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Hyper-Hamiltonian circulants

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Abstract

A Hamiltonian graph $G = (V, E)$ is called hyper-Hamiltonian if $G - v$ is Hamiltonian for any $v \in V(G)$. G is called a *circulant* if its automorphism group contains a $|V(G)|$-cycle. First, we give the necessary and sufficient conditions for any undirected connected circulant to be hyper-Hamiltonian. Second, we give necessary and sufficient conditions for a connected circulant digraph with two jumps to be hyper-Hamiltonian. In addition, we specify some sufficient conditions for a circulant digraph with arbitrary number of jumps to be hyper-Hamiltonian.

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1. Introduction

A *Hamilton cycle* in an undirected graph $G = (V, E)$ is a cycle that passes through every vertex $v \in V(G)$ exactly once. A graph G is called $Hamiltonian$ if it contains such a cycle, and a Hamiltonian graph G is *hyper-Hamiltonian* if $G - v$ is Hamiltonian for any $v \in V(G)$. It is well known that the problem of determining whether or not an arbitrary graph contains a Hamilton cycle is *NP-complete*. This implies that the problem of determining if an arbitrary graph is hyper-Hamiltonian is at least as hard as *NP-complete* problems. For special families of graphs, however, a Hamilton cycle can be computed efficiently. In particular, it is known that a class of undirected vertex-symmetric graphs called circulants, considered in the third section of this paper, always contains a Hamilton cycle [7]. Furthermore, since all circulants (undirected and directed) are vertex-transitive then the condition for hyper-Hamiltonicity in these graphs has to be checked

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for only one vertex. Hence, circulants represent a family of graphs for which it is natural to consider if they are hyper-Hamiltonian. Other families of graphs with respect to hyper-Hamiltonian cycles have been also investigated in the literature, as this problem seems to be relevant to network survivability with respect to a single node failure [9]. For example, Araki examined hyper-Hamiltonian laceability of Cayley graphs, taking into consideration networks for parallel and distributed systems [1]. Mai et al. considered sufficient conditions for a hyper-Hamiltonian cycle in generalized Petersen graphs [8]. Del-Vecchio et al. also focused on some sufficient conditions for a graph to be hyper-Hamiltonian by providing spectral and non-spectral conditions for the hyper-Hamiltonian cycle [6]. In this paper, we give both necessary and sufficient conditions for circulants to be hyper-Hamiltonian.

Let $(a_1, a_2, \ldots, a_k)$ be a sequence of k pairwise distinct positive integers. The undirected *circulant graph* $G_n(a_1, a_2, \ldots, a_k)$ has vertices $i \pm a_1, i \pm a_2, \ldots, i \pm a_k \pmod{n}$ adjacent to each vertex i, where $a_j < \frac{n+1}{2}$ for $1 \leq j \leq k$. Similarly, the *circulant digraph* $G_n(a_1, a_2, \ldots, a_k)$ has vertices $i + a_1, i + a_2, \ldots, i + a_k \pmod{n}$ adjacent to each vertex i, where $a_j < n$ for $1 \leq j \leq k$. The sequence $\{a_j\}$ is called the *jump sequence*, and the a_js are called the jumps [2]. Hence, a circulant digraph $G_n(a_1, a_2, \ldots, a_k)$ contains an edge (i.e., two opposite arcs) if it contains a_i such that $a_i = n - a_j$ for some a_j. In particular, a circulant digraph $G_n(a_1, a_2, \ldots, a_k)$ is equivalent to an undirected circulant H_n if for every a_i there exists a_j such that $a_i = n - a_j$. Consequently, a Hamilton cycle in $G_n(a_1, a_2, \ldots, a_k)$ needs to adhere to the directions of arcs.

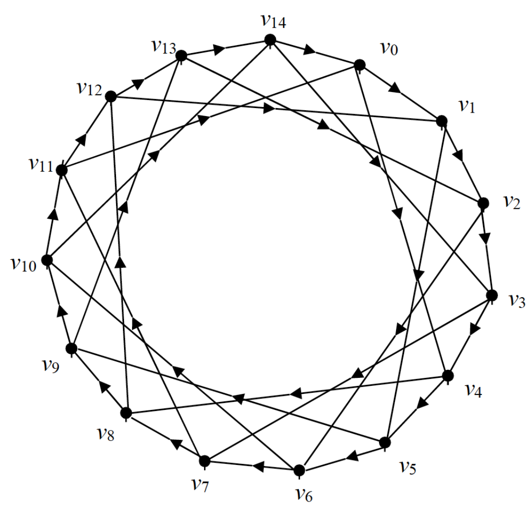

Figure 1. Example of a circulant digraph $G_n(a_1, a_2) = G_{15}(1, 4) \simeq G_{15}(4, 1)$.

This paper is organized as follows. In the next Section 2, we cover prior published results that we will leverage in the proofs of our theorems in the following sections. In Section 3, we give necessary and sufficient conditions in Theorem 3.1 for an arbitrary undirected circulant to be hyper-Hamiltonian. Section 4 covers directed circulants. First, in Theorem 4.1 we give the necessary and sufficient conditions for a circulant digraph with 2 jumps to be hyper-Hamiltonian. In the second part of Section 4 we focus and give some sufficient conditions for hyper-Hamiltonicity in circulant digraph with arbitrary number of jumps.

2. Preliminary Results

There are five results that will be useful in proving our main theorem in the next section for undirected circulants, and one result (i.e., Theorem 2.4) pertaining to circulant digraphs that we will leverage in the last section of this paper. First, the following result was proved by Lovász.

Theorem 2.1 ([7])**.** *Every connected circulant is Hamiltonian.*

Second, we examined pancyclic and edge-bipancyclic properties in undirected circulants [5]. $G(V, E)$ is *edge-bipancyclic* if every edge e, $e \in E(G)$, is included in a cycle of length $2k$ for every value of $2k$, where $|V(G)| \geq 2k \geq 4$. Furthermore, $G(V, E)$ is *pancyclic* if it contains cycles of length k for every value of k, where $|V(G)| \geq k \geq 3$. We obtained the following two results:

Theorem 2.2 ([5])**.** *Let G be a connected circulant of order n with at least two jumps. Then, G is edge-bipancyclic.*

Theorem 2.3 ([5])**.** *Let G be a connected circulant of order n and girth 3. Then, G is pancyclic.*

Both results are related to hyper-Hamiltonicity of undirected circulants, which will be exploited in the proof of our main theorem of Section 3.

We also need the following result due to Boesch and Tindell that pertains to undirected as well as directed circulants.

Theorem 2.4 ([2])**.** *Circulant $G_n(a_1, a_2, \ldots, a_k)$ is connected if and only if*

$$\gcd(n, a_1, a_2, \ldots, a_k) = 1.$$

Furthermore, we extended Theorem 2.4 as follows.

Theorem 2.5 ([3])**.** *Circulant $G_n(a_1, a_2, \ldots, a_k)$ consists of r connected components if and only if*

$$\gcd(n, a_1, a_2, \ldots, a_k) = r.$$

Since each connected component of any circulant is of identical size, having r from Theorem 2.5 establishes their size equal $\frac{n}{r}$, which will be useful in the proofs in Sections 3-4.

3. Hyper-Hamiltonian Undirected Circulants

We now prove necessary and sufficient conditions for an arbitrary undirected and connected circulant to be hyper-Hamiltonian.

Theorem 3.1. *A connected circulant $G_n(a_1, a_2, \ldots, a_k)$ for $k \geq 2$ is hyper-Hamiltonian if and only if either n is odd or at least one a_i is even.*

Proof. Recall that $G = G_n(a_1, a_2, \dots, a_k)$ has vertices $v_0, v_1, \dots, v_{n-1}$. As G is vertex transitive and Hamiltonian by Theorem 2.1, it suffices to show that $H = G - v_0$ has a Hamilton cycle.

We first assume that n is even and $a_1, a_2, \dots, a_k$ are odd, and we show that $G = G_n(a_1, a_2, \dots, a_k)$ is not hyper-Hamiltonian. Suppose G is hyper-Hamiltonian, so there exists a cycle C of length $n-1$. Without loss of generality assume we walk along W starting at v_0. Since every jump a_i is odd then the following two conditions are satisfied in W: (1) if a current vertex v in W is even then the next vertex $w = v \pm a_i \pmod n$ in W is odd, and (2) if a current vertex w is odd then the next vertex $v = w \pm a_i \pmod n$ in W is even. Since n is even then there does not exist a closed walk W that ends with $v_j = v_0$, which results in a simple cycle C of odd length – a contradiction, which proves our necessary condition of hypothesis.

For a sufficient condition of hyper-Hamiltonicity of G we consider the following two cases.

Case 1. n odd.

By Theorem 2.1 G contains a Hamilton cycle, and by Theorem 2.2 G is edge-bipancyclic. So, G also contains a Hamilton cycle in $H = G - v_i$ for any vertex v_i in G. Hence, G is hyper-Hamiltonian in this case.

Case 2. n even and at least one of the jumps a_i is even.

Since n is even then according to Theorem 2.4 also at least one of the jumps a_j is odd. Without loss of generality assume a_1 odd and a_k even. Let $\gcd(n, a_1, a_2, \dots, a_{k-1}) = r_1$ and $\gcd(n, a_k) = r_2$. Let v_i^j denote ith vertex in jth cycle formed by a_k in G. So, according to Theorem 2.5 $v_1^j v_2^j \cdots v_{\frac{n}{r_2}}^j$ denotes jth cycle formed by a_k in G, where $r_2 \geq j \geq 1$.

Let H be the subgraph of G consisting of all the edges $(i, i + a_k \pmod n)$. Since n, a_k are even, H must be the union of at least two pairwise disjoint cycles C_j. Let G' be obtain from G by contracting each cycle C_j of H into a vertex. If j is the smallest number such that some cycle in H contains vertices 1 and $1 + j \pmod n$, then the cycles of H containing vertices $1, 2, \dots, j$ are pairwise distinct and constitute all the cycles of H. Then the isomorphism $i \to i + 1 \pmod n$ of G induces an isomorphism of G', which proves that G' is a connected circulant. By Theorem 2.1, G' has a Hamilton cycle. This implies existence of a simple path $P = v_1^1 v_1^2 \cdots v_1^{r_2} v_q^1$ in G. Consequently, according to Theorem 2.5 there are $\gcd(n, a_1, a_2, \dots, a_{k-1})$, which is odd, cycles C_i corresponding to a Hamilton cycle of G' formed by jumps $a_1, \dots, a_{k-1}$ in G as follows:

$$C_1 = v_1^1 v_1^2 \cdots v_1^{r_2}\, v_{q_1}^1 v_{q_1}^2 \cdots v_{q_1}^{r_2} \cdots v_{z_1}^1 v_{z_1}^2 \cdots v_{z_1}^{r_2} v_1^1,$$

$$C_2 = v_2^1 v_2^2 \cdots v_2^{r_2}\, v_{q_2}^1 v_{q_2}^2 \cdots v_{q_2}^{r_2} \cdots v_{z_2}^1 v_{z_2}^2 \cdots v_{z_2}^{r_2} v_2^1,$$

$$\dots$$

$$C_{r_1} = v_{r_1}^1 v_{r_1}^2 \cdots v_{r_1}^{r_2}\, v_{q_{r_1}}^1 v_{q_{r_1}}^2 \cdots v_{q_{r_1}}^{r_2} \cdots v_{z_{r_1}}^1 v_{z_{r_1}}^2 \cdots v_{z_{r_1}}^{r_2} v_{r_1}^1,$$

where $q_1 = q$. Hence,

$$q = b \cdot \gcd(n, a_1, a_2, \dots, a_{k-1}) \pmod{\frac{n}{\gcd(n, a_k)}} + 1$$

for some positive odd integer b. Furthermore, since r_1 is odd then q is either 1 or can be assumed even. In particular, if $\frac{n}{\gcd(n,a_k)}$ is odd and q greater than 1 then q is even based on a convenient orientation of cycles formed by a_k (i.e., if q is odd in one orientation then it is even in opposite orientation). On the other hand, if $\frac{n}{\gcd(n,a_k)}$ is even then q has to be even as well.

If $\frac{n}{\gcd(n,a_k)} = 3$ then a_k forms triangles in G and by Theorem 2.3 G is pancyclic. This in turn implies that G is hyper-Hamiltonian and we are done. Otherwise, without loss of generality consider a walk W in G from vertex v_1^1.

Subcase 2.1. $q = 1$.

We start walk W with

$$W_2 = v_1^1 v_1^2 \cdots v_1^{r_2} v_2^{r_2},$$

which is followed by walks:

$$W_3 = v_2^1 v_3^1\ v_3^2 v_2^2 \cdots v_3^{r_2-2} v_2^{r_2-2}\ v_2^{r_2-1} v_3^{r_2-1}\ v_3^{r_2} v_4^{r_2},$$

$$W_4 = v_4^1 v_5^1\ v_5^2 v_4^2 \cdots v_5^{r_2-2} v_4^{r_2-2}\ v_4^{r_2-1} v_5^{r_2-1}\ v_5^{r_2} v_4^{r_2},$$

$$\cdots$$

$$W_t = v_{2t-4}^1 v_{2t-3}^1\ v_{2t-3}^2 v_{2t-4}^2 \cdots v_{2t-3}^{r_2-2} v_{2t-4}^{r_2-2}\ v_{2t-4}^{r_2-1} v_{2t-3}^{r_2-1}\ v_{2t-3}^{r_2} v_{2t-2}^{r_2},$$

where $2t - 1 = \frac{n}{r_2}$, and followed by walk,

$$W_{t+1} = v_{2t-1}^{r_2} v_{2t-1}^1 v_1^1.$$

Hence, $W = W_2 W_3 \cdots W_t W_{t+1}$ is a closed walk representing a simple cycle of length $n - 1$ in G that skips the vertex v_{2t-2}^1. So, in this subcase G is hyper-Hamiltonian.

Subcase 2.2. q even.

If either $q = 2$ or $q = \frac{n}{\gcd(n,a_2)}$ then a_2 forms a Hamilton cycle in G – a contradiction. So, we may assume $\frac{n}{\gcd(n,a_2)} > q > 2$. Let $t_1 = \frac{n}{r_1}$ be the size of a cycle induced by a_1. We start walk W with

$$W_1 = v_1^1 v_1^2 \cdots v_1^{r_2} v_q^1,$$

followed by walks:

$$W_2 = v_{q+1}^1 v_{q+2}^1 \cdots v_{t_1}^1\ v_{t_1}^2 v_{t_1-1}^2 \cdots v_{q+1}^2 v_q^2,$$

$$W_3 = v_{q+1}^2 v_{q+2}^2 \cdots v_{t_1}^2\ v_{t_1}^3 v_{t_1-1}^3 \cdots v_{q+1}^3 v_q^3,$$

$$\cdots$$

$$W_{r_2} = v_{q+1}^{r_2-1} v_{q+2}^{r_2-1} \cdots v_{t_1}^{r_2-1}\ v_{t_1}^{r_2} v_{t_1-1}^{r_2} \cdots v_{q+1}^{r_2} v_q^{r_2},$$

and if $r_2 \geq 4$ then subsequently followed by walks:

$$W_{r_2+1} = v_{q-1}^{r_2} v_{q-2}^{r_2} \cdots v_2^{r_2}\ v_2^{r_2-1} v_3^{r_2-1} \cdots v_{q-1}^{r_2-1},$$

$$W_{r_2+2} = v_{q-1}^{r_2-2} v_{q-2}^{r_2-2} \cdots v_2^{r_2-2}\ v_2^{r_2-3} v_3^{r_2-3} \cdots v_{q-1}^{r_2-3},$$

$$\dots$$

$$W_{r_2+\frac{r_2-2}{2}} = v^4_{q-1}v^4_{q-2}\cdots v^4_2\; v^3_2v^3_3\cdots v^3_{q-1}.$$

If $q-1>3$ then $W_1W_2\cdots W_{r_2+\frac{r_2-2}{2}}$ is followed by walk:

$$W_{r_2+\frac{r_2-2}{2}+1} = v^2_{q-1}v^1_{q-1}v^1_{q-2}v^2_{q-2}\; v^2_{q-3}v^1_{q-3}v^1_{q-4}v^2_{q-4}\cdots v^2_5v^1_5v^1_4v^2_4.$$

Otherwise, $W_{r_2+\frac{r_2-2}{2}+1}$ is skiped. We end walk $W_1W_2\cdots W_{r_2+\frac{r_2-2}{2}}(W_{r_2+\frac{r_2-2}{2}+1})$ with

$$W_{r_2+\frac{r_2-2}{2}+2} = v^2_3v^1_3v^1_2v^1_1.$$

So, in this subcase W is a closed walk representing a simple cycle of length $n-1$ in G that skips the vertex v^2_2. Hence, in this case G is hyper-Hamiltonian. Consequently, Cases 1-2 prove a sufficient condition for hyper-Hamiltonicity of G. $\square$

4. Hyper-Hamiltonian Circulant Digraphs

In this section, we first prove the necessary and sufficient conditions for a connected circulant digraphs with two jumps to be hyper-Hamiltonian. For convenience and better clarity in this section let $v_{i\pm j}$ denote vertex $i\pm j$ taken modulo n.

Theorem 4.1. *A connected circulant digraph $G_n(a_1,a_2)$ is hyper-Hamiltonian if and only if either $a_1\equiv 2a_2 \pmod n$ or $a_2\equiv 2a_1 \pmod n$.*

Proof. Without loss of generality, we first assume $a_2\equiv 2a_1 \pmod n$ for a sufficient condition. If $\gcd(n,a_1)>1$ then $\gcd(n,a_1,a_2)>1$ and by Theorem 2.4 G is disconnected. So, assume $\gcd(n,a_1)=1$. Then the isomorphism $i\to i+a_1 \pmod n$ of G induces circulant digraph $G'_n(1,2)\simeq G_n(a_1,a_2)$. Let $C_{G'}, C_{G'-v_1}$ denote the Hamilton cycles in G' and $G'-v_1$, respectively. Consequently, $C_{G'}=v_0v_1v_2\cdots v_{n-1}v_0$ and $C_{G'-v_1}=v_0v_2\cdots v_{n-1}v_0$, which proves our sufficient condition.

For a necessary condition, assume $a_1\neq 2a_2 \pmod n$, $a_2\neq 2a_1 \pmod n$, and G is hyper-Hamiltonian. Without loss of generality assume $H=G-v_{a_1}$ to have a Hamilton cycle C_H. Such a cycle C_H must be induced by both jumps. Then, H contains either path $Q_1=v_0v_{a_2}v_{a_2+a_1}v_{a_2+2a_1}$ or path $Q_2=v_{a_1-a_2}v_{2a_1-a_2}v_{2a_1}v_{2a_1+a_2}$.

Case 1. H contains Q_1.

If C_H is of the form $v_0v_{a_2}v_{a_2+a_1}v_{a_2+2a_1}\cdots v_{a_2+z_1a_1}$ where $a_2+z_1a_1\equiv 0 \pmod n$ then $z_1=n-2$, which implies $a_2\equiv 2a_1 \pmod n$ – a contradiction. Otherwise, let z_1 be the largest positive integer for which C_H contains path $P_0=v_0v_{a_2}v_{a_2+a_1}v_{a_2+2a_1}\cdots v_{a_2+z_1a_1}$. Then P_0 implies

$$P_1 = v_{a_2-a_1}v_{(a_2-a_1)+a_2}v_{(a_2-a_1)+a_2+a_1}v_{(a_2-a_1)+a_2+2a_1}\cdots v_{(a_2-a_1)+a_2+z_1a_1}$$

in C_H, since otherwise the vertices $v_{a_1}, v_{a_2-a_1}$ would not be visited in C_H. No successive path P_i in C_H can start with v_{a_1} because it would imply all (i.e., at least 3) vertices in P_i to be unvisited. Hence, by induction every path P_i in C_H must be of the following form

$$P_i = v_{i(a_2-a_1)}v_{i(a_2-a_1)+a_2}v_{i(a_2-a_1)+a_2+a_1}v_{i(a_2-a_1)+a_2+2a_1}\cdots v_{i(a_2-a_1)+a_2+z_1a_1},$$

consisting of exactly z_1 arcs of the form $(j, j + a_1 \pmod n)$. On the other hand, the unvisited vertex v_{a_1} and P_0 imply a path $P_x = v_{2a_1} v_{2a_1+a_1} v_{2a_1+2a_1} \cdots v_{2a_1+(z_1-1)a_1}$, with $z_1 - 1$ arcs of the form $(j, j + a_1 \pmod n)$ in C_H – a contradiction.

Case 2. H contains Q_2.

If C_H is of the form $v_{a_1-a_2} v_{2a_1-a_2} v_{2a_1} v_{2a_1+a_2} \cdots v_{2a_1+z_2a_2}$ where $2a_1 + z_2 a_2 \equiv a_1 - a_2 \pmod n$ then $z_2 = n-2$, which implies $2(a_1 - a_2) \equiv a_1 - a_2 \pmod n$ – a contradiction. Otherwise, let z_2 be the largest positive integer for which C_H contains path $P_0 = v_{a_1-a_2} v_{2a_1-a_2} v_{2a_1} v_{2a_1+a_2} \cdots v_{2a_1+z_2a_2}$. Then P_0 implies

$$P_1 = v_{(a_1-a_2)+a_1-a_2} v_{(a_1-a_2)+2a_1-a_2} v_{(a_1-a_2)+2a_1} v_{(a_1-a_2)+2a_1+a_2} \cdots v_{(a_1-a_2)+2a_1+z_2a_2}$$

in C_H, since otherwise the vertices $v_{a_1}, v_{2(a_1-a_2)}$ would not be visited in C_H. No successive path P_i in C_H can start with v_{a_1} because it would imply all vertices in P_i to be unvisited. Consequently by induction every P_i in C_H has to be of the following form

$$P_i = v_{i(a_1-a_2)+a_1-a_2} v_{i(a_1-a_2)+2a_1-a_2} v_{i(a_1-a_2)+2a_1} v_{i(a_1-a_2)+2a_1+a_2} \cdots v_{i(a_1-a_2)+2a_1+z_2a_2}$$

consisting of exactly $z_2 + 1$ arcs of the form $(j, j + a_2 \pmod n)$. On the other hand, the unvisited v_{a_1} and P_0 imply a path $P_y = v_{a_1+a_2} v_{a_1+2a_2} v_{a_1+3a_2} \cdots v_{a_1+(z_2+1)a_2}$, with z_2 arcs of the form $(j, j + a_2 \pmod n)$ in C_H – a contradiction.
Consequently, the contradictions of Cases 1-2 prove a necessary condition. □

For the general case of a circulant digraph $G = G_n(a_1, a_2, \ldots, a_k)$ the problem of deciding whether or not G is hyper-Hamiltonian is more challenging since it is an open problem in respect to which circulant digraphs are Hamiltonian in general. Hence, in the rest of this section we focus on some sufficient conditions for hyper-Hamiltonicity of G.

For given integers $s_1, s_2, \ldots, s_k$ define P_i as follows:

$$\begin{aligned} P_i = {} & v_{i+a_1} v_{i+2a_1} \cdots v_{i+s_1a_1} \\ & v_{i+s_1a_1+a_2} v_{i+s_1a_1+2a_2} \cdots v_{i+s_1a_1+s_2a_2} \\ & \cdots \\ & v_{i+s_1a_1+s_2a_2+\cdots+s_{k-1}a_{k-1}+a_k} v_{i+s_1a_1+s_2a_2+\cdots+s_{k-1}a_{k-1}+2a_k} \cdots v_{i+s_1a_1+s_2a_2+\cdots+s_{k-1}a_{k-1}+s_ka_k}. \end{aligned}$$

Let $C_G(s_1, s_2, \ldots, s_k)$ denote a Hamilton cycle in G of the form $P_0 P_c P_{2c} \cdots P_m \pmod n$ for some positive integers c, m, where $m + s_1a_1 + s_2a_2 + \cdots + s_ka_k \equiv 0 \pmod n$. The following theorem from [4] will be useful in obtaining some sufficient conditions for hyper-Hamiltonicity in circulant digraphs.

Theorem 4.2 ([4]). *Let $G_{rs}(a_1, a_2, \ldots, a_k)$ be a connected circulant digraph. Let s_i be integers such that $\frac{rs}{a_i} > s_i \geq 1$ and $s = s_1 + s_2 + \cdots + s_k$. Let t be a positive integer and $ts = \frac{rs}{\gcd(rs, aj)}$ for some a_j. If $\gcd(rs, s_1a_1 + s_2a_2 + \cdots + s_ka_k) = s$ and $a_i \equiv a_j \pmod s$ then $G_{rs}(a_1, a_2, \ldots, a_k)$ has a Hamilton cycle $C_G(s_1, s_2, \ldots, s_k)$.*

Since a Hamilton cycle is explicitly identified in Theorem 4.2, we can leverage that in determining sufficient conditions for hyper-Hamiltonicity in circulant digraphs as follows.

Theorem 4.3. *Let $G_{rs}(a_1, a_2, \ldots, a_k)$ be a connected circulant digraph. Let s_i be integers such that $\frac{rs}{a_i} > s_i \geq 1$ and $s = s_1 + s_2 + \cdots + s_k$. Let t be a positive integer such that $ts = \frac{rs}{\gcd(rs, aj)}$ for some a_j. $G_{rs}(a_1, a_2, \ldots, a_k)$ is hyper-Hamiltonian if the following conditions are satisfied:*

(1) $\gcd(rs, s_1a_1 + s_2a_2 + \cdots + s_ka_k) = s$,
(2) $a_i \equiv a_j \pmod{s}$,
(3) $a_i + a_j \equiv a_t \pmod{rs}$ *for at least one combination of i, j, t,*
or $a_i \equiv 2a_q \pmod{rs}$ *and* $s_q \geq 2$ *for some* i, q.

Proof. If (1) and (2) are satisfied then by Theorem 4.2 $G = G_{rs}(a_1, a_2, \ldots, a_k)$ has a Hamilton cycle of form $C_G(s_1, s_2, \ldots, s_k)$. Hence, it suffices to show that $G - v$ also has a Hamilton cycle for some arbitrary $v \in V(G)$ since G is vertex transitive. If $s_q \geq 2$ for some q, $k \geq q \geq 1$, then by definition corresponding $C_G = C_G(s_1, s_2, \ldots, s_k)$ contains two consecutive arcs a_q, i.e., $C_G = \cdots a_qa_q \cdots$. So, if there exists a_i in G such that $a_i \equiv 2a_q \pmod{rs}$ then a_qa_q in C_G can be substituted with a_i resulting in a Hamilton cycle in $G - v$. Otherwise, assume $a_i + a_j \equiv a_t \pmod{rs}$ for some combination of i, j, t. Let $(a'_1, a'_2, \ldots, a'_k)$ be any permutation of $(a_1, a_2, \ldots, a_k)$. If (1) and (2) are satisfied then $\gcd(rs, s'_1a'_1 + s'_2a'_2 + \cdots + s'_ka'_k) = s$ and $a'_i \equiv a'_j \pmod{s}$ are satisfied for some permutation $(s'_1, s'_2, \ldots, s'_k)$ of $(s_1, s_2, \ldots, s_k)$ too. Furthermore, according to Theorem 4.2 $C_G(s_1, s_2, \ldots, s_k)$ exists, which implies that $C_{G'}(s'_1, s'_2, \ldots, s'_k)$ also exists in $G' = G_{rs}(a'_1, a'_2, \ldots, a'_k)$. Hence, for $a_i + a_j \equiv a_t \pmod{rs}$ we conveniently choose $(a'_1, a'_2, \ldots, a'_k)$, so $a'_1 = a_i$, $a'_2 = a_j$, and $a'_k = a_t$. Consequently, $C_{G'}(s'_1, s'_2, \ldots, s'_k)$ induces a Hamilton cycle in $G' - v$ by substituting one occurrence of two arcs $a'_1a'_2$ with a single arc a'_k, which completes this proof. □

Finally, the sufficient condition for the hyper-Hamiltonicity of an arbitrary circulant digraph can be extended from circulant digraph with 2 jumps based on Theorem 4.1, as follows.

Theorem 4.4. *A connected circulant digraph $G_n(a_1, a_2, \ldots, a_k)$ is hyper-Hamiltonian if it contains two jumps a_i, a_j such that $a_i \equiv 2a_j \pmod{n}$ and $\gcd(n, a_j) = 1$.*

Proof. If $G = G_n(a_1, a_2, \ldots, a_k)$ contains a_j such that $\gcd(n, a_j) = 1$ then according to Theorem 2.4 our G contains a spanning connected circulant digraph $G'_n(a_i, a_j)$ for any a_i in G. Furthermore, if $a_i \equiv 2a_j \pmod{n}$ then $a_i \neq a_j$ and by Theorem 4.1 G is hyper-Hamiltonian. □

In closing, we note that $a_i \equiv 2a_j \pmod{n}$ imply hyper-Hamiltonicity of $G_n(a_1, a_2, \ldots, a_k)$ in Theorems 4.3-4.4 under different additional conditions. In particular, Theorem 4.4 also requires $\gcd(n, a_j) = 1$ while Theorem 4.3 does not require $\gcd(n, a_j) = 1$, but other conditions.

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