-1}\right)^n P_{DS_A(\overline{G})}\left[-\left(\frac{1+2rt}{t}\right)\left(\frac{n-r-1}{r}\right)\right]}{P_{DS_A(G)}\left(\frac{1}{t}\right)} - 1\right\}.
\end{aligned}$$

Hence we get the required generating function. □

Theorem 2.2. *If G is a regular graph with degree r and n vertices, then DS_A polynomial of the complememt $\overline{G}$ is*

$$P_{DS_A(\overline{G})}(\beta) = (-1)^n\left(\frac{n-r-1}{r}\right)^n\left[\frac{\beta - 2(n-r-1)^2}{\beta + (n-r-1)(2r+2)}\right]P_{DS_A(G)}\left(\frac{-r\beta - 2r(n-r-1)}{n-r-1}\right). \tag{12}$$

Proof. Since G is a r regular graph, a walk can begin at any one vertex of G and may continue in r ways. Therefore, number of walks of length k in G is $N_k = nr^k$.

Thus, for $DS_A(G)$ we have $\frac{N_k}{(2r)^k} = nr^k$.
Hence for the generating function $H_G(t)$ we have,

$$\begin{aligned} H_{DS_A(G)}(t) &= \sum_{k=0}^{\infty} N_k t^k = \sum_{k=0}^{\infty} (DS_A)^k t^k \\ &= \sum_{k=0}^{\infty} n.r^k.(2r)^k t^k = \sum_{k=0}^{\infty} n(2r^2t)^k \\ &= \frac{n}{1-2r^2t}. \end{aligned} \tag{13}$$

Using Eq.(4) we get

$$\frac{1}{2rt}\left\{\frac{\left(\frac{r}{n-r-1}\right)^n (-1)^n P_{DS_A(\overline{G})}\left[-\left(\frac{1+2rt}{t}\right)\left(\frac{n-r-1}{r}\right)\right]}{P_{DS_A(G)}\left(\frac{1}{t}\right)}\right\} = \frac{n}{1-2r^2t}. \tag{14}$$

Substituting $-\left(\frac{1+2rt}{t}\right)\left(\frac{n-r-1}{r}\right) = \beta$ in Eq.(14) we get

$$\begin{aligned} \left\{\frac{(-1)^n\left(\frac{r}{n-r-1}\right)^n P_{DS_A(\overline{G})}(\beta)}{P_{DS_A(G)}\left[\frac{-r\beta-2r(n-r-1)}{n-r-1}\right]} - 1\right\} &= \frac{-n}{1+\frac{2r(n-r-1)}{\beta+2(n-r-1)}}.\frac{2(n-r-1)}{\beta+2(n-r-1)} \\ &= \frac{-2n(n-r-1)}{\beta+(n-r-1)(2r+2)} \\ \frac{(-1)^n\left(\frac{r}{n-r-1}\right)^n P_{DS_A(\overline{G})}(\beta)}{P_{DS_A(G)}\left[\frac{-r\beta-2r(n-r-1)}{n-r-1}\right]} &= \frac{-2n(n-r-1)}{\beta+(n-r-1)(2r+2)} + 1 \\ &= \frac{\beta+(n-r-1)[-2n+2r+2]}{\beta+(n-r-1)(2r+2)}. \end{aligned}$$

Simplifying we get the required DS_A polynomial for $\overline{G}$ in terms of DS_A polynomial of G. □

3. DS_A polynomials and spectra of some regular graphs

Theorem 3.1. *[14] The degree sum adjacency polynomial of a complete graph K_n with n vertices is*

$$P_{DS_A(K_n)}(\beta) = [\beta + 2(n-1)]^{n-1}\left[\beta - 2(n-1)^2\right]. \tag{15}$$

This result can also be obtained by using lemma (1.1).

Theorem 3.2. *The DS_A-polynomial for a 1-regular graph G with k vertices is*

$$P_{DS_A(K_2)}(\beta) = (\beta^2 - 4)^k. \tag{16}$$

Proof. As each component of a 1-regular graph is isomorphic to K_2, by substituting $n = 2$ in Eq. (15) we obtain

$$P_{DS_A(K_2)}(\beta) = (\beta - 2)(\beta + 2) = (\beta^2 - 4)^k.$$

□

A cocktail-party graph is a complementary graph of 1-regular graph.

Corollary 3.1. *The DS_A- polynomial of the cocktail-party graph with $2k$ vertices is*

$$P_{DS_A(CP(k))}(\beta) = \beta^k[\beta - 2(2k-2)^2][\beta + 4(2k-2)]^{k-1}. \tag{17}$$

Proof. Let G be a 1-regular graph, then $P_{DS_A(\overline{G})} = P_{DS_A(CP(k))}$. To obtain DS_A polynomial for cocktail-party graph, substitute $n = 2k$ and $r = 1$ in Eq. (12)

$$\begin{aligned} P_{DS_A(CP(k))}(\beta) &= (-1)^{2k}(2k-2)^{2k}\left[\frac{\beta - 2(2k-2)^2}{\beta + (2k-2)4}\right] P_{DS_A(G)}\left[\frac{-\beta - 2(2k-2)}{2k-2}\right] \\ &= (2k-2)^{2k}\left[\frac{\beta - 2(2k-2)^2}{\beta + (2k-2)4}\right]\left\{\left[\frac{-\beta - 2(2k-2)}{2k-2}\right]^2 - 4\right\}^k \\ &= \beta^k[\beta - 2(2k-2)^2][\beta + 4(2k-2)]^{k-1}. \end{aligned}$$

□

Theorem 3.3. *If C_n is a cycle with n vertices, then eigenvalues of degree sum matrix of C_n are*

$$\beta_k = 8cos\left(\frac{2\pi k}{n}\right) \qquad k = 0, 1, \ldots, n-1. \tag{18}$$

Proof. The eigenvalues of $A(C_n)$ are $\lambda_k = 2cos\frac{2\pi k}{n}$ where $k = 0, 1, \ldots, n-1$. As $DS_A(G) = 4A(G)$, the eigenvalues of $DS_A(C_n)$ are $\beta_k = 8cos\frac{2\pi k}{n}$ where $k = 0, 1, \ldots, n-1$. □

A crown graph S_n^0 is obtained from the complete bipartite graph $K_{n,n}$ by deleting the perfect matching edges.

Theorem 3.4. *The DS_A-polynomial of a $2n$-vertex crown graph S_n^0 is*

$$P_{DS_A(S_n^0)}(\beta) = \left[\beta^2 - 4(n-1)^2\right]^{n-1}\left[\beta^2 - 4(n-1)^4\right] \tag{19}$$

Proof. The DS_A-matrix of crown graph will be of the form $\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$. The DS_A matrix can be reduced to the form $(X - Y)(X + Y)$, where X is a matrix of all zeros and Y is a matrix with all non diagonal entries as $2(n-1)$ and the diagonl entries as zero. The matrix Y is of the form $2(n-1)J - 2(n-1)I$. Separately evaluating $(X - Y)$ and $(X + Y)$ by applying lemma (1.1) and then multiplying, we get the required result. □

4. DS_A polynomial of some graph operations

Line Graph $L(G)$ of a graph G is the graph which has one-to-one correspondence between the vertex set and the set of edges of the graph G, with two vertices of $L(G)$ being adjacent iff the corresponding edges are adjacent in G [8].

Theorem 4.1. *If G is a r regular graph having n vertices and $m = \frac{1}{2}nr$ edges and $L(G)$ is a line graph, then DS_A polynomial of $L(G)$ in terms of DS_A polynomial of G is*

$$P_{DS_A(L(G))}(\beta) = (\beta + 8r - 8)^{m-n} \left(\frac{2r-2}{r}\right)^n P_{DS_A(G)} \left[\frac{r}{2r-2}\left(\beta - 4r^2 + 12r - 8\right)\right]. \quad (20)$$

Proof. Let A be an adjacency matrix of graph G, B be an adjacency matrix of graph $L(G)$ and R be the incidence matrix of G with D as the degree matrix. Then for G, we have

$$RR^T = A + D = \frac{DS_A(G)}{2r} + D \text{ and } R^T R = B + 2I = \frac{DS_A L(G)}{4r-4} + 2I$$

Taking $r' = 4r - 4$ we have,

$$\begin{aligned}
\beta^m P_{RR^T}(\beta) &= \beta^n P_{R^T R}(\beta) \\
\beta^{m-n}\left|\beta I - RR^T\right| &= \left|\beta I - R^T R\right| \\
\beta^{m-n}\left|\beta - \frac{DS_A(G)}{2r} - D\right| &= \left|\beta I - \frac{DS_A L(G)}{r'} - 2I\right| \\
\beta^{m-n}\left|(\beta - r)I - \frac{DS_A(G)}{2r}\right| &= \left|(\beta - 2)I - \frac{DS_A L(G)}{r'}\right| \\
\frac{\beta^{m-n}}{(2r)^n} \cdot (r')^m P_{DS_A(G)}[2r(\beta - r)] &= P_{DS_A(L(G))}[r'(\beta - 2)]
\end{aligned}$$

substituting $r'(\beta - 2) = \beta$ and $r' = 4r - 4$ we get the required result as shown in Eq. (20). □

$Subdivision\ graph$ $s(G)$ of a simple graph G is the graph which is obtained by adding (inserting) a new vertex onto every edge of G [8].

Theorem 4.2. *If G is a regular graph of degree r with n vertices and $m \left(= \frac{nr}{2}\right)$ edges and $s(G)$ is a subdivision graph, then DS_A polynomial $P_{DS_A(s(G))}$ of $s(G)$ in terms of its adjacency polynomial $\phi(G)$ is,*

$$P_{DS_A(s(G))}(\beta) = \beta^{m-n}(r+2)^{2n}\phi\left(G : \left[\frac{\beta^2}{(r+2)^2} - r\right]\right). \quad (21)$$

Proof. For a $r-$regular graph G having n vertices, its degree sum adjacency matrix of subdivision graph $s(G)$ of graph G is $DS_A(s(G))$. As vertex set of $s(G)$ is partitioned into two sets, one with

n vertices of degree r and the other with m vertices of degree 2, the characteristic polynomial of $DE(s(G))$ is obtained as follows.

$$\begin{aligned} P_{DS_A(s(G))}(\beta) &= \begin{vmatrix} \beta I_m & -(r+2)R^T \\ -(r+2)R & \beta I_n \end{vmatrix} \\ &= \beta^{m-n} \left| \beta^2 I_n - RR^T(r+2)^2 \right| \\ &= \beta^{m-n} \left| \beta^2 I_n - (r+2)^2(A + rI_n) \right| \\ &= \beta^{m-n}(r+2)^{2n} \left| \left(\frac{\beta^2}{(r+2)^2} - r \right) I_n - A \right| \\ &= \beta^{m-n}(r+2)^{2n} \phi \left(G : \left[\frac{\beta^2}{(r+2)^2} - r \right] \right). \end{aligned}$$

□

Semi total point graph $T_1(G)$ is a graph which is derived from graph G by inserting (adding) a new vertex into every edge of G and each new inserted vertex is then joined to the end points of the corresponding edge [3].

Theorem 4.3. *The DS_A polynomial $P_{DS_A(T_1(G))}$ of semi total point graph $T_1(G)$ of a n ordered r-regular graph G in terms of its adjacency polynomial $\phi(G)$ is*

$$P_{DS_A(T_1(G))}(\beta) = \beta^{m-n} \left[4r\beta + (2r+2)^2 \right]^n \phi \left(G : \left[\frac{\beta^2 - r(2r+2)^2}{4r\beta + (2r+2)^2} \right] \right). \tag{22}$$

Proof. Let G be a r-regular graph with n vertices, where $m = nr/2$ new vertices are added to construct a $T_1(G)$ graph. Then the DS_A polynomial of $T_1(G)$ is $DS_A(T_1(G)) = det(\beta I - DS_A(T_1(G)))$.

$$\begin{aligned} P_{T_1(G)}(\beta) &= \begin{vmatrix} \beta I_m & -(2r+2)R^T \\ -R(2r+2) & \beta I_n - 4rA \end{vmatrix} \\ &= \beta^m \left| \beta I_n - 4rA - \frac{(2r+2)^2 RR^T I_m}{\beta} \right| \\ &= \beta^{m-n} \left| \beta^2 I_n - 4rA\beta - (2r+2)^2(A + rI) \right| \\ &= \beta^{m-n} \left| \left(\beta^2 - r(2r+2)^2 \right) I_n - \left[4r\beta + (2r+2)^2 \right] A \right| \\ &= \beta^{m-n} \left[4r\beta + (2r+2)^2 \right]^n \left| \frac{\left(\beta^2 - r(2r+2)^2 \right) I_n}{\left[4r\beta + (2r+2)^2 \right]} - A \right| \\ &= \beta^{m-n} \left[4r\beta + (2r+2)^2 \right]^n \phi \left(G : \left[\frac{\beta^2 - r(2r+2)^2}{4r\beta + (2r+2)^2} \right] \right). \end{aligned}$$

□

Semi total line graph $T_2(G)$ of a graph G, is the graph with vertex set $V(T_2(G)) = V(G) \cup E(G)$ in which two vertices are adjacent if they are on adjacent edges of G or one is a vertex of G and the other is an edge of G, incident to it [3].

Theorem 4.4. *Let G be a r-regular graph having n vertices and m edges and let $T_2(G)$ be a semi total line graph of G. Then the DS_A polynomial $P_{DS_A(T_2(G))}$ of semi total line graph $T_2(G)$ of a graph G in terms of its adjacency polynomial of line graph $\phi(L(G))$ is*

$$P_{DS_A(T_2(G))}(\beta) = \beta^{n-m}(4r\beta + 9r^2)^m \phi\left(L(G) : \left[\frac{\beta^2 - 18r^2}{4r\beta + 9r^2}\right]\right). \tag{23}$$

Proof. For a r-regular graph G, the DS_A polynomial of $T_2(G)$ is

$$\begin{aligned} P_{DS_A(T_2(G))}(\beta) &= \begin{vmatrix} \beta I_n & 3rR \\ 3rR^T & \beta I_m - 4rB \end{vmatrix} \\ &= \beta^{n-m}\left|(\beta I_m - 4rB)\beta - 9r^2R^TR\right| \\ &= \beta^{n-m}\left|(\beta I_m - 4rB)\beta - 9r^2(B+2I)\right| \\ &= \beta^{n-m}\left|(\beta^2 - 18r^2)I_m - (4r\beta + 9r^2)B\right| \\ &= \beta^{n-m}(4r\beta + 9r^2)^m \phi\left(L(G) : \left[\frac{\beta^2 - 18r^2}{4r\beta + 9r^2}\right]\right). \end{aligned}$$

. □

$Thorn\ graph$ G^{+k} is a graph which is obtained from graph G by attaching k pendent vertices to every edge of G. If G is a graph with n vertices and m edges, then G^{+k} has $n + nk$ vertices and $m + nk$ edges.

Theorem 4.5. *The DS_A polynomial $P_{DS_A(G^{+k})}$ of Thorn graph G^{+k} of a n ordered r-regular graph G in terms of its adjacency polynomial $\phi(G)$ is*

$$P_{DS_A(G^{+k})}(\beta) = \beta^{nk}[2(r+k)]^n \phi\left(G : \left[\frac{\beta}{2(r+k)} - \frac{k(r+k+1)^2}{2(r+k)\beta}\right]\right). \tag{24}$$

Proof. The DS_A polynomial of Thorn graph can be written as,

$$P_{DS_A(G^{+k})}(\beta) = \begin{vmatrix} \beta I_n - 2(r+k)A & -(r+k+1)J & -(r+k+1)J & \cdots & -(r+k+1)J \\ -(r+k+1)J' & \beta I_k & 0 & \cdots & 0 \\ -(r+k+1)J' & 0 & \beta I_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(r+k+1)J' & 0 & 0 & \cdots & \beta I_k \end{vmatrix}$$

where A is the adjacency matrix of G, I is the unit matrix and J is a block matrix of order (n, k). For $\beta \neq 0$, multiply the rows (consisting of block matrices) numbered $2, 3, \ldots, k+1$ by $\frac{1}{\beta}(r+k+1)$

and add the resulting rows to the first row. This reduces the determinant as follows.

$$\begin{aligned} P_{G^{+k}}(\beta) &= \begin{vmatrix} [\beta I_n - 2(r+k)A] - \dfrac{k(r+k+1)^2}{\beta} & 0 & 0 & \cdots & 0 \\ -(r+k+1)J' & \beta I_k & 0 & \cdots & 0 \\ -(r+k+1)J' & 0 & \beta I_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -(r+k+1)J' & 0 & 0 & \cdots & \beta I_k \end{vmatrix} \\ &= \beta^{nk} \left| \left(\beta - \frac{k(r+k+1)^2}{\beta} \right) I_n - 2(r+k)A \right| \\ &= \beta^{nk}[2(r+k)]^n \phi \left(G : \left[\frac{\beta}{2(r+k)} - \frac{k(r+k+1)^2}{2(r+k)\beta} \right] \right). \end{aligned}$$

Hence the result. □

$Total\ graph$ $T(G)$ of G is a graph with vertex set $V(T(G)) = V(G) \cup E(G)$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident [8].

Theorem 4.6. *If G is a regular graph of degree r having n vertices and m edges, then the Total graph $T(G)$ has $(m-n)$ DS_A eigenvalues equal to $-8r$ and the other $2n$ eigenvalues are given by,*

$$\frac{1}{2}\left(4r^2 - 8r + 8r\lambda_i\right) \pm 4r\sqrt{r^2 + 4 + 4\lambda_i}$$

where λ_i $(i = 1, 2, \ldots, n)$ being the adjacency eigenvalues of G.

Proof. Let G be a r regular graph with n vertices and m edges. As $DS_A(T(G))$ can be expressed in terms of its adjacency matrix A, adjacency matrix of line graph B and the incidence matrix R of a graph G, we get

$$DS_A(T(G)) = \begin{pmatrix} 4rA & 4rR \\ 4rR^T & 4rB \end{pmatrix}.$$

Its DS_A polynomial can be expressed as

$$P_{DS_A(T(G))}(\beta) = \begin{vmatrix} \beta I - 4rA & -4rR \\ -4rR^T & \beta I - 4rB \end{vmatrix}.$$

As $A + D = RR^T$ and $B + 2I = R^T R$
$-4rA = 4r^2 I - 4rRR^T$ and $-4rB = 8rI - 4rR^T R$

$$P_{DS_A(T(G))}(\beta) = \begin{vmatrix} \beta I + 4r^2 I - 4rRR^T & -4rR \\ -4rR^T & \beta I + 8rI - 4rR^T R \end{vmatrix}.$$

Applying series of elementary transformation,

- Second row = second row - R^T first row

- First row = First row + $\frac{4rR}{\beta+8r}$ second row

the determinant can be expressed as follows.

$$\begin{aligned}
P_{DS_A(T(G))}(\beta) &= \begin{vmatrix} (\beta+4r^2)I - 4rRR^T & -4rR \\ -4rR^T - (\beta+4r^2)IR^T - 4rR^TRR^T & (\beta+8r)I \end{vmatrix} \\
&= \begin{vmatrix} \beta I - 4rA & -4rR \\ -(\beta+4r^2+4r)R^T + 4rR^TRR^T & (\beta+8r)I \end{vmatrix} \\
&= \left|(\beta+8r)I_m \left\{(\beta I - 4rA) + \left[-(\beta+4r^2+4r) + 4rRR^T\right]\frac{4rR}{\beta+8r}\right\}\right| \\
&= (\beta I + 8r)^{m-n}\left|(\beta I - 4rA)(\beta+8r) + [-(\beta+4r^2+4r)+4rRR^T]4rRR^T\right| \\
&= (\beta I + 8r)^{m-n}\left|(\beta I - 4rA)(\beta+8r) + [4rA - (\beta+4r)I](4rA+4r^2I)\right| \\
&= (\beta I + 8r)^{m-n}\left|16r^2A^2 + (16r^3 - 48r^2 - 8r\beta)A + (\beta^2 + 8r\beta - 4r^2\beta - 16r^3)\right| \\
&= (\beta I + 8r)^{m-n} \times \\
&\quad \times \prod_{i=1}^{n}\left\{\beta^2 - \beta(4r^2 - 8r + 8r\lambda_i) + \left[16r^2\lambda_i^2 + \lambda_i(16r^3 - 48r^2) - 16r^3\right]\right\}
\end{aligned}$$

where λ_i $(i = 1, 2, \ldots, n)$ are the eigenvalues of A. Thus we have proved that there are exactly $(m-n)$ DS_A eigenvalues of $T(G)$ equal to $\beta = -8r$.
Using $b^2 - 4ac$, we find that the roots of the polynomial $a\beta^2 + b\beta + c$ where $a = 1$, $b = -(4r^2 - 8r + 8r\lambda_i)$ and $c = 16r^2\lambda_i^2 + \lambda_i(16r^3 - 48r^2) - 16r^3$.
On solving we get $2n$ eigenvalues of $T(G)$ as

$$\frac{1}{2}\left\{(4r^2 - 8r + 8r\lambda_i) \pm 4r\sqrt{r^2 + 4 + 4\lambda_i}\right\}.$$

□

The *join* $G_1 \nabla G_2$ of (disjoint) graphs G_1 and G_2 is the graph that is obtained from $G_1 \cup G_2$, by joining every vertex of G_1 to all vertices of G_2.

Theorem 4.7. *Let G_1 and G_2 be two regular graphs with regularity r_1 and r_2 and with orders n_1 and n_2 respectively. Then the DS_A-polynomial of $G_1 \nabla G_2$ is given by the relation,*

$$P_{DS_A(G_1\nabla G_2)}(\beta) = \frac{P_{G_1}\left(\frac{r_1\beta}{r_1+n_2}\right) P_{G_2}\left(\frac{r_2\beta}{r_2+n_2}\right)}{[\beta - 2r_1(r_1+n_2)][\beta - 2r_2(r_2+n_1)]}\{[\beta - 2r_1(r_1+n_2)] [\beta - 2r_2(r_2+n_1)] - n_1n_2x^2\}. \tag{25}$$

where $x = n_1 + n_2 + r_1 + r_2$.

Proof. The DS_A-polynomial of $G_1 \nabla G_2$ is obtained as

$$\begin{aligned} P_{DS_A(G_1\nabla G_2)}(\beta) &= det(\beta I - DS_A(G_1\nabla G_2)) \\ &= \begin{vmatrix} \beta I_{n_1} - \left(\frac{r_1+n_2}{r_1}\right) DS_A(G_1) & -xJ_{n_1\times n_2} \\ -xJ_{n_2\times n_1} & \beta I_{n_2} - \left(\frac{r_2+n_1}{r_2}\right) DS_A(G_2) \end{vmatrix} \end{aligned}$$

where $x = n_1 + n_2 + r_1 + r_2$ and J is a matrix whose all entries are equal to unity. The above determinant can be written as,

$$\begin{vmatrix} \beta & -ds_{12} & \cdots & -ds_{1n_1} & -x & -x & \cdots & -x \\ -ds_{21} & \beta & \cdots & -ds_{2n_1} & -x & -x & \cdots & -x \\ \vdots & & \vdots & & & & \vdots & \\ -ds_{n_11} & -ds_{n_12} & \cdots & \beta & -x & -x & \cdots & -x \\ -x & -x & \cdots & -x & \beta & -ds'_{12} & \cdots & -ds'_{1n_2} \\ -x & -x & \cdots & -x & -ds'_{21} & \beta & \cdots & -ds'_{2n_2} \\ \vdots & & \vdots & & & & \vdots & \\ -x & -x & \cdots & -x & -ds'_{n_21} & -ds'_{n_22} & \cdots & \beta \end{vmatrix}. \tag{26}$$

Where ds_{ij} is the ij^{th} entry DS_A matrix of G_1 and ds'_{ij} is the ij^{th} entry DS_A matrix of G_2. In G_1 each vertex is adjacent to all vertices of G_2, so its new vertex degree is $r_1 + n_2$ and as there are r_1 vertices adjacent to a vertex v_i in G_1, therefore

$$\sum_{j=1}^{n_1} ds_{ij} = 2r_1(r_1+n_2) \qquad \text{for} \quad i = 1, 2, \ldots, n_1. \tag{27}$$

Similarly for G_2

$$\sum_{j=1}^{n_2} ds'_{ij} = 2r_2(r_2+n_1) \qquad \text{for} \quad i = 1, 2, \ldots, n_2. \tag{28}$$

We carry out a series of elementary transformations so that the determinant remains unchanged. Subtracting $(n_1+1)^{th}$ row from the rows $(n_1+2), (n_1+3), \ldots, (n_1+n_2)$ of determinant (26), we get

$$\begin{vmatrix} \beta & -ds_{12} & \cdots & -ds_{1n_1} & -x & -x & \cdots & -x \\ -ds_{21} & \beta & \cdots & -ds_{2n_1} & -x & -x & \cdots & -x \\ \vdots & & \vdots & & & & \vdots & \\ -ds_{n_11} & -ds_{n_12} & \cdots & \beta & -x & -x & \cdots & -x \\ -x & -x & \cdots & -x & \beta & -ds'_{12} & \cdots & -ds'_{1n_2} \\ 0 & 0 & \cdots & 0 & -ds'_{21}-\beta & \beta+ds'_{12} & \cdots & -ds'_{2n_2}+ds'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & -ds'_{n_21}-\beta & -ds'_{n_22}+ds'_{12} & \cdots & \beta+ds'_{1n_2} \end{vmatrix}$$

Add the columns $(n_1+2),(n_1+3),\ldots,(n_1+n_2)$ to the $(n_1+1)^{th}$ column, using Eq. (28), and also taking into consideration $ds'_{ij}=ds'_{ji}$ we arrive at the following determinant,

$$\begin{vmatrix} \beta & -ds_{12} & \cdots & -ds_{1n_1} & -n_2 & -x & \cdots & -x \\ -ds_{21} & \beta & \cdots & -ds_{2n_1} & -n_2x & -x & \cdots & -x \\ \vdots & & \vdots & & & & \vdots & \\ -ds_{n_11} & -ds_{n_12} & \cdots & \beta & -n_2x & -x & \cdots & -x \\ -x & -x & \cdots & -x & \beta-2r_2(r_2+n_1) & -ds'_{12} & \cdots & -ds'_{1n_2} \\ 0 & 0 & \cdots & 0 & 0 & \beta+ds'_{12} & \cdots & -ds'_{2n_2}+ds'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & -ds'_{n_22}+ds'_{12} & \cdots & \beta+ds'_{1n_2} \end{vmatrix}.$$

On simplifying, the determinant reduces to

$$\begin{vmatrix} \beta & -ds_{12} & \cdots & -ds_{1n_1} & -n_2x \\ -ds_{21} & \beta & \cdots & -ds_{2n_1} & -n_2x \\ \vdots & & \vdots & & \\ -ds_{n_11} & -ds_{n_12} & \cdots & \beta & -n_2x \\ -x & -x & \cdots & -x & \beta-2r_2(r_2+n_1) \end{vmatrix} |X|, \tag{29}$$

where

$$|X| = \begin{vmatrix} \beta+ds'_{12} & -ds'_{23}+ds'_{13} & \cdots & -ds'_{2n_2}+ds'_{1n_2} \\ -ds'_{32}+ds'_{12} & \beta+ds'_{13} & \cdots & -ds'_{3n_2}+ds'_{1n_2} \\ \vdots & & \vdots & \\ -ds'_{n_22}+ds'_{12} & -ds'_{n_23}+ds'_{13} & \cdots & \beta+ds'_{1n_2} \end{vmatrix}. \tag{30}$$

Subtracting first rows of determinant (29) from all other rows, we get

$$\begin{vmatrix} \beta & -ds_{12} & \cdots & -ds_{1n_1} & -n_2x \\ -ds_{21}-\beta & \beta+ds_{12} & \cdots & -ds_{2n_1}+ds_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -ds_{n_11}-\beta & -ds_{n_12}+ds_{12} & \cdots & \beta+ds_{1n_1} & 0 \\ -1 & -1 & \ldots & -1 & \beta-2r_2(r_2+n_1) \end{vmatrix} |X| \,. \tag{31}$$

Adding columns $2,3,\ldots,n_1$ to the first column and using Eq. (27) we get

$$\begin{vmatrix} \beta-2r_1(r_1+n_2) & -ds_{12} & \cdots & -ds_{1n_1} & -n_2x \\ 0 & \beta+ds_{12} & \cdots & -ds_{2n_1}+ds_{1n_1} & 0 \\ \vdots & & \vdots & & \\ 0 & -ds_{n_12}+ds_{12} & \cdots & \beta+ds_{1n_1} & 0 \\ -n_1x & -x & \cdots & -x & \beta-2r_2(r_2+n_1) \end{vmatrix} |X|.$$

Expanding the determinant along its first column we get

$$\{[\beta - 2r_1(r_1 + n_2)]\Delta_1 - (-1)^{n_1} n_1 \Delta_2\}\,|X|. \tag{32}$$

Where

$$\Delta_1 = \begin{vmatrix} \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} & 0 \\ -ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -ds_{n_1 2} + ds_{12} & -ds_{n_1 3} + ds_{13} & \cdots & \beta + ds_{1n_1} & 0 \\ -x & -x & \cdots & -x & \beta - 2r_2(r_2 + n_1) \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} -ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} & -n_2 x \\ \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} & 0 \\ -ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -ds_{n_1 2} + ds_{12} & -ds_{n_1 3} + ds_{13} & \cdots & \beta + ds_{1n_1} & 0 \end{vmatrix}.$$

The expression in (32) can be rewritten as

$$\begin{aligned} & \{[\beta - 2r_1(r_1 + n_2)][\beta - 2r_2(r_2 + n_1)]|Y| - n_1 n_2 |Y|\}|X| \\ = \; & |X||Y|\{[\beta - 2r_1(r_1 + n_2)][\beta - 2r_2(r_2 + n_1)] - n_1 n_2\} \end{aligned} \tag{33}$$

where

$$|Y| = \begin{vmatrix} \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} \\ -ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} \\ \vdots & & \vdots & \\ -ds_{n_1 2} + ds_{12} & -ds_{n_1 3} + ds_{13} & \cdots & \beta + ds_{1n_1} \end{vmatrix}.$$

The above determinant can be written as

$$|Y| = \frac{1}{[\beta - 2r_1(r_1 + n_2)]} \times$$

$$\begin{vmatrix} \beta - 2r_1(r_1 + n_2) & -ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} \\ 0 & \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} \\ 0 & -d_{32} + d_{12} & \mu + d_{13} & \ldots & -d_{3n_1} + d_{1n_1} \\ \vdots & & & \vdots & \\ 0 & -ds_{n_1 2} + ds_{12} & -ds_{n_1 3} + ds_{13} & \cdots & \beta + ds_{1n_1} \end{vmatrix}.$$

Using Eq. (27), the sum of the i-th row in the above determinant is $\beta + ds_{i1}$ for $i = 2, 3, \ldots, n_1$. Therefore, by subtracting the columns $2, 3, \ldots, n_1$ of above determinant from the first column, we obtain

$$|Y| = \frac{1}{[\beta - 2r_1(r_1+n_2)]} \times$$

$$\begin{vmatrix} \beta & -ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} \\ -\beta - ds_{21} & \beta + ds_{12} & -ds_{23} + ds_{13} & \cdots & -ds_{2n_1} + ds_{1n_1} \\ -\beta - ds_{31} & -ds_{32} + ds_{12} & \beta + ds_{13} & \cdots & -ds_{3n_1} + ds_{1n_1} \\ \vdots & & & \vdots & \\ -\beta - ds_{n_11} & -ds_{n_12} + ds_{12} & -ds_{n_13} + ds_{13} & \cdots & \beta + ds_{1n_1} \end{vmatrix}.$$

Adding first row to all other rows of the determinant, we get

$$\begin{aligned} |Y| &= \frac{1}{[\beta - 2r_1(r_1+n_2)]} \begin{vmatrix} \beta & -ds_{12} & -ds_{13} & \cdots & -ds_{1n_1} \\ -ds_{21} & \beta & -ds_{23} & \cdots & -ds_{2n_1} \\ -ds_{31} & -ds_{32} & \beta & \cdots & -ds_{3n_1} \\ \vdots & & & \vdots & \\ -ds_{n_11} & -ds_{n_12} & -ds_{n_13} & \cdots & \beta \end{vmatrix} \\ &= \frac{1}{[\beta - 2r_1(r_1+n_2)]} P_{G_1}(\beta) . \end{aligned} \tag{34}$$

Similarly, we can show that from Eq. (30) we get

$$|X| = \frac{1}{[\beta - 2r_2(r_2+n_1)]} P_{G_2}(\beta) . \tag{35}$$

Substituting Eq. (34) and Eq. (35) into Eq. (33) results to Eq. (25).

□

Let G be a graph with n_1 vertices and let H be a graph with n_2 vertices. Then the $corona$ $G \circ H$ is the graph with $n_1 + n_1 n_2$ vertices, which is obtained by taking graph G and n copies of graph H and by joining i^{th} vertex of G to each vertex in the i-copy of H $(i = 1, \cdots, n_1)$.

Theorem 4.8. *Let G and H be regular graphs with n_1 and n_2 vertices respectively. Then the DS_A polynomial $P_{DS_A(G\circ H)}$ of the corona $G \circ H$ in terms of its adjacency polynomials $\phi(G)$ and $\phi(H)$ is*

$$\begin{aligned} P_{DS_A(G\circ H)}(\beta) = & 2^{n_1 n_2 + n_1}(r_1+n_2)^{n_1}(r_2+1)^{n_1 n_2} \left\{ \phi\left(H : \left[\frac{\beta}{2(r_2+1)} \right] \right) \right\}^{n_1} \\ & \phi\left(G : \left[\frac{\beta}{2(r_1+n_2)} - \frac{m(r_1+r_2+n_2+1)^2}{2(r_1+n_2)(\beta - 2(r_2+1))} \right] \right) . \end{aligned} \tag{36}$$

Proof. Let A be adjacency matrix of the r_1 regular graph G with n_1 vertices and let B be the adjacency matrix of the r_2 regular graph H with n_2 vertices. Its DS_A polynomial can be obtained as follows,

$$P_{DS_A(G\circ H)} = \begin{vmatrix} \beta I - 2(r_1+n_2)A & -(r_1+r_2+1+n_2)J & \cdots & -(r_1+r_2+1+n_2)J \\ -(r_1+r_2+1+n_2)J^T & \beta I - 2(r_2+1)B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(r_1+r_2+1+n_2)J^T & 0 & \cdots & \beta I - 2(r_2+1)B \end{vmatrix}.$$

Multiply the rows (consisting of block matrices) numbered $2, 3, \ldots, n_1$ by $\dfrac{(r_1+r_2+1+n_2)}{\beta I - 2(r_2+1)}$, then the sum of rows of the block matrices to the respective row of the first block matrix. This reduces the determinant to

$$P_{DS_A(G\circ H)}(\beta) =$$
$$\begin{vmatrix} \beta I - \dfrac{m(r_1+r_2+1+n_2)^2}{\beta I - 2(r_2+1)} - 2(r_1+n_2)A & 0 & \cdots & 0 \\ -(r_1+r_2+1+n_2)J^T & \beta I - 2(r_2+1)B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(r_1+r_2+1+n_2)J^T & 0 & \cdots & \beta I - 2(r_2+1)B \end{vmatrix}.$$

On simplifying the determinant, we get

$$\begin{aligned} P_{DS_A(G\circ H)}(\beta) &= |\beta I - 2(r_2+1)B|^{n_1} \left| \left(\beta - \frac{m(r_1+r_2+1+n_2)^2}{\beta I - 2(r_2+1)}\right) I - 2(r_1+n_2)A \right| \\ &= 2^{n_1 n_2}(r_2+1)^{n_1 n_2} \left| \frac{\beta}{2(r_2+1)} I - B \right|^{n_1} 2^{n_1}(r_1+n_2)^{n_1} \\ &\quad \left| \left(\frac{\beta}{2(r_1+n_2)} - \frac{m(r_1+r_2+1+n_2)^2}{(r_1+n_2)\beta I - 2(r_2+1)} \right) I - A \right| \\ &= 2^{n_1 n_2 + n_1}(r_2+1)^{n_1 n_2}(r_1+n_2)^{n_1} \left\{ \phi\left(H : \left[\frac{\beta}{2(r_2+1)} \right] \right) \right\}^{n_1} \\ &\quad \phi\left(G : \left[\frac{\beta}{2(r_1+n_2)} - \frac{m(r_1+r_2+1+n_2)^2}{(r_1+n_2)\beta I - 2(r_2+1)} \right] \right). \end{aligned}$$

□

Theorem 4.9. *The DS_A-polynomial of cartesian product of complete graphs K_2 and K_n, $K_2 \square K_n$ is*

$$P_{DS_A(K_2\square K_n)}(\beta) = \left(\frac{\beta - 2n^2}{\beta + 2n^2}\right) \left[(\beta+2n)^2 - 4n^2\right]^{n-1} \left[(\beta+2n)^2 - 4n^2(n-1)^2\right] \quad (37)$$

Proof. As complement of $2n$-vertex crown graph S_n^0 is the cartesian product of K_2 and K_n, $K_2 \square K_n$. Applying the result of Theorem (3.4) in Eq. (12) of Theorem(2.2) we get the required result. □

References

[1] C. Adiga and M. Smitha, On maximum degree energy of a graph, *Int. J. Contemp. Math. Sci*, **4**, (2009), 385–396.

[2] R.B. Bapat, *Graphs and Matrices*, Springer, 2010.

[3] D.M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs Theory and Applications*, Academic Press, New York, 1980.

[4] T. Došlic, B. Furtula, A. Graovac, I. Gutman, S. Moradi, and Z. Yarahmadi, On vertex–degree–based molecular structure descriptors, *MATCH Commun. Math. Comput. Chem*, **66**, (2011), 613–626.

[5] K. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem*, **52**, (2004), 103–112.

[6] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons, *Chemical Physics Letters*, **17(4)**, (1972), 535–538.

[7] I. Gutman and K. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem*, **50** (2004) 83–92.

[8] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.

[9] H. Minc and M. Marcus, *A Survey of Matrix Theory and Matrix Inequalities*, Prindle, Weber & Schmidt, Boston, 1964.

[10] H.S. Ramane, D.S. Revenkar and J.B. Patil, Bounds for the Degree sum eigenvalue and Degree sum energy of a graph, *Int. J. Pure App. Sci.*, **6(2)**, (2013), 161–167.

[11] H.S. Ramane and S.S. Shinde, Degree exponent polynomial and degree exponent energy of graphs, *Indian J. Discrete Math.*, **2(1)** (2016),01 – 07.

[12] H.S. Ramane and S.S. Shinde, Degree exponent polynomial of graphs obtained by some graph operations, *Electronic Notes in Discrete Math.*, **63** (2017), 161–168.

[13] H.S. Ramane, K.C. Nandeesh, G.A. Gudodagi and B. Zhou, Degree subtraction eigenvalues and energy of graphs, *Computer Science Journal of Moldova*,**26(2)**, (2018).

[14] R.K. Zaferani, *A study of some topics in the theory of graphs*, (Ph.D Thesis), University of Mysore, Mysore, 2009.

[15] B. Zhou and I. Gutman, Further properties of Zagreb indices, *MATCH Commun. Math. Comput. Chem*, **54**,(2005), 233–239.