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Total vertex irregularity strength for trees with many vertices of degree two

Rinovia Simanjuntak[a], Susilawati[b], Edy Tri Baskoro[a]

[a] *Combinatorial Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha 10 Bandung, Indonesia*
[b] *Mathematics Department, Faculty of Mathematics and Natural Sciences, Universitas Riau, Kampus Bina Widya KM. 12,5, Pekanbaru, Indonesia*

rino@math.itb.ac.id, susilawati_nurdin@yahoo.com, ebaskoro@math.itb.ac.id

Abstract

For a simple graph $G = (V, E)$, a mapping $\phi : V \cup E \rightarrow \{1, 2, \ldots, k\}$ is defined as a vertex irregular total k-labeling of G if for every two different vertices x and y, $wt(x) \neq wt(y)$, where $wt(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$. The minimum k for which the graph G has a vertex irregular total k-labeling is called the total vertex irregularity strength of G. In this paper, we provide three possible values of total vertex irregularity strength for trees with many vertices of degree two. For each of the possible values, sufficient conditions for trees with corresponding total vertex irregularity strength are presented.

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1. Introduction

The concept of total vertex irregularity strength of graphs was first introduced by Baca *et.al* [2] in 2007. They defined a mapping $\phi : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ to be a *vertex irregular total k-labeling of G* if for every two different vertices x and y, $wt(x) \neq wt(y)$, where $wt(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$. The minimum k for which the graph G has a vertex irregular

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total k-labeling is called the *total vertex irregularity strength* of G, denoted by $tvs(G)$. Baca *et.al* determined the total vertex irregularity strength of some well-known classes of graphs, *i.e.* paths, cycles, and stars. Other authors (for instance, [1], [3]) determined the total vertex irregularity strength of some other classes of graphs, however results are still limited.

In the original paper of Baca *et.al* [2], it was proved that for a tree T with m pendant vertices and no vertex of degree 2, $\lceil \frac{m+1}{2} \rceil \leq tvs(T) \leq m$. In 2010, Nurdin *et.al* [4] settled the total vertex irregularity strength for a tree T with m pendant vertices and no vertices of degree 2, i.e. $tvs(T) = \lceil \frac{m+1}{2} \rceil$. They also improved the lower bound of Baca *et.al* as in the following.

Theorem 1.1. *[4] Let T be any tree having n_i vertices of degree $i(i = 1, 2, \ldots, \Delta)$, where Δ is the maximum degree in T. Then*

$$tvs(T) \geq max\left\{ \lceil \frac{1+n_1}{2} \rceil, \lceil \frac{1+n_1+n_2}{3} \rceil, \ldots, \lceil \frac{1+n_1+n_2+\cdots+n_\Delta}{\Delta+1} \rceil \right\}.$$

The lower bound in Theorem 1.1 remains the most general bound known for trees. However, it was conjectured that the total vertex irregularity strength of a tree is only determined by the number of vertices of degrees at most 3.

Conjecture 1.1. *[4] Let T be a tree with maximum degree Δ. Let n_i be the number of vertices of degree $i(i = 1, 2, \ldots, \Delta)$ and $t_i = \lceil \frac{1+\sum_{k=1}^{i} n_k}{(i+1)} \rceil (i = 1, 2, \ldots, \Delta)$. Then*

$$tvs(T) = max\{t_1, t_2, t_3\}.$$

To date, the conjecture has been confirmed for some types of trees, i.e. paths and stars, trees with maximum degree up to 5 [4, 6, 7] and subdivision of some classes of trees [5, 8].

In this paper, our aim is to determine the total vertex irregularity strength of trees with many vertices of degree 2 which include subdivision of trees. This result could somewhat be viewed as generalization of our result in [8], where we presented sufficient conditions for subdivision of trees to admit total vertex irregularity strength of t_2.

Throughout the paper, we consider T as a tree with maximum degree Δ. We denote by n_i the number of vertices of degree $i(i = 1, 2, \ldots, \Delta)$ and $t_i = \lceil \frac{1+\sum_{k=1}^{i} n_k}{(i+1)} \rceil (i = 1, 2, \ldots, \Delta)$.

2. Basic Properties of Trees

In this section, we shall provide properties of trees, in term on n_1, n_2, and n_3, having t_1, t_2 or t_3 as the maximum among all t_is. We start by quoting a useful property proved in [2].

Lemma 2.1. *[2]*

$$n_1 = 2 + \sum_{i \geq 2} (i-2) n_i.$$

Lemma 2.2. *If $n_1 \geq 2n_2 - 1$ and $n_2 = n_3$ then $t_1 \geq max\{t_1, t_2, \ldots, t_\Delta\}$.*

Proof. Utilising Lemma 2.1 in the definition of t_i, we have $t_i = \lceil \frac{3+\sum_{k=2}^{i}(k-1)n_k + \sum_{j=i+1}^{\Delta}(j-2)n_j}{(i+1)} \rceil$.

Consider $t_1 - t_2 = \lceil \frac{1+n_1}{2} \rceil - \lceil \frac{1+n_1+n_2}{3} \rceil = \lceil \frac{(2n_1+2n_2+2)+(n_1+1-2n_2)}{6} \rceil - \lceil \frac{2+2n_1+2n_2}{6} \rceil$. Since $n_1 \geq 2n_2 - 1$, we have $n_1 + 1 - 2n_2 \geq 0$ and thus $t_1 \geq t_2$.

On the other hand,

$$\begin{aligned} t_1 - t_3 &= \lceil \frac{1+n_1}{2} \rceil - \lceil \frac{1+n_1+n_2+n_3}{4} \rceil \\ &= \lceil \frac{(2+2n_1+2n_2+2n_3)+(2n_1+2-2n_3-2n_2)}{8} \rceil - \lceil \frac{2+2n_1+2n_2+2n_3}{8} \rceil. \end{aligned}$$

Since $n_1 \geq 2n_2 - 1$ and $n_2 = n_3$ then $2n_1 + 2 - 2n_3 - 2n_2 \geq 0$, which yields $t_1 \geq t_3$.

For $i \geq 4$,

$$\begin{aligned} t_1 - t_i &= \lceil \frac{1+n_1}{2} \rceil - \lceil \frac{3+\sum_{k=2}^{i}(k-1)n_k + \sum_{j=i+1}^{\Delta}(j-2)n_j}{i+1} \rceil \\ &\geq \lceil \frac{5+5n_1}{2(i+1)} \rceil - \lceil \frac{6+2n_2+4n_3+6n_4+2\sum_{j=5}^{\Delta}(j-2)n_j}{2(i+1)} \rceil. \end{aligned}$$

Since $n_1 \geq 2n_2 - 1$ and $n_2 = n_3$, $9+n_3+4n_4+3\sum_{i=5}^{\Delta}(i-2)n_i - 2n_2 \geq 6+2n_4+2\sum_{i=5}^{\Delta}(i-2) > 0$, which leads to $t_1 - t_i \geq 0$. □

Using similar proof of Lemma 2.2, we could prove the following lemmas.

Lemma 2.3. *If $n_2 \geq \frac{1}{2}(n_1 + 1)$ and $n_1 \geq 2n_3 - 1$ then $t_2 \geq max\{t_1, t_2, \ldots, t_\Delta\}$.*

Lemma 2.4. *If $n_2 = n_1$ and $n_3 \geq \frac{1}{3}(2n_2 + 1)$ then $t_3 \geq max\{t_1, t_2, \ldots, t_\Delta\}$.*

3. Trees with Many Vertices of Degree 2

In this section, we provide sufficient conditions, in term on n_1, n_2, and n_3, for a tree T with many vertices of degree 2 admitting $tvs(T) = t_1, t_2$ or t_3.

We start by defining several notions that will be frequently utilized in our labeling algorithms. Let v be a vertex of T. A *branch* of T at v is defined as maximal subtree of T containing v as an end point. That is, a branch of T at v is the subgraph induced by v and one of the components of $T - v$. If the degree of v is k, then v has k different branches. A branch of T at v which isomorphic to a path will be called a *branch path* at v, provided that the degree of v is at least 3. The vertex v, in this case, will be called a *stem* of the branch path at v. We define an *interior path* in T as a path whose both of end vertices are stem vertices. A vertex of degree one in T is called a *pendant vertex*. A vertex incident to a pendant vertex in T is called an *exterior vertex*. The vertices other than exterior and pendant vertices are called *interior vertices*. An edge incident with a pendant vertex is called a *pendant edge*. We denote by $E^p(v)$ the set of pendant edges incident to an exterior vertex v.

Theorem 3.1. *If* $n_1 \geq 2n_2 - 1$ *and* $n_2 = n_3 > 0$ *then* $tvs(T) = t_1$.

Proof. By Lemma 2.2 and Theorema 1.1, $tvs(T) \geq t_1$. We define a total labeling $\alpha : V(G) \cup E(G) \to \{1, 2, \dots, t_1\}$ according to the following algorithm.

Algorithm 1: Labeling α with tvs t_1

1. Let $W = \{w_1, w_2, \dots, w_k\}$ be the set of exterior vertices in T such that either $d(w_i) \geq d(w_{i+1})$ or $|E^p(w_i)| \geq |E^p(w_{i+1})|$.
2. Let $V_1 = \{w_{ij} | i = 1, 2, \dots, k$ and $j = 1, 2, \dots, |E^p(w_i)|\}$ be the ordered set of pendant vertices adjacent to all exterior vertices. Label the first t_1 pendant vertices in V_1 with 1 and the remaining $(n_1 - t_1)$ pendant vertices with $2, 3, \dots, n_1 - t_1 + 1$, respectively.
3. Let $E_1 = \{e_{ij} | i = 1, 2, \dots, k, j = 1, 2, \dots, |E^p(w_i)|\}$ be the ordered set of pendant edges incident to w_{ij}. Label the first t_1 pendant edges in E_1 with $\{1, 2, \dots, t_1\}$ and the remaining edges with t_1.
4. Let $y_1, y_2, \dots, y_N$ be vertices in $V \backslash V_1$. For all $y \in V \backslash V_1$, define $wt'(y) = \alpha(y) + \sum_{yz \in E(T)} \alpha(yz)$, as a temporary weight of a vertex y, where $wt'(y_i) \leq wt(y_{i+1})$. Label y_1 with $n_1 + 2 - wt'(y_1)$. For $2 \leq i \leq N$, label y_i with $\max\{1, wt(y_{i-1}) + 1 - wt'(y_i)\}$.

We observe that α is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \dots, t_1\}$ where the weights of n_1 pendant vertices are $2, 3, \dots, n_1 + 1$ and the weights of all remaining vertices are $n_1 + 2 = wt(y_1) < wt(y_2) < wt(y_3) < \dots < wt(y_N)$ where $N = \sum_{i=2}^{\Delta} n_i$. Therefore, $tvs(T) \leq t_1$. □

Theorem 3.2. *If* $n_2 \geq \frac{1}{2}(n_1 + 1)$ *and* $n_1 \geq 2n_3 - 1$ *then* $tvs(T) = t_2$.

Proof. By Lemma 2.3 and Theorem 1.1, $tvs(T) \geq t_2$. We show that $tvs(T) \leq t_2$ through a total labeling $\beta : V(T) \cup E(T) \to \{1, 2, \dots, t_2\}$ according to the following algorithm.

Algorithm 2: Labeling β with tvs t_2

1. **If** T has more interior paths than branch paths **then**

   (a) Let $W = \{w_1, w_2, \ldots, w_k\}$ be the set of stem vertices where $d(w_i) \geq d(w_{i+1})$.

   (b) Let $V_1 = \{w_{ij} | i = 1, 2, \ldots, k, j = 1, 2, \ldots, j_i\}$ be the set of all pendant vertices w_{ij} in the branch path of w_i. Label n_1 pendant vertices in V_1 with $\lceil \frac{i}{2} \rceil$.

   (c) Let $E_1 = \{e_{ij}\}$ be the set of all pendant edges e_{ij} incident to w_{ij}. Label n_1 pendant edges $e \in E_1$ with $\lceil \frac{i+1}{2} \rceil$.

   (d) Label all edges incident to stem vertices with t_2.

   (e) Let $E_2 = \{e_1, e_2, \ldots, e_k\}$ be the set of edges where both of end vertices of e_i are of degree two. Label e_i with $\lceil \frac{n_1+1+i}{3} \rceil$.

   else

   (a) Let $P = \{P^1, P^2, \ldots, P^k\}$ be the ordered set of branch paths, where $|P^i| \geq |P^{i+1}|$.

   (b) Let $W = \{w_1, w_2, \ldots, w_k\}$ be the set of stem vertices where $d(w_i) \geq d(w_{i+1})$.

   (c) Let $E_1 = \bigcup_{i=1} E(w_i)$ be an ordered set of all pendant edges in the path P^i. Label n_1 pendant edges in E_1 with $\lceil \frac{i+1}{2} \rceil$.

   (d) Label n_1 pendant vertices incident to e_i with $\lceil \frac{i}{2} \rceil$.

   (e) Label all edges incident to stem vertices with t_2.

   (f) Let $E_2 = \{e_1, e_2, \ldots, e_k\}$ be the ordered set of edges in $P^1 \cup P^2 \cup \cdots \cup P^k$. Label $e_i \in E_2$ with $\beta(e_i) = \lceil \frac{1+n_1+i}{3} \rceil$.

   (g) Let $L = \{L_1, L_2, \ldots, L_k\}$ be the set of interior paths where $|L_i| \geq |L_{i+1}|$.

   (h) Let $E_3 = \{f_1, f_2, \ldots, f_k\}$ be the ordered set of edges in path $L_1 \cup L_2 \cup \cdots \cup L_k$. Label $f_i \in E_3$ with $\lceil \frac{n_1+1+i}{3} \rceil$.

2. Denote all vertices not in V_1 by $y_1, y_2, \ldots, y_N$ such that $wt'(y_1) \leq wt'(y_2) \leq \cdots \leq wt'(y_N)$, where $wt'(y) = \sum_{yz \in E} \beta(yz)$ can be considered as a temporary weight of y. Label y_1 with $n_1 + 2 - s(y_1)$. For $2 \leq i \leq N$, label y_i with $\max\{1, wt(y_i + 1 - s(y_i))\}$.

We observe that β is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$, the weight of all pendant vertices form a sequence $1, 2, 3, \ldots, n_1 + 1$, and the weight of all remaining vertices are $n_1 + 2 = wt(y_1) < wt(y_2) < \cdots < wt(y_N)$. Therefore, $tvs(T) \leq t_2$. □

Examples of families of trees admitting total vertex irregularity strength of t_2 are special cases of subdivision of tress that could be found in [8].

Theorem 3.3. *If* $n_2 = n_1 > 0$ *and* $n_3 \geq \frac{1}{3}(2n_2 + 1)$ *then* $tvs(T) = t_3$.

Proof. By Lemma 2.4 and Theorem 1.1, $tvs(T) \geq t_3$. A total labeling $\gamma : V(T) \cup E(T) \rightarrow \{1, 2, 3, \ldots, t_3\}$ is defined according to the following algorithm.

Algorithm 3: Labeling γ with tvs t_3

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1. Let $W = \{w_1, w_2, w_3, \ldots, w_k\}$ be the set of all exterior vertices in T such that either $d(w_i) \geq d(w_{i+1})$ or $|E^p(w_i)| \geq |E^p(w_{i+1})|$.
2. Let $V_1 = \{w_{ij}|i = 1, 2, \ldots, k, j = 1, 2, \ldots, |E^p(w_i)|\}$ be the ordered set of pendant vertices adjacent to w_i. Label the first t_3 pendant vertices in V_1 with 1 and the remaining pendant vertices with $2, 3, \ldots, n_1 - t_3 + 1$, respectively.
3. Let $E_1 = \{e_{ij}|i = 1, 2, \ldots, k, j = 1, 2, \ldots, |E^p(w_i)|\}$ be the ordered set of pendant edges. Label the first t_3 pendant edges in E_1 with $\{1, 2, 3, \ldots, t_3\}$ and the remaining pendant edges with t_3.
4. **If** T has at least t_3 interior vertices of degree 2 **then**
   (a) Let $Y = \{y_1, y_2, \ldots, y_N\}$ be the set of exterior and interior vertices where either $wt'(y_i) \leq wt(y_{i+1})$ ($wt'(y) = \gamma(y) + \sum_{yz \in E(T)} \gamma(yz)$ is the temporary weight of y) or $deg(y_i) \leq deg(y_{i+1})$. Then $y_1, y_2, \ldots, y_{n_2}$ are the interior vertices of degree 2 where $wt'(y) = 0$.
   (b) **for** $i = 1, 2, \ldots, N$ **do** label y_i and all its adjacent edges (almost) evenly such that $wt(y_i) = n_1 + i + 1$ and the labels of edges are at least the label of y_i.
   (c) Let $S = \{s_1, s_2, \ldots, s_k\}$ be the set of exterior and interior vertices where $wt'(s) \neq 0$ and $wt'(s_i) \leq wt'(s_{i+1})$.
   (d) **for** $i = 1, 2, \ldots, k$ **do** label s_i and all its adjacent edges (almost) evenly such that $wt(s_i) = n_1 + 1/2n_2 + i + 1$ and the labels of edges are at least the label of s_i.

   else

   (a) Label all edges not in E_1 with t_3.
   (b) Let $y_1, y_2, \ldots, y_N$ be the vertices in V/V_1. For all $y \in V/V_1$, define $wt'(y) = \gamma(y) + \sum_{yz \in E(T)} \gamma(yz)$ as the temporary weight y. Label y_1 with $n_1 + 2 - wt'(y_1)$. For $2 \leq i \leq N$, label y_i with $\max\{1, wt(y_{i-1}) + 1 - wt'(y_i)\}$.

We observe that γ is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_3\}$ where the weights of n_1 pendant vertices are $\{2, 3, 4, \ldots, n_1 + 1\}$ and the weights of all the remaining vertices are $n_1 + 2 = wt(y_1) < wt(y_2) < \cdots < wt(y_N)$. This yields $t_3 \leq tvs(T)$. $\square$

4. Conclusion

Our results provide sufficient conditions for trees containing many vertices of degree 2 where the total vertex irregularity strength is either t_1, t_2 or t_3. These results strengthens the conjecture Nurdin *et.al*.

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