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The geodetic-dominating number of comb product graphs

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Abstract

A set of vertices S is called a *geodetic-dominating set* of G if every vertex outside S is adjacent to a vertex in S, and also is located inside a shortest path between two vertices in S. The *geodetic-dominating number* of G is the minimum cardinality of geodetic-dominating sets of G. In this paper, we determine an exact value of the geodetic-dominating number of comb product graphs of any connected graphs of order at least two.

Keywords: comb product, domination number, geodetic-dominating number, geodetic number
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1. Introduction

In this paper, all graphs are assumed to be connected, finite, simple, and undirected. Let G be a graph. For a vertex $z \in V(G)$, we recall that the *open neighborhood* and the *closed neighborhood* of z in G is defined as $N_G(z) = \{w \in V(G) \mid zw \in E(G)\}$ and $N_G[z] = N_G(z) \cup \{z\}$, respectively. A set $D \subseteq V(G)$ is called a *dominating set* if $N_G[D] = V(G)$. The *domination number* of G is the minimum cardinality of dominating sets of G. This concept provides several applications especially in protection strategies and business networking [10]. Interested readers are referred to a number of relevant literature mentioned in the references, including [16, 24].

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There are several modifications on domination concept in graph. Some of them are locating-dominating set [2, 6, 19, 23], independent dominating set [4, 14], Roman dominating set [9, 13]. In this paper, we are interested to study another variant of domination in graph, namely geodetic-dominating set.

A *walk* in G is a finite non-empty sequence $W = v_0e_1v_1e_2v_2...e_kv_k$ where for $1 \leq j \leq k$, v_j is a vertex and for $1 \leq i \leq k$, e_i is an edge where v_{i-1} and v_i are its end points. We can say that W is a $v_0 - v_k$ walk. A walk W is called a *trail* in case all edges of W are different. If all vertices of a trail W are also different, then W is called a *path*. The *distance* between vertices $a, b \in V(G)$, denoted by $d_G(a, b)$, is the minimum number of edges of $a - b$ paths in G. An $a - b$ path with $d_G(a, b)$ edges is called an $a - b$ *geodesic*. We denote $I_G[a, b]$ as the set of vertices which are located inside some $a - b$ geodesics of G. For a non-empty set $B \subseteq V(G)$, we define $I_G[B] = \bigcup_{a,b \in B} I_G[a, b]$. The set B then we called as a *geodetic set* of G in case $I_G[B] = V(G)$. The minimum cardinality of geodetic sets of G is called as the *geodetic number* of G, denoted by $g(G)$. For references on geodetic number in graphs, see [3, 5].

In this paper, let a set $B \subseteq V(G)$ be both geodetic and dominating in G. The set B then we call as a *geodetic-dominating set* of G. The *geodetic-dominating number* of G, denoted by $\gamma_g(G)$, is the minimum cardinality of geodetic-dominating sets of G.

This topic was firstly introduced by Escuadro *et al.* [12]. They proved that for a connected graph G or order at least $n \geq 2$, $\max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n$. They also characterized all graphs of order $n \geq 2$ with geodetic-dominating number 2, n, and $n - 1$. Some authors consider this topic to certain classes of graph. Hansberg and Volkmann [15] have shown that the geodetic-dominating problem for chordal graphs is NP-complete. Meanwhile the geodetic-dominating number of tree graphs and triangle-free graphs, can be seen in [12]. Some other references on geodetic-dominating number in graphs, see [7, 8, 18].

In this paper, we are interested to apply the geodetic-dominating concept to a product graphs. In this paper, we consider the *comb product* of connected graphs G and H. In chemistry [1], some classes of chemical graphs can be considered as the comb product graphs. The *comb product* of connected graphs G and H at vertex $o \in V(H)$, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the *i*-th copy of H at the vertex o to the *i*-th vertex of G. The vertex $o \in V(H)$ then we call as the *identifying vertex*. This product graphs have been widely investigated in many areas, including metric distance problems [11, 21, 22] and graph labeling problems [17, 20].

In this paper, we use some definitions in order to determine the geodetic-dominating number of $G \rhd_o H$. Let $V(G) = \{g_1, g_2, \ldots, g_n\}$ and $V(H) = \{h_1, h_2, \ldots, h_m\}$. For the identifying vertex $o \in V(H)$, we also define $K_o = G \rhd_o H$, $V(K_o) = \{(g_i, h_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, $V_0 = \{(g_l, o) | 1 \leq l \leq n\}$, and for $l \in \{1, 2, \ldots, n\}$, $V_l = \{(g_l, h_f) | 1 \leq f \leq m\}$. For $S \subseteq V(G)$, we also use the notation $G[S]$ which is a maximal subgraph of G induced by all vertices of S.

2. Geodetic-domination number of comb product graphs

In two lemmas below, we provide some properties of a dominating set and a geodetic set in two isomorphic graphs.

Lemma 2.1. *Let $\theta : V(A) \to V(B)$ be an isomorphism between graphs A and B. The set S is a dominating set of A if and only if $\{\theta(x)|x \in S\}$ is a dominating set of B.*

Proof. Let $x, y \in V(A)$. Thus by isomorphism $\theta(x), \theta(y) \in V(B)$. We define $T \subseteq V(B)$ such that $T = \{\theta(x)|x \in S\}$. Note that x and y are adjacent in A if and only if $\theta(x)$ and $\theta(y)$ are adjacent in B. Therefore, $N_B[\theta(x)] = \{\theta(y)|y \in N_A[x]\}$ and $N_A[x] = \{y|\theta(y) \in N_B[\theta(x)]\}$.

If S dominates A, then we obtain

$$\begin{aligned} N_B[T] &= \bigcup_{t\in T} N_B[t] = \bigcup_{t\in\{\theta(s):s\in S\}} N_B[t] = \bigcup_{s\in S} N_B[\theta(s)] \\ &= \{\theta(s)|s \in N_A[S]\} = \{\theta(s)|s \in A\} = B. \end{aligned}$$

If T dominates B, then we obtain

$$\begin{aligned} N_A[S] &= \bigcup_{s\in S} N_A[s] = \bigcup_{s\in\{t|\theta(t)\in T\}} N_A[s] = \bigcup_{\theta(t)\in T} N_A[t] \\ &= \{t|\theta(t) \in N_B[T]\} = \{t|\theta(t) \in B\} = A. \end{aligned}$$

□

Lemma 2.2. *Let $\theta : V(A) \to V(B)$ be an isomorphism between graphs A and B. The set S is a geodetic set of A if and only if $\{\theta(x)|x \in S\}$ is a geodetic set of B.*

Proof. Let $x, y \in V(A)$. Thus by isomorphism $\theta(x), \theta(y) \in V(B)$. We define $T \subseteq V(B)$ such that $T = \{\theta(x)|x \in S\}$. Note that if $z \in V(A)$ is contained in $x - y$ path in A, then $\theta(z) \in V(B)$ is also contained in $\theta(x) - \theta(y)$ path in B, and vice versa. So, z belongs to $x - y$ geodesic if and only if $\theta(z)$ belongs to $\theta(x) - \theta(y)$ geodesic. Therefore, $I_B[\theta(x), \theta(y)] = \{\theta(z)|z \in I_A[x, y]\}$ and $I_A[x, y] = \{z|\theta(z) \in I_B[\theta(x), \theta(y)]\}$

If S is a geodetic set of A, then we obtain

$$\begin{aligned} I_B[T] &= \bigcup_{i,j\in T} I_B[i, j] = \bigcup_{i,j\in\{\theta(s):s\in S\}} I_B[i, j] = \bigcup_{k,l\in S} I_B[\theta(k), \theta(l)] \\ &= \{\theta(s)|s \in I_A[S]\} = \{\theta(s)|s \in A\} = B. \end{aligned}$$

If T is a geodetic set of B, then we obtain

$$\begin{aligned} I_A[S] &= \bigcup_{k,l\in S} I_A[k, l] = \bigcup_{k,l\in\{t|\theta(t)\in T\}} I_A[k, l] = \bigcup_{\theta(j),\theta(k)\in T} I_A[j, k] \\ &= \{t|\theta(t) \in I_B[T]\} = \{t|\theta(t) \in B\} = A \end{aligned}$$

□

Therefore, we obtain a direct consequences of Lemmas 2.1 and 2.2 in corollary below.

Corollary 2.1. *Let $\theta : V(A) \to V(B)$ be an isomorphism between graphs A and B. The set S is a geodetic-dominating set of A if and only if $\{\theta(x)|x \in S\}$ is a geodetic-dominating set of B.*

Now, we investigate the geodetic properties of a geodetic-dominating set of a comb graph K_o with the identifying vertex $o \in V(H)$.

Lemma 2.3. *Let $o \in V(H)$ be the identifying vertex and u, v be two distinct vertices of K_o. For $l \in \{1, 2, ..., n\}$, if $u \in V_l$ and $v \notin V_l$, then every $u - v$ path in K_o consists of (g_l, o).*

Proof. The only vertex in V_l which is adjacent to a vertex in $V(K_o) \setminus V_l$ is (g_l, o). So, (g_l, o) must belong to every $u - v$ path in K_o. □

Lemma 2.4. *Let $o \in V(H)$ be the identifying vertex and a, b, v be distinct vertices in K_o. For $l \in \{1, 2, ..., n\}$, let $A_l = V_l \backslash \{(g_l, o)\}$. If $v \in A_l$ and $a, b \notin A_l$, then v does not belong to any $a - b$ paths in K_o.*

Proof. By Lemma 2.3, the vertex (g_l, o) in K_o always belongs to any $a - v$ walks and $b - v$ walks. So, $a - b$ walk always has the form $a...(g_l, h_o)...v...(g_l, h_o)...b$. In the other hand, v does not belong to any $a - b$ paths. □

Lemma 2.5. *Let $o \in V(H)$ be the identifying vertex and S be a geodetic set of K_o. Then for $l \in \{1, 2, ..., n\}$, $(S \cap V_l) \cup \{(g_l, h_o)\}$ is a geodetic set of $K_o[V_l]$.*

Proof. Suppose that $(S \cap V_l) \cup \{(g_l, o)\}$ is not a geodetic set of $K_o[V_l]$. Then, there exists a vertex $b \in V_l$ such that $b \notin I_{K_o}[(S \cap V_l) \cup \{(g_l, o)\}]$. Note that,

$$\begin{aligned} I_{K_o}[S] &= \bigcup_{x,y \in S} I_{K_o}[x, y] \\ &= \bigcup_{x,y \in S \cap V_l} I_{K_o}[x, y] \cup \bigcup_{x,y \in S \setminus V_l} I_{K_o}[x, y] \cup \bigcup_{x \in S \cap V_l, y \in S \setminus V_l} I_{K_o}[x, y]. \end{aligned}$$

By Lemma 2.3, we have

$$\bigcup_{x \in S \cap V_l, y \in S \setminus V_l} I_{K_o}[x, y] = \bigcup_{x \in S \cap V_l} I_{K_o}[x, (g_l, o)] \cup \bigcup_{y \in S \setminus V_l} I_{K_o}[y, (g_l, o)].$$

Since $\bigcup_{x,y \in S \cap V_l} I_{K_o}[x, y] \cup \bigcup_{x \in S \cap V_l} I_{K_o}[x, (g_l, o)] = \bigcup_{x,y \in (S \cap V_l) \cup \{(g_l, o)\}} I_{K_o}[x, y]$ and $\bigcup_{x,y \in S \setminus V_l} I_{K_o}[x, y] \cup \bigcup_{y \in S \setminus V_l} I_{K_o}[y, (g_l, o)] = \bigcup_{x,y \in (S \setminus V_l) \cup \{(g_l, o)\}} I_{K_o}[x, y]$, we obtain $I_{K_o}[S] = \bigcup_{x,y \in (S \cap V_l) \cup \{(g_l, o)\}} I_{K_o}[x, y] \cup \bigcup_{x,y \in (S \setminus V_l) \cup \{(g_l, o)\}} I_{K_o}[x, y]$.

Because $b \neq (g_l, o)$, then $b \notin I_{K_o}[(S \setminus V_l) \cup \{(g_l, o)\}]$. By considering Lemma 2.4, we have that S is not a geodetic set of K_o, a contradiction. □

In some lemmas below, we consider some properties of the geodetic-dominating set of an induced subgraph of K_o.

Lemma 2.6. *Let $o \in V(H)$ be the identifying vertex, $S \subseteq V(H)$, and $\Gamma_l = \{(g_l, x) | x \in S\}$ for $l \in \{1, 2, ..., n\}$. Then, S is a geodetic-dominating set of H if and only if Γ_l is a geodetic-dominating set of $K_o[V_l]$.*

Proof. By considering Corollary 2.1, we choose an isomorphism $\theta : V(H) \to V_l$ between graphs H and $K_o[V_l]$. Thus for $h \in V(H)$, $\theta(h) = (g_l, h)$. For $l \in \{1, 2, ..., n\}$ then $\Gamma_l = \{(g_l, x)|x \in S\} = \{\theta(x)|x \in S\}$. □

Lemma 2.7. *Let $o \in V(H)$ be the identifying vertex, and S be a dominating set of K_o. Then for $l \in \{1, 2, ..., n\}$, $S \cap V_l$ is a dominating set of $K_o[V_l \backslash \{(g_l, o)\}]$.*

Proof. Suppose that $S \cap V_l$ is not a dominating set of $K_o[V_l \backslash \{(g_l, o)\}]$. Then, there exists a vertex $b \in V_l \backslash \{(g_l, o)\}$ such that $b \notin N_{K_o}[S \cap V_l]$. Note that, $N_{K_o}[S] = N_{K_o}[S \cap V_l] \cup N_{K_o}[S \setminus V_l]$. Since $b \notin N_{K_o}[S \setminus V_l]$, then S is not a dominating set of K_o, a contradiction. □

By Lemmas 2.5 and 2.7, we obtain a property of geodetic-dominating set of an induced subgraph of K_o, which can be seen in corollary below.

Corollary 2.2. *Let $o \in V(H)$ be the identifying vertex, and S be a geodetic-dominating set of K_o. Then for $l \in \{1, 2, ..., n\}$, $(S \cap V_l) \cup \{(g_l, h_o)\}$ is a geodetic-dominating set of $K_o[V_l]$.*

Proof. By Lemma 2.5, $(S \cap V_l) \cup \{(g_l, o)\}$ is a geodetic set of $K_o[V_l]$. By considering Lemma 2.7, note that $N_{K_o}[(S \cap V_l) \cup \{(g_l, o)\}] = N_{K_o}[S \cap V_l] \cup N[(g_l, o)] \supseteq N_{K_o}(V_l \backslash \{(g_l, o)\}) \cup \{(g_l, o)\} = V_l$. So, $(S \cap V_l) \cup \{(g_l, o)\}$ is also a dominating set of $K_o[V_l]$. □

Now, let us consider a connected graph H of order at least 2. Let o be vertex in H. We define $\mathcal{B}$ as a collection of geodetic-dominating sets of graph H with cardinality $\gamma_g(H)$ containing o. The collection $\mathcal{B}$ can be written as

$$\mathcal{B} = \{B|B \subseteq V(H), N_H[B] = I_H[B] = V(H), o \in B, |B| = \gamma_g(H)\}.$$

We say that the graph H is of:

- type A_o if there exists a set $S \in \mathcal{B}$ such that $N_H[S \setminus \{o\}] = V(H)$.
- type B_o if there exists a set $S \in \mathcal{B}$ such that $N_H[S \setminus \{o\}] = V(H) - \{o\}$.

By above definitions, note that a graph H with the identifying vertex $o \in V(H)$ can be both of type A_o and B_o. Now, we are ready to determine the geodetic-dominating number of $G \rhd_o H$.

Theorem 2.1. *Let G and H be connected graphs of order at least 2. Let $o \in V(H)$. Then*

$$\gamma_g(G \rhd_o H) = \begin{cases} \gamma_g(H) \cdot |V(G)|, & \text{if } H \text{ is neither of type } A_o \text{ nor } B_o, \\ (\gamma_g(H) - 1) \cdot |V(G)|, & \text{if } H \text{ is of type } A_o, \\ \gamma(G) + (\gamma_g(H) - 1) \cdot |V(G)|, & \text{otherwise}. \end{cases}$$

Proof. For the identifying vertex $o \in V(H)$, we recall the notation $K_o = G \rhd_o H$. We distinguish three cases.

Case 1. H is neither of type A_o nor B_o

Let C be a geodetic-dominating set of H with $|C| = \gamma_g(H)$. We define $\Lambda = \{(g, h)|g \in V(G), h \in C\}$. By considering Lemma 2.6, we obtain that Λ is a geodetic-dominating set of K_o. Therefore, $\gamma_g(K_o) \leq |\Lambda| = |C| \cdot |V(G)| = \gamma_g(H) \cdot |V(G)|$.

For the lower bound, let us consider Corollary 2.2. Let S be a geodetic-dominating set of K_o. Then for $l \in \{1, 2, \dots, n\}$, $(S \cap V_l) \cup \{(g_l, o)\}$ is a geodetic-dominating set of $K_o[V_l]$. Let $B \in \mathcal{B}$. For $l \in \{1, 2, \dots, n\}$, we define $T_{l,B} = \{(g_l, b)|b \in B\}$ and $\mathcal{B}_l = \{T_{l,B}|B \in \mathcal{B}\}$. Note that $|T_{l,B}| = \gamma_g(H)$.

If $(S \cap V_l) \cup \{(g_l, o)\} \in \mathcal{B}_l$, then by considering Corollary 2.2, we have

$$|S \cap V_l| = |(S \cap V_l) \cup \{(g_l, o)\}| \geq \gamma_g(K_o[V_l]) = \gamma_g(H).$$

Otherwise, we have

$$|(S \cap V_l) \cup \{(g_l, o)\}| \geq \gamma_g(K_o[V_l]) + 1 = \gamma_g(H) + 1.$$

It follows that

$$|S \cap V_l| \geq \gamma_g(H).$$

Therefore, $|S \cap V_l| \geq \gamma_g(H)$ for $1 \leq l \leq n$.

Since $S = \bigcup_{l=1}^{n} S \cap V_l$ and $V_i \cap V_j = \emptyset$ for $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$, we obtain that

$$|S| \geq n \cdot |S \cap V_l| \geq n \cdot \gamma_g(H) = |V(G)| \cdot \gamma_g(H).$$

Case 2. H is of type A_o

Let $C \in \mathcal{B}$ such that $N_H[C \setminus \{o\}] = V(H)$. We define $\Lambda = \{(g, h)|g \in V(G), h \in C \setminus \{o\}\}$. Since $N_{K_o}[\Lambda] = I_{K_o}[A] = V(K_o)$, we obtain that Λ is a geodetic-dominating set of K_o. Therefore, $\gamma_g(K_o) \leq |\Lambda| = (|C| - 1) \cdot |V(G)| = (\gamma_g(H) - 1) \cdot |V(G)|$.

For the lower bound, let us consider Corollary 2.2. Let S be a geodetic-dominating set of K_o. Then for $l \in \{1, 2, \dots, n\}$, $(S \cap V_l) \cup \{(g_l, o)\}$ is a geodetic-dominating set of $K_o[V_l]$. Then we have that,

$$|(S \cap V_l) \cup \{(g_l, o)\}| \geq \gamma_g(K_o[V_l]) = \gamma_g(H).$$

It follows that

$$|S \cap V_l| \geq \gamma_g(H) - 1.$$

Since $S = \bigcup_{l=1}^{n} S \cap V_l$ and $V_i \cap V_j = \emptyset$ for $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$, we obtain that

$$|S| \geq n \cdot |S \cap V_l| \geq n \cdot (\gamma_g(H) - 1) = |V(G)| \cdot (\gamma_g(H) - 1).$$

Case 3. H is of type B_o and is not of type A_o

Let $C \in \mathcal{B}$ such that $N_H[C \setminus \{o\}] = V(H)$ and D be a dominating set of G with $|D| = \gamma(G)$. We define $\Lambda = \{(g, h)|g \in V(G), h \in C \setminus \{o\}\} \cup \{(g, o)|g \in D\}$. Since $N_{K_o}[\Lambda] = I_{K_o}[A] = V(K_o)$, we obtain that Λ is a geodetic-dominating set of K_o. Therefore, $\gamma_g(K_o) \leq |\Lambda| = (|C| - 1) \cdot |V(G)| + |D| = (\gamma_g(H) - 1) \cdot |V(G)| + \gamma(G)$.

For the lower bound, suppose that $\gamma_g(K_o) < (\gamma_g(H) - 1) \cdot |V(G)| + \gamma(G)$. Let S be a geodetic-dominating set of K_o with $|S| = \gamma_g(K_o)$. By Corollary 2.2, for $l \in \{1, 2, \dots, n\}$, $(S \cap V_l) \cup \{(g_l, o)\}$

is a geodetic-dominating set of $K_o[V_l]$. Note that

$$\begin{aligned} S &= \bigcup_{1\le l\le n} S\cap V_l \\ &= \bigcup_{1\le l\le n} S\cap\{(g_l,o)\}\cup \bigcup_{1\le l\le n} S\cap (V_l\setminus\{(g_l,o)\}) \\ &= (S\cap V_0)\cup \bigcup_{1\le l\le n} S\cap (V_l\setminus\{(g_l,o)\}). \end{aligned}$$

So, we obtain that there exists $l\in\{1,2,\dots,n\}$ such that $|S\cap(V_l\setminus\{(g_l,o)\})|<\gamma_g(H)-1$ or $|S\cap V_0|<\gamma(G)$. However,

$$\begin{aligned} |(S\cap(V_l\setminus\{(g_l,o)\}))\cup\{(g_l,o)\}| &= |(S\cap V_l)\cup\{(g_l,o)\}| \\ &\ge \gamma_g(K_o[V_l])=\gamma_g(H), \end{aligned}$$

which implies

$$|S\cap(V_l\setminus\{(g_l,o)\})|\ge\gamma_g(H)-1.$$

Therefore, $|S\cap V_0|<\gamma(G)$. By considering that $K_o[V_0]=G$, there exists a vertex $x\in V_0$ such that $x\notin N_{K_o}[S\cap V_0]$. It is clear that $x\notin S$.

If $x\notin N_{K_o}[S\cap(V_l\setminus\{(g_l,o)\})]$ for $1\le l\le n$, then we have a contradiction with S is a geodetic-dominating set of K_o. So, we assume that there exists $l\in\{1,2,\dots,n\}$ such that $x\in N_{K_o}[S\cap(V_l\setminus\{(g_l,o)\})]$. Since $x\in V_0$, thus $x=(g_l,o)$.

Let $B\in\mathcal{B}$. For $l\in\{1,2,\dots,n\}$, we define $T_{l,B}=\{(g_l,b)|b\in B\}$ and $\mathcal{B}_l=\{T_{l,B}|B\in\mathcal{B}\}$. Note that $|T_{l,B}|=\gamma_g(H)$.

If $(S\cap V_l)\cup\{(g_l,o)\}=(S\cap V_l)\cup\{x\}\in\mathcal{B}_l$, then

$$|N_{K_o}[S\cap V_l]|=|N_{K_o}[S\cap(V_l\setminus\{x\})]|\le|V(K_o[V_l])|-1$$

So, there is at least one vertex z in $K_o[V_l]$ such that $z\notin N_{K_o}[S\cap V_l]$. If $z=x$ then it will contradict to $x\in N[S\cap(V_l\setminus(g_l,o))]$. Otherwise, we have a contradiction to Lemma 2.7.

If $(S\cap V_l)\cup\{(g_l,o)\}=(S\cap V_l)\cup\{x\}\notin\mathcal{B}_l$, then

$$|(S\cap V_l)\cup\{x\}|\ge\gamma_g(K_o[V_l])+1=\gamma_g(H)+1,$$

which implies $|S\cap V_l|\ge\gamma_g(H)$. Since $S=\bigcup_{l=1}^n S\cap V_l$, $V_i\cap V_j=\emptyset$ for $i,j\in\{1,2,\dots,n\}$ and $i\ne j$, and $\gamma(G)\le|V(G)|$, we obtain that

$$\begin{aligned} |S| &\ge n\cdot|S\cap V_l|\ge n\cdot\gamma_g(H)=|V(G)|\cdot\gamma_g(H) \\ &\ge |V(G)|\cdot\gamma_g(H)-|V(G)|+\gamma(G) \\ &= (\gamma_g(H)-1)\cdot|V(G)|+\gamma(G). \end{aligned}$$

A contradiction. □

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