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Zeroth-order general Randić index of trees with given distance k-domination number

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Abstract

The zeroth-order general Randić index of a graph G is defined as $R_a(G) = \sum_{v\in V(G)} d_G^a(v)$, where $a \in \mathbb{R}$, $V(G)$ is the vertex set of G and $d_G(v)$ is the degree of a vertex v in G. We obtain bounds on the zeroth-order general Randić index for trees of given order and distance k-domination number, where $k \geq 1$. Lower bounds are given for $0 < a < 1$ and upper bounds are given for $a < 0$ and $a > 1$. All the extremal graphs are presented which means that our bounds are the best possible.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph G is called the order of G. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident with v. A pendant vertex is a vertex having degree one. The distance $d_G(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path connecting them. The diameter of G is the

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distance between any two furthest vertices in G. A diametral path is a shortest path in G connecting two furthest vertices in G. A tree is a connected graph containing no cycles.

For a positive integer k, a set $D \subset V(G)$ is a distance k-dominating set of G if every vertex in G is of distance at most k from some vertex in D. The cardinality of a smallest distance k-dominating set of G is called the distance k-domination number of G, denoted by $\gamma_k(G)$ (or γ_k). We say that $v \in V(G)$ is dominated by $u \in D$ if $d_G(u, v) \leq k$. We also say that the vertex u dominates the vertex v. Note that the distance 1-domination number is the classical domination number of G.

For $a \in \mathbb{R}$, the zeroth-order general Randić index of a graph G is defined as

$$R_a(G) = \sum_{v \in V(G)} d_G^a(v).$$

Some other important indices are special cases of the R_a index. If $a = 2$, we get the first Zagreb index and if $a = -\frac{1}{2}$, we obtain the classical zeroth-order Randić index.

Indices of graphs have a lot of applications, especially in chemistry. Mathematical properties of the zeroth-order general Randić index have been extensively studied. The minimum and maximum zeroth-order general Randić index for tress of prescribed order and number of leaves/branching vertices/segments were obtained by Khalid and Ali [6], the maximum R_a index of trees in terms of order and diameter/radius was presented by Yamaguchi [14], an upper bound for trees with respect to order and independence number was given by Tomescu and Jamil [11], upper and lower bounds for trees with prescribed order and independence number/domination number were given in [12], bounds for graphs of given order and size were obtained by Milošević et al. [8], sharp upper and lower bounds for bicyclic graphs of given order were presented by Chen and Deng [3], tight bounds for cactus graphs were given by Ahmed, Bhatti and Ali [1], graphs of given chromatic number, clique number, connectivity and number of cut edges were studied by Jamil et al. [5]. Some related indices were studied in [2], [4] and [10].

We obtain sharp bounds on the zeroth-order general Randić index for trees of given order and distance k-domination number. Bounds on the zeroth-order general Randić index for trees of given order and classical domination number, and bounds on the first Zagreb index for trees of given order and distance k-domination number are special cases of our results.

2. Preliminary results

Results given in this section are used in the proofs of our theorems presented in Section 3. Lemma 2.1 was proved in [13].

Lemma 2.1. *Let $a, x, y, c \in \mathbb{R}$, where $1 \leq x < y$ and $c > 0$. For $0 < a < 1$, we have*

$$(x + c)^a - x^a > (y + c)^a - y^a.$$

For $a > 1$ and $a < 0$, we have

$$(x + c)^a - x^a < (y + c)^a - y^a.$$

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For a tree T with a pendant vertex $u \in V(T)$, we denote by $T - u$ a tree obtained by the removal of u and the edge incident with u from T. Let A_T be the set containing each pendant vertex u of T for which $\gamma_k(T-u) = \gamma_k(T)$. So, a pendant vertex $u \in A_T$ if $\gamma_k(T-u) = \gamma_k(T)$ and $u \notin A_T$ if $\gamma_k(T-u) = \gamma_k(T) - 1$. Let $N_T(A_T) = \bigcup_{u \in A_T} N_T(u)$, where $N_T(u)$ is the set containing the vertex adjacent to u in T.

Lemma 2.2. *Let T' be a tree with the maximum zeroth-order general Randić index for $a < 0$ and $a > 1$ (the minimum zeroth-order general Randić index for $0 < a < 1$) among trees of order n and distance k-domination number $\gamma_k(T') \geq 2$, where $k \geq 1$. Then $|N_{T'}(A_{T'})| \leq 1$.*

Proof. Suppose to the contrary that $|N_{T'}(A_{T'})| \geq 2$; say $v, v' \in N_{T'}(A_{T'})$. Without loss of generality, assume that $r = d_{T'}(v) \leq d_{T'}(v') = s$. Let us denote any pendant vertex adjacent to v in T' by u (adjacent to v' by u'), where $u, u' \in A_{T'}$.

Let D be any distance k-dominating set of T' which does not contain u. We define T'' with $E(T'') = \{uv'\} \cup E(T') \setminus \{uv\}$. Since $\gamma_k(T'-u) = \gamma_k(T')$, we get $\gamma_k(T'') = \gamma_k(T')$.

We obtain $d_{T''}(v) = r - 1$, $d_{T''}(v') = s + 1$ and

$$\begin{aligned} R_a(T') - R_a(T'') &= d^a_{T'}(v) - d^a_{T''}(v) + d^a_{T'}(v') - d^a_{T''}(v') \\ &= r^a - (r-1)^a + s^a - (s+1)^a. \end{aligned}$$

For $a < 0$ and $a > 1$, by Lemma 2.1, we have $r^a - (r-1)^a < (s+1)^a - s^a$, thus $r^a - (r-1)^a + s^a - (s+1)^a < 0$ which implies that $R_a(T') < R_a(T'')$, so T' does not have the maximum R_a index.

For $0 < a < 1$, by Lemma 2.1, we have $r^a - (r-1)^a > (s+1)^a - s^a$, thus $r^a - (r-1)^a + s^a - (s+1)^a > 0$ and $R_a(T') > R_a(T'')$, so T' does not have the minimum R_a index, which is a contradiction. □

Pei and Pan [9] (who presented bounds for the Zagreb indices) proved that for any tree with n vertices, maximum degree Δ and distance k-domination number $\gamma_k \geq 2$, where $k \geq 1$, we have

$$\Delta \leq n - k\gamma_k. \tag{1}$$

3. Main results

For positive integers n and k with $n \geq 2(k+1)$, let $T^j_n(k,2)$ be the tree obtained from the path $v_0 v_1 \ldots v_{2k+1}$ by joining v_j to $n - 2(k+1)$ pendant vertices, where $j \in \{1, 2, \ldots, 2k\}$. Let $\mathbb{T}_n(k,2)$ be the set containing the trees $T^1_n(k,2), T^2_n(k,2), \ldots, T^{2k}_n(k,2)$. Since $T^i_n(k,2)$ and $T^{2k-i}_n(k,2)$ are isomorphic for every $i = 1, 2, \ldots, k-1$, we have

$$\mathbb{T}_n(k,2) = \{T^1_n(k,2), T^2_n(k,2), \ldots, T^k_n(k,2)\}.$$

Every tree in $\mathbb{T}_n(k,2)$ has one vertex of degree $n - 2k$, $2k - 1$ vertices of degree 2 and $n - 2k$ vertices of degree 1. Therefore,

$$R_a(T^i_n(k,2)) = (n-2k)^a + (2k-1)2^a + n - 2k.$$

for each $i = 1, 2, \ldots, k$. We show that any tree in the set $\mathbb{T}_n(k,2)$ is a tree with the maximum R_a index for $a < 0$ and $a > 1$ (the minimum R_a index for $0 < a < 1$) among trees of order n and distance k-domination number 2.

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Theorem 3.1. *Let T be any tree with n vertices and distance k-domination number 2, where $k \geq 1$. For $a < 0$ and $a > 1$, we have $n \geq 2(k+1)$ and*

$$R_a(T) \leq (n-2k)^a + (2k-1)2^a + n - 2k$$

with equality if and only if T is in $\mathbb{T}_n(k, 2)$.

Proof. Let T' be a tree with the maximum R_a index among trees of order n and distance k-domination number 2.

Let $v_0v_1 \dots v_d$ be a diametral path of T'. Note that $d \geq 2k+1$, otherwise if $d \leq 2k$, then T' can be dominated by its central vertex $v_{\lfloor \frac{d}{2} \rfloor}$. By Lemma 2.2, we have $|N_{T'}(A_{T'})| \leq 1$. Therefore, $v_0 \notin A_{T'}$ or $v_d \notin A_{T'}$. Without loss of generality, assume that $v_0 \notin A_{T'}$ which means that $\gamma_k(T' - v_0) = \gamma_k(T') - 1 = 1$. This implies that the diameter of $T' - v_0$ is at most $2k$, therefore the diameter of T' is at most $2k+1$ which yields $d = 2k+1$ and $n \geq 2(k+1)$.

It follows that if $n = 2(k+1)$, then T' is the path $v_0v_1 \dots v_{2k+1}$ which is the only tree in the set $\mathbb{T}_{2(k+1)}(k, 2)$ and

$$R_a(T') = (2k)2^a + 2.$$

Thus, Theorem 3.1 holds for $n = 2(k+1)$. For $n > 2(k+1)$, we prove Theorem 3.1 by induction on n.

Assume that $n > 2(k+1)$ and the result holds for $n-1$. Thus T' contains a pendant vertex, say w other than v_0 and v_{2k+1}. Note that $A_{T'} \neq \emptyset$, otherwise if $A_{T'} = \emptyset$, then $\gamma_k(T' - w) = \gamma_k(T') - 1 = 1$ which is not possible, since v_0 and v_{2k+1} cannot be dominated by one vertex. Since $A_{T'} \neq \emptyset$, we get $N_{T'}(A_{T'}) \neq \emptyset$.

Therefore, by Lemma 2.2, we obtain $|N_{T'}(A_{T'})| = 1$. Let v be the vertex in $N_{T'}(A_{T'})$ and let $u \in A_{T'}$. So $\gamma_k(T' - u) = \gamma_k(T') = 2$.

Then by (1), we get $d_{T'}(v) \leq n - 2k$. Thus by Lemma 2.1, for $a < 0$ and $a > 1$, we obtain

$$d^a_{T'}(v) - (d_{T'}(v) - 1)^a \leq (n-2k)^a - (n-2k-1)^a.$$

The equality holds if and only if $d_{T'}(v) = n - 2k$. Note that

$$R_a(T') - R_a(T' - u) = d^a_{T'}(v) - (d_{T'}(v) - 1)^a + d^a_{T'}(u).$$

By the induction hypothesis,

$$R_a(T' - u) \leq (n-2k-1)^a + (2k-1)2^a + n - 2k - 1$$

with equality if and only if T is in $\mathbb{T}_{n-1}(k, 2)$. Thus

$$\begin{aligned}
R_a(T') &= R_a(T'-u) + d^a_{T'}(v) - (d_{T'}(v) - 1)^a + d^a_{T'}(u) \\
&\leq (n-2k-1)^a + (2k-1)2^a + n - 2k - 1 + (n-2k)^a - (n-2k-1)^a + 1 \\
&= (n-2k)^a + (2k-1)2^a + n - 2k
\end{aligned}$$

with equality if and only if $T' - u$ is any tree in $\mathbb{T}_{n-1}(k, 2)$ and $d_{T'}(v) = n - 2k$. Hence, T' is any tree in $\mathbb{T}_n(k, 2)$. $\square$

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Theorem 3.2. *Let T be any tree with n vertices and distance k-domination number 2, where $k \geq 1$. For $0 < a < 1$, we have $n \geq 2(k+1)$ and*

$$R_a(T) \geq (n-2k)^a + (2k-1)2^a + n - 2k$$

with equality if and only if T is in $\mathbb{T}_n(k, 2)$.

Proof. The proofs of Theorems 3.1 and 3.2 are similar. We present only those parts of the proof of Theorem 3.2 which are different from the proof of Theorem 3.1.

Let T' be a tree with the minimum R_a index among trees of order n and distance k-domination number 2.

By Lemma 2.1, for $0 < a < 1$, we obtain

$$d_{T'}^a(v) - (d_{T'}(v) - 1)^a \geq (n-2k)^a - (n-2k-1)^a.$$

By the induction hypothesis,

$$R_a(T' - u) \geq (n-2k-1)^a + (2k-1)2^a + n - 2k - 1.$$

Thus

$$\begin{aligned} R_a(T') &= R_a(T'-u) + d_{T'}^a(v) - (d_{T'}(v)-1)^a + d_{T'}^a(u) \\ &\geq (n-2k-1)^a + (2k-1)2^a + n - 2k - 1 + (n-2k)^a - (n-2k-1)^a + 1 \\ &= (n-2k)^a + (2k-1)2^a + n - 2k. \end{aligned}$$

□

Let T be a tree with a diametral path $v_0 v_1 \dots v_d$ and let T_i be the subtree of T induced by v_i and the vertices which are closer to v_i than to all the other vertices in $v_0 v_1 \dots v_d$, where $i = 0, 1, \dots, d$. Note that T_0 contains only v_0 and no edge, and T_d contains only v_d and no edge. For $i = 1, 2, \dots, d-1$, T_i is the component of F containing v_i, where $V(F) = V(T)$ and $E(F) = E(T) \setminus \{v_{i-1}v_i, v_i v_{i+1}\}$.

For integers n, k, γ_k such that $k \geq 1$, $\gamma_k \geq 3$ and $n \geq (k+1)\gamma_k$, let $T_n(k, \gamma_k)$ be a tree containing a vertex of degree $n - k\gamma_k$ which is attached to one path of length k, $\gamma_k - 1$ paths of length $k+1$ and $n - (k+1)\gamma_k$ paths of length 1. The tree $T_n(k, \gamma_k)$ has one vertex of degree $n - k\gamma_k$, $k\gamma_k - 1$ vertices of degree 2 and $n - k\gamma_k$ vertices of degree 1, thus

$$R_a(T_n(k, \gamma_k)) = (n - k\gamma_k)^a + (k\gamma_k - 1)2^a + n - k\gamma_k.$$

We show that $T_n(k, \gamma_k)$ is the tree with the maximum R_a index for $a < 0$ and $a > 1$ (the minimum R_a index for $0 < a < 1$) among trees of order n and distance k-domination number $\gamma_k \geq 3$. First, we study the case $\gamma_k = 3$.

Theorem 3.3. *Let T be any tree with n vertices and distance k-domination number 3, where $k \geq 1$. For $a < 0$ and $a > 1$, we have $n \geq 3(k+1)$ and*

$$R_a(T) \leq (n-3k)^a + (3k-1)2^a + n - 3k$$

with equality if and only if T is $T_n(k, 3)$.

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Proof. Let T' be a tree with the maximum R_a index among trees of order n and distance k-domination number 3.

Let $v_0v_1\dots v_d$ be a diametral path of T'. Note that $d \geq 2k+2$, otherwise T' can be dominated by its one or two central vertices. We can include the vertices v_k and v_{d-k} in a distance k-dominating set D of T' having cardinality $\gamma_k(T') = 3$, where no vertex of $\bigcup_{j=0}^{k} V(T'_j) \setminus \{v_k\}$ and no vertex of $\bigcup_{j=d-k}^{d} V(T'_j) \setminus \{v_{d-k}\}$ is in D, since v_k dominates all the vertices in $\bigcup_{j=0}^{k} V(T'_j)$ and v_{d-k} dominates all the vertices in $\bigcup_{j=d-k}^{d} V(T'_j)$.

First, we solve Theorem 3.3 for $n \leq 3(k+1)$. Since both $\bigcup_{j=0}^{k} V(T'_j)$ and $\bigcup_{j=d-k}^{d} V(T'_j)$ contain at least $k+1$ vertices, $\bigcup_{j=k+1}^{d-k-1} V(T'_j)$ contains at most $k+1$ vertices. It is easy to check that the only tree satisfying this condition not dominated by 2 vertices v_k and v_{d-k} is $T_{3(k+1)}(k,3)$ which is the tree of order $3(k+1)$ and diameter $2k+2$, where T'_{k+1} is the path of length k. Hence, $n \geq 3(k+1)$ and the result holds for $n = 3(k+1)$.

For $n > 3(k+1)$, we prove Theorem 3.3 by induction on n. Assume that $n > 3(k+1)$ and the result holds for $n-1$.

Claim 1: $A_{T'} \neq \emptyset$.

Suppose to the contrary that $A_{T'} = \emptyset$. Then $\gamma_k(T'-u) = \gamma_k(T') - 1$ for every pendant vertex u of T'. We have $d_{T'}(v_i) = 2$ for each $i \in \{1,2,\dots,k\} \cup \{d-k, d-k+1, \dots, d-1\}$, otherwise if $d_{T'}(v_i) \geq 3$ for some $i \in \{1,2,\dots,k\} \cup \{d-k, d-k+1,\dots,d-1\}$, then $V(T'_i)$ contains a pendant vertex, say w, such that $\gamma_k(T'-w) = \gamma_k(T')$, since all the vertices in D are necessary also for the tree $T'-w$.

Note that $\gamma_k(T'-v_0) = \gamma_k(T') - 1$. This means that v_k is in D only because of v_0. We can include the vertices v_{k+1} and v_{d-k} in a distance k-dominating set D' of $T'-v_0$ having cardinality $\gamma_k(T') - 1$, where no vertex of $\bigcup_{j=1}^{k} V(T'_j)$ and no vertex of $\bigcup_{j=d-k}^{d} V(T'_j) \setminus \{v_{d-k}\}$ is in D'. It follows that $D' \cup \{v_k\}$ is a distance k-dominating set of T', therefore we can include v_{k+1} in D. Similarly, we can show that we can include v_{d-k-1} in D.

It follows that $d = 2k+2$, otherwise if $d > 2k+2$, then $k+1 < d-k-1$ and $v_{k+1} \neq v_{d-k+1}$, which means that $\gamma_k(T') \geq 4$. So $D = \{v_k, v_{k+1}, v_{d-k}\}$. We have $d_{T'}(v_{k+1}) \geq 3$, otherwise if $d_{T'}(v_{k+1}) = 2$, then T' is the path $v_0v_1\dots v_{2k+2}$, which has distance k-domination number 2.

Moreover, the tree T'_{k+1} contains one pendant vertex, otherwise if T'_{k+1} contains (at least) two pendant vertices, say u_1 and u_2, such that $\gamma_k(T'-u_i) = \gamma_k(T')-1$ for $i = 1,2$, then $d_{T'}(u_i, v_{k+1}) = k$ for $i = 1,2$, which implies that the vertices v_0, v_d, u_2 cannot be dominated by two vertices in $T'-u_1$. So, T'_{k+1} is a path of length at most k, which gives $n \leq 3(k+1)$, which is a contradiction. Hence, Claim 1 is proved.

Since $A_{T'} \neq \emptyset$, we obtain $N_{T'}(A_{T'}) \neq \emptyset$. Therefore, by Lemma 2.2, we have $|N_{T'}(A_{T'})| = 1$. Let v be the vertex in $N_{T'}(A_{T'})$ and let $u \in A_{T'}$. So $\gamma_k(T'-u) = \gamma_k(T') = 3$.

Then by (1), we get $d_{T'}(v) \leq n-3k$. Thus by Lemma 2.1, for $a<0$ and $a>1$, we obtain

$$d^a_{T'}(v) - (d_{T'}(v)-1)^a \leq (n-3k)^a - (n-3k-1)^a.$$

The equality holds if and only if $d_{T'}(v) = n-3k$. Note that

$$R_a(T') - R_a(T'-u) = d^a_{T'}(v) - (d_{T'}(v)-1)^a + d^a_{T'}(u).$$

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By the induction hypothesis,

$$R_a(T'-u) \leq (n-3k-1)^a + (3k-1)2^a + n - 3k - 1$$

with equality if and only if $T'-u$ is $T_{n-1}(k,3)$. Thus

$$\begin{aligned} R_a(T') &= R_a(T'-u) + d^a_{T'}(v) - (d_{T'}(v)-1)^a + d^a_{T'}(u) \\ &\leq (n-3k-1)^a + (3k-1)2^a + n - 3k - 1 + (n-3k)^a - (n-3k-1)^a + 1 \\ &= (n-3k)^a + (3k-1)2^a + n - 3k \end{aligned}$$

with equality if and only if $T'-u$ is $T_{n-1}(k,3)$ and $d_{T'}(v) = n-3k$. Hence, T' is $T_n(k,3)$. □

Theorem 3.4. *Let T be any tree with n vertices and distance k-domination number 3, where $k \geq 1$. For $0 < a < 1$, we have $n \geq 3(k+1)$ and*

$$R_a(T) \geq (n-3k)^a + (3k-1)2^a + n - 3k$$

with equality if and only if T is $T_n(k,3)$.

Proof. We present the parts of the proof of Theorem 3.4 which are different from the proof of Theorem 3.3.

Let T' be a tree with the minimum R_a index among trees of order n and distance k-domination number 3. By Lemma 2.1,

$$d^a_{T'}(v) - (d_{T'}(v)-1)^a \geq (n-3k)^a - (n-3k-1)^a$$

for $0 < a < 1$. By the induction hypothesis,

$$R_a(T'-u) \geq (n-3k-1)^a + (3k-1)2^a + n - 3k - 1.$$

Therefore,

$$\begin{aligned} R_a(T') &= R_a(T'-u) + d^a_{T'}(v) - (d_{T'}(v)-1)^a + d^a_{T'}(u) \\ &\geq (n-3k-1)^a + (3k-1)2^a + n - 3k - 1 + (n-3k)^a - (n-3k-1)^a + 1 \\ &= (n-3k)^a + (3k-1)2^a + n - 3k. \end{aligned}$$

□

In Theorems 3.5 and 3.6, we obtain bounds on the R_a index for trees of given order and distance k-domination number $\gamma_k \geq 3$.

Theorem 3.5. *Let T be any tree with n vertices and distance k-domination number $\gamma_k \geq 3$, where $k \geq 1$. For $a < 0$ and $a > 1$, we have $n \geq \gamma_k(k+1)$ and*

$$R_a(T) \leq (n-k\gamma_k)^a + (k\gamma_k - 1)2^a + n - k\gamma_k$$

with equality if and only if T is $T_n(k,\gamma_k)$.

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Proof. Let T' be a tree with the maximum R_a index among trees with n vertices and distance k-domination number $\gamma_k(T') \geq 3$. Let $v_0v_1 \dots v_d$ be a diametral path of T'. Note that $d \geq 2k+2$, otherwise T' can be dominated by its one or two central vertices. We can include the vertices v_k and v_{d-k} in a distance k-dominating set D of T' having cardinality $\gamma_k(T')$, where no vertex of $\bigcup_{j=0}^{k} V(T'_j) \setminus \{v_k\}$ and no vertex of $\bigcup_{j=d-k}^{d} V(T'_j) \setminus \{v_{d-k}\}$ is in D, since v_k dominates all the vertices in $\bigcup_{j=0}^{k} V(T'_j)$ and v_{d-k} dominates all the vertices in $\bigcup_{j=d-k}^{d} V(T'_j)$.

By Lemma 2.2, we have $|N_{T'}(A_{T'})| \leq 1$. Therefore, if $N_{T'}(A_{T'}) \neq \emptyset$, then the vertex of $N_{T'}(A_{T'})$ is not in the set $\bigcup_{j=0}^{k} V(T'_j)$ or it is not in the set $\bigcup_{j=d-k}^{d} V(T'_j)$. Without loss of generality, assume that $\bigcup_{j=0}^{k} V(T'_j) \cap N_{T'}(A_{T'}) = \emptyset$. Obviously, this holds also if $N_{T'}(A_{T'}) = \emptyset$. Since the set $A_{T'}$ can contain only pendant vertices, it follows that $\bigcup_{j=0}^{k} V(T'_j) \cap A_{T'} = \emptyset$.

So $v_0 \notin A_{T'}$ and thus $\gamma_k(T' - v_0) = \gamma_k(T') - 1$. This means that v_k is in D only because of v_0. We can include the vertices v_{k+1} and v_{d-k} in a distance k-dominating set D' of $T' - v_0$ having cardinality $\gamma_k(T') - 1$, where no vertex of $\bigcup_{j=1}^{k} V(T'_j)$ and no vertex of $\bigcup_{j=d-k}^{d} V(T'_j) \setminus \{v_{d-k}\}$ is in D'. It follows that $D' \cup \{v_k\}$ is a distance k-dominating set of T', therefore we can include v_{k+1} in D. Thus, we have $\{v_k, v_{k+1}, v_{d-k}\} \subset D$.

For $i = 1, 2, \dots, k$, we have $d_{T'}(v_i) = 2$, otherwise if $d_{T'}(v_i) \geq 3$, then T'_i contains a pendant vertex, say u, such that $u \notin A_{T'}$, which means that $\gamma_k(T'-u) = \gamma_k(T')-1$ and that is not possible, since all the vertices in D are needed also for $T' - u$.

Let T_0 be the tree obtained from T' by the removal of the vertices $v_0, v_1, \dots, v_k$ (which means that the edges $v_0v_1, v_1v_2, \dots, v_kv_{k+1}$ are also removed). Since the only vertex dominated only by v_k in T' is v_0, the set $D \setminus \{v_k\}$ is a distance k-dominating set of T_0. Thus $\gamma_k(T_0) \leq \gamma_k(T') - 1$. Note that the purpose of the vertices of $D \setminus \{v_k\}$ in T' is to dominate the vertices in T_0, since $v_0, v_1, \dots, v_k$ are dominated by v_k in T'. Therefore, $\gamma_k(T') - 1$ vertices are necessary also for a minimum distance k-dominating set of T_0. Thus $\gamma_k(T_0) = \gamma_k(T') - 1$.

Using this equality, we prove Theorem 3.5 by induction on $\gamma_k(T') = \gamma_k$. Theorem 3.5 holds for $\gamma_k = 3$ by Theorem 3.3. Assume that $\gamma_k \geq 4$ and Theorem 3.5 holds for $\gamma_k - 1$.

The tree T_0 has order $n(T_0) = n-k-1$ and $\gamma_k(T_0) = \gamma_k - 1$. Note that $n-k-1 = |V(T_0)| \geq \gamma_k(T_0)(k+1) = (\gamma_k - 1)(k+1)$ is equivalent to $n \geq \gamma_k(k+1)$. By the induction hypothesis, Theorem 3.5 holds for T_0, so

$$R_a(T_0) \leq (n-k-1-k(\gamma_k - 1))^a + (k(\gamma_k - 1) - 1)2^a + n - k - 1 - k(\gamma_k - 1)$$

with equality if and only if T_0 is $T_{n-k-1}(k, \gamma_k - 1)$.

By (1), every vertex of T' has degree at most $n - k\gamma_k$. so $d_{T'}(v_{k+1}) \leq n - k\gamma_k$. Thus by Lemma 2.1, for $a < 0$ and $a > 1$, we obtain

$$d^a_{T'}(v_{k+1}) - (d_{T'}(v_{k+1}) - 1)^a \leq (n - k\gamma_k)^a - (n - k\gamma_k - 1)^a.$$

The equality holds if and only if $d^a_{T'}(v_{k+1}) = n - k\gamma_k$. Note that

$$R_a(T') - R_a(T_0) = \sum_{i=0}^{k} d^a_{T'}(v_i) + d^a_{T'}(v_{k+1}) - (d_{T'}(v_{k+1}) - 1)^a.$$

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Thus

$$\begin{aligned}
R_a(T') &= R_a(T_0) + \sum_{i=0}^{k} d^a_{T'}(v_i) + d^a_{T'}(v_{k+1}) - (d_{T'}(v_{k+1}) - 1)^a \\
&\leq (n-k-1-k(\gamma_k-1))^a + (k(\gamma_k-1)-1)2^a + n-k-1-k(\gamma_k-1) \\
&\quad +k2^a + 1 + (n-k\gamma_k)^a - (n-k\gamma_k-1)^a \\
&= (n-k\gamma_k)^a + (k\gamma_k-1)2^a + n - k\gamma_k,
\end{aligned}$$

with equality if and only if T_0 is $T_{n-k-1}(k, \gamma_k - 1)$ and $d_{T'}(v_{k+1}) = n - k\gamma_k$, which implies that T' is $T_n(k, \gamma_k)$. □

Theorem 3.6. *Let T be any tree with n vertices and distance k-domination number $\gamma_k \geq 3$, where $k \geq 1$. For $0 < a < 1$, we have $n \geq \gamma_k(k+1)$ and*

$$R_a(T) \geq (n-k\gamma_k)^a + (k\gamma_k-1)2^a + n - k\gamma_k$$

with equality if and only if T is $T_n(k, \gamma_k)$.

Proof. We present the parts of the proof of Theorem 3.6 which are different from the proof of Theorem 3.5.

Let T' be a tree with the minimum R_a index among trees with n vertices and distance k-domination number $\gamma_k(T') \geq 3$.

By the induction hypothesis,

$$R_a(T_0) \geq (n-k-1-k(\gamma_k-1))^a + (k(\gamma_k-1)-1)2^a + n-k-1-k(\gamma_k-1).$$

By Lemma 2.1, for $0 < a < 1$,

$$d^a_{T'}(v_{k+1}) - (d_{T'}(v_{k+1}) - 1)^a \geq (n-k\gamma_k)^a - (n-k\gamma_k-1)^a.$$

Therefore

$$\begin{aligned}
R_a(T') &= R_a(T_0) + \sum_{i=0}^{k} d^a_{T'}(v_i) + d^a_{T'}(v_{k+1}) - (d_{T'}(v_{k+1}) - 1)^a \\
&\geq (n-k-1-k(\gamma_k-1))^a + (k(\gamma_k-1)-1)2^a + n-k-1-k(\gamma_k-1) \\
&\quad +k2^a + 1 + (n-k\gamma_k)^a - (n-k\gamma_k-1)^a \\
&= (n-k\gamma_k)^a + (k\gamma_k-1)2^a + n - k\gamma_k.
\end{aligned}$$

□

4. Conclusion

We presented bounds on the zeroth-order general Randić index for trees of given order and distance k-domination number $\gamma_k \geq 2$, where $k \geq 1$. Lower bounds were given for $0 < a < 1$ and upper bounds were given for $a < 0$ and $a > 1$. All the extremal graphs were also presented.

Let us note that the problem is easy if $\gamma_k = 1$. From [7] it follows that the star S_n is the tree having the maximum R_a index for $a < 0$ and $a > 1$ (the minimum R_a index for $0 < a < 1$) among trees of order n. Obviously, S_n has distance k-domination number 1. Therefore, if T is a tree with n vertices and distance k-domination number 1, then

$$R_a(T) \leq (n-1)^a + (n-1)$$

for $a < 0$ and $a > 1$, and

$$R_a(T) \geq (n-1)^a + (n-1)$$

for $0 < a < 1$, with equalities if and only if T is S_n.

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