On the restricted size Ramsey number for $P_3$ versus dense connected graphs

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Abstract

Let $F$, $G$ and $H$ be simple graphs. A graph $F$ is said a $(G, H)$—arrowing graph if in any red-blue coloring of edges of $F$ we can find a red $G$ or a blue $H$. The size Ramsey number of $G$ and $H$, $\hat{r}(G, H)$, is the minimum size of $F$. If the order of $F$ equals to the Ramsey number of $G$ and $H$, $r(G, H)$, then the minimum size of $F$ is called the restricted size Ramsey number of $G$ and $H$, $r^*(G, H)$. The Ramsey number of $G$ and $H$, $r(G, H)$, is the minimum order of $F$. In this paper, we study the restricted size number involving a $P_3$. The value of $r^*(P_3, K_n)$ has been given by Faudree and Sheehan. Here, we examine $r^*(P_3, H)$ where $H$ is dense connected graph.

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1. Introduction

Let $G$ be a graph with the vertex and edge set $V(G)$ and $E(G)$, respectively. We denote the order of $G$ by $v(G)$ and and the size of $G$ by $e(G)$. A $\delta(G)$ (resp. $\Delta(G)$) denotes the minimum (resp. maximum) degree of vertices in $G$. If $G$ is a graph and $H$ is a subgraph of $G$, then graph...
$G - H$ has $V(G - H) = V(G)$ and $E(G - H) = E(G) \setminus E(H)$. Further terminologies in graphs can be found in [3].

A graph $F$ is a $(G, H)$—arrowing graph if in any red-blue coloring of the edges of $F$ we can find a red $G$ or a blue $H$. Let $F$ be $(G, H)$—arrowing graph. The Ramsey number of $G$ and $H$, $r(G, H)$, is the smallest order of $F$ and the size Ramsey number of $G$ and $H$, $\hat{r}(G, H)$, is the smallest size of $F$. The restricted size Ramsey number of $G$ and $H$, $r^*(G, H)$ is the smallest size of a $F$ when its order equals the Ramsey number $r(G, H)$.

The size Ramsey number for a pair of graphs was introduced by Erdős et al. in 1978 [4], while the restricted size Ramsey number for a pair of graphs is a direct consequence of the concept of Ramsey and size Ramsey number in graphs. Some previous results on the (restricted) size Ramsey number of graphs was given in [1, 5] and the previous results on the restricted size Ramsey number involving a $P_3$ can be found in [9, 10, 11, 12, 13].

In 1972, Chvátal and Harary [2] introduced the off-diagonal Ramsey number, where the pair of graphs involved are from different classes. One of their results is the Ramsey number for $P_3$ and any graph without isolated vertices. In 1983, Faudree and Sheehan [7] investigated the size and the restricted Ramsey numbers involving stars. One of their results is the size and the restricted size Ramsey number for $P_3$ and $K_n$ and they found that these both values are the same, namely, $\hat{r}(P_3, K_n) = r^*(P_3, K_n)$.

Furthermore, it was known that the lower and upper bounds of the size and the restricted size Ramsey number for any pair of graph $G$ and $H$ as follows.

$$e(G) + e(H) - 1 \leq \hat{r}(G, H) \leq r^*(G, H) \leq \left(\frac{r(G, H)}{2}\right). \quad (1)$$

The first inequality was given by Harary and Miller [8].

In our previous work in [10], we have characterized all graphs $H$ such that $r^*(P_3, H)$ attains the upper and lower bounds of (1). In this paper, we continue the investigation on the restricted size Ramsey number involving a $P_3$. We give $r^*(P_3, H)$ with $H$ a dense graph. Since $H$ is dense, we can obtain it by removing some edge from a complete graphs.

2. Preliminaries

The size and the restricted size Ramsey numbers for a path $P_3$ and a complete graph $K_n$ was given by Faudree and Sheehan [7], as stated in Theorem 2.1. From the proof of Theorem 2.1 [7], we have Lemma 2.2 and Lemma 2.3.

**Theorem 2.1.** [7] For a positive integer $n \geq 2$,

$$\hat{r}(P_3, K_n) = r^*(P_3, K_n) = 2(n - 1)^2.$$

**Lemma 2.2.** [7] For a positive integer $n \geq 2$, $F = K_{2n-1} - M$ is a $(P_3, K_n)$—arrowing graph, with $M$ is a maximal matching in $K_{2n-1}$.

**Lemma 2.3.** [7] For a positive integer $n \geq 2$, let $F$ be a graph with $\nu(F) = 2n - 1$. If $F$ is a $(P_3, K_n)$—arrowing graph, then $\delta(F) \geq 2n - 3$. 

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The Ramsey number for $P_3$ and any graph $H$ without isolated vertices was given by Chvátal and Harary [2], as stated in Theorem 2.4. This result gives the order of $(P_3, H)$-arrowing graph to find $r^*(P_3, H)$.

**Theorem 2.4.** [2] For any graph $H$ with no isolates,

$$r(P_3, H) = \begin{cases} v(H), & \text{if } \overline{H} \text{ has } 1 \text{-factor}, \\ 2v(H) - 2\beta(\overline{H}) - 1, & \text{otherwise,} \end{cases}$$

with $\beta(\overline{H})$ the maximum number of independent edges in the complement of $H$.

Let $H$ be a connected graph with $v(H) = n$. From Theorem 2.4 we have $r(P_3, H) = n$ if $\beta(\overline{H}) = \lceil \frac{n}{2} \rceil$ and $r(P_3, H) > n$ otherwise. In [10], we showed that $r^*(P_3, H)$ is less than the upper bound of (1) for all $H$ with $r(P_3, H) > n$. Here, we find the exact value of $r^*(P_3, H)$ for some $H$ with $r(P_3, H) > n$.

The following monotonicity property is clear from the definition of the (restricted) size Ramsey number of graphs. If $F_0 \subset F_1$ and $F_0 \subset F_2$, then

$$\hat{r}(F'_1, F'_2) \leq \hat{r}(F_1, F_2).$$

(2)

and

$$r^*(F'_1, F'_2) \leq r^*(F_1, F_2).$$

(3)

Note that Chvátal and Harary [2] also gave the same monotonicity property for Ramsey number of graphs.

**3. Main Results**

First, we investigate the restricted size Ramsey number $r^*(P_3, H)$ for $H$ a connected graph obtained by deleting some edges incident to a vertex from a complete graph. The results are given in Theorem 3.2 and 3.3.

Second, we investigate the restricted size Ramsey number $r^*(P_3, H)$ for $H$ a connected graph obtained by deleting some edges incident to two vertices from a complete graph. The results are given in Theorem 3.4, 3.5, and 3.6.

To prove those theorems, we adopt the idea from Faudree and Sheehan in [7] by using a graph $G_F$ which is defined as follows. Let $F$ be a $(G, H)$-arrowing graph with all edges colored by red and blue. A $G_F$ is a graph with $V(G_F) = V(F)$ and $E(G_F)$ consists of red edges in $F$ and edges in $\overline{F}$. Notice that $G_F$ is the blue subgraph of $F$.

Additionally, we will also use Observation 3.1.

**Observation 3.1.** Let $F$, $F'$, $G$, and $H$ be graphs. If $F$ is a $(G, H)$-arrowing graph and $F \subseteq F'$, then $F'$ is a $(G, H)$-arrowing graph.

**Proof.** Suppose to the contrary $F'$ is not a $(G, F)$-arrowing graph. It means there is a red-blue coloring $\phi$ of all edges in $F'$ not containing a red $H$ or a blue $G$. However, since $F \subseteq F'$, $\phi$ is also a red-blue coloring of all edges in $F$ not containing a red $H$ or a blue $G$. Thus, $F$ is not a $(G, F)$-arrowing graph. A contradiction. $\square$

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Theorem 3.2. Let $n$ and $t$ be integers with $2 \leq t \leq n - 2$. For $n \geq 3$, 

$$r^*(P_3, K_n - K_{1,t}) = 2(n - 2)^2.$$ 

Proof. Note that $K_{n-1} \subseteq K_n - K_{1,t}$ for any $t$. Since $\beta(K_n - K_{1,t}) = 1$, Theorem 2.4 implies the Ramsey number $r(P_3, K_n - K_{1,t}) = 2n - 3$ for $2 \leq t \leq n - 2$. However, the Ramsey number $r(P_3, K_{n-1}) = 2n - 3$ too. From Theorem 2.1 we have $r^*(P_3, K_{n-1}) = 2(n - 2)^2$. Since $K_{n-1} \subseteq K_n - K_{1,t}$, by (3) we get $r^*(P_3, K_n - K_{1,t}) \geq 2(n - 2)^2$ for all $t$ in $2 \leq t \leq n - 2$. This completes the proof for the lower bound.

For the upper bound, let $F = K_{2n-3} - M$ with $M$ a maximal matching in $K_{2n-3}$. Note that $e(F) = 2(n - 2)^2$. Consider any red-blue coloring $\phi$ of all edges in $F$ not containing a red $P_3$. The graph $G_F$ will consist of even cycles and paths. It means $\Delta(G_F) \leq 2$. Furthermore, according to Lemma 2.2, $F$ is a $(P_3, K_{n-1})$–arrowing graph. Since $\Delta(G_F) \leq 2$, we can extend $K_{n-1}$ to $K_n - K_{1,t}$ for $2 \leq t \leq n - 2$ in $G_F$. Thus, $F$ is a $(P_3, K_n - K_{1,t})$–arrowing graph and $r^*(P_3, K_n - K_{1,t}) \leq 2(n - 2)^2$ for $2 \leq t \leq n - 2$. 

The restricted size Ramsey number $r^*(P_3, H)$ for $H = K_n - K_{1,1}$ is given in Theorem 3.3. We use $K_2$ in terms of $K_{1,1}$.

Theorem 3.3. For $n \geq 4$, 

$$r^*(P_3, K_n - K_2) = 2(n - 2)^2 + 1.$$ 

Proof. Note that $K_{n-1} \subseteq K_n - K_2$. Since $\beta(K_n - K_2) = 1$, Theorem 2.4 implies $r(P_3, K_n - K_2) = 2n - 3$. Note again that the Ramsey number $r(P_3, K_{n-1}) = 2n - 3$ too. For the lower bound, we consider all graphs $F$ with $v(F) = 2n - 3$ and $e(F) = 2(n - 2)^2$. However, if $F$ is a $(P_3, K_{n-1})$–arrowing graph, Lemma 2.3 implies $\delta(F) \geq 2n - 5$. The only graph satisfies the above conditions is $F = K_{2n-3} - M$, with $M$ a maximal matching in $K_{2n-3}$. Take a red-blue coloring of all edges in $F$ not containing a red $P_3$ such that $G_F \cong P_{2n-3}$. According to Lemma 2.2, $F$ is a $(P_3, K_{n-1})$–arrowing graph. But, each vertex which does not belong to subgraph $K_{n-1}$ in $G_F$ is adjacent to exactly two vertices that induced $K_{n-1}$ in $G_F$. It means we cannot extend $K_{n-1}$ to $K_n - K_2$ in $G_F$. Thus, $F$ is not a $(P_3, K_n - K_2)$–arrowing graph and $r^*(P_3, K_n - K_2) \geq 2(n - 2)^2 + 1$.

For the upper bound, let $F = K_{2n-3} - (|M| - 1)K_2$ with $M$ a maximal matching in $K_{2n-3}$. Note that $e(F) = 2(n - 2)^2 + 1$. We will show that $F$ is a $(P_3, K_n - K_2)$–arrowing graph. According to Lemma 2.2, $K_{2n-3} - M$ is a $(P_3, K_{n-1})$–arrowing graph. Since $K_{2n-3} - M \subseteq F$, Observation 3.1 implies $F$ is also a $(P_3, K_{n-1})$–arrowing graph. Consider any red-blue coloring $\phi$ of all edges in $F$ not containing a red $P_3$. The graph $G_F$ will consist of even cycles and paths with at least one path of even order. Suppose $V'$ is the set of vertices that induces a $K_{n-1}$ in $G_F$. Since there is a path of even order in $G_F$, there must be at least one vertex $v \in V \setminus V'$ that is adjacent to exactly one vertex $v' \in V'$. It means we can extend $K_{n-1}$ to a $K_n - K_2$ in $G_F$. Thus, $F$ is a $(P_3, K_n - K_2)$–arrowing graph and $r^*(P_3, K_n - K_2) \leq 2(n - 2)^2 + 1$. 

The next result is $r^*(P_3, H)$ with $H$ a connected graph obtained by deleting some edges incident to two vertices in $K_n$. First, we consider collection of graphs obtained by deleting edges in $K_{2s} \cup K_{1,t}$ for $2 \leq s \leq n - 2$ and $3 \leq t \leq n - 2$, as given in Theorem 3.4. When $1 \leq s, t \leq 2$,
the graph $K_{1,s} \cup K_{1,t}$ is one of $2P_3, P_3 \cup K_2, P_4, 2K_2, P_3,$ or $K_2$. The values of $r^*(P_3, H)$ with $H$ either $K_n - P_3$ or $K_n - K_2$ already include in Theorems 3.2 and 3.3. Theorems 3.5 and 3.6 give $r^*(P_3, H)$ for $H$ a graph obtained by deleting edges in $2P_3, P_3 \cup K_2, P_4,$ or $2K_2$ from $K_n$.

**Theorem 3.4.** Let $n, s, t$ be integers with $2 \leq s \leq n - 2$ and $3 \leq t \leq n - 2$. For $n \geq 5$,

$$r^*(P_3, K_n - (K_{1,s} \cup K_{1,t})) = 2(n - 3)^2.$$

**Proof.** Note that $K_{n-2} \subseteq K_n - (K_{1,s} \cup K_{1,t})$ for $2 \leq s \leq n - 2$ and $3 \leq t \leq n - 2$. Since $\beta(K_n - (K_{1,s} \cup K_{1,t})) = 2$, Theorem 2.4 implies $r(P_3, K_n - (K_{1,s} \cup K_{1,t})) = 2n - 5$. Note that the Ramsey number $r(P_3, K_{n-2}) = 2n - 5$ too. From Theorem 2.1 we have $r^*(P_3, K_{n-2}) = 2(n - 3)^2$. Since $K_{n-2} \subseteq K_n - (K_{1,s} \cup K_{1,t})$, by (3) we get $r^*(P_3, K_n - (K_{1,s} \cup K_{1,t})) \geq 2(n - 3)^2$ for $2 \leq s \leq n - 2$ and $3 \leq t \leq n - 2$. This completes the proof for the lower bound.

For the upper bound, let $F = K_{2n-5} - M$ with $M$ a maximal matching in $K_{2n-5}$. Note that $e(F) = 2(n - 3)^2$. Lemma 2.2 implies $F$ is a $(P_3, K_{n-2})$-arrowing graph. Consider any red-blue coloring of all edges of $F$ not containing a red $P_3$. The graph $G_F$ will satisfy the above conditions. However, if $F$ is a $(P_3, K_{n-2})$-arrowing graph, then Lemma 2.3 implies $d(F) \geq 2n - 7$. The only graph that satisfies the above conditions is $F = K_{2n-5} - M$, with $M$ a maximal matching in $K_{2n-5}$. Take a red-blue coloring $\phi$ of all edges in $F$ such that $G_F \cong P_{2n-5}$. According to Lemma 2.2, $F$ is a $(P_3, K_{n-2})$-arrowing graph. Suppose $V'$ is the set of vertices that induces $K_{n-2}$ in $G_F$. Since $G_F \cong P_{2n-5}$, each vertex $v \in V \setminus V'$ is adjacent to exactly two vertices that belong to $V'$ in $G_F$. It means we cannot extend the subgraph $K_{n-2}$ to have a subgraph $K_n - 2P_3$ in $G_F$. As a consequence, $r^*(P_3, K_n - 2P_3) \geq 2(n - 3)^2 + 1$. Since $K_n - 2P_3 \subseteq K_n - (P_3 \cup K_2)$, by (3), $r^*(P_3, K_n - (P_3 \cup K_2)) \geq 2(n - 3)^2 + 1$.

For the upper bound, let $F = K_{2n-5} - (|M| - 1)K_2$ with $M$ a maximal matching in $K_{2n-5}$. Note that $e(F) = 2(n - 3)^2 + 1$. We will show that $F$ is a $(P_3, K_n - (P_3 \cup K_2))$-arrowing graph. According to Lemma 2.2, $K_{2n-3} - M$ is a $(P_3, K_{n-2})$-arrowing graph. Since $K_{2n-5} - M \subseteq F$, Observation 3.1 implies $F$ is also a $(P_3, K_{n-2})$-arrowing graph. Consider any red-blue coloring $\phi$ of all edges in $F$ not containing a red $P_3$. The graph $G_F$ will consist of even cycles and paths with at least one path of even order. Suppose $V'$ is the set of vertices that induces a $K_{n-2}$ in $G_F$. Since there is a path of even order in $G_F$, there must be at least one vertex $v \in V \setminus V'$ that adjacent to exactly one vertex $v' \in V'$ in $G_F$. It means we can extend the subgraph $K_{n-2}$ to have
a subgraph $K_n - 2K_2$ in $\overline{G}_F$. As a consequence, $F$ is a $(P_3, K_n - (P_3 \cup K_2))$-arrowing graph and $r^*(P_3, K_n - (P_3 \cup K_2)) \leq 2(n - 3)^2 + 1$. Since $K_n - 2P_3 \subseteq K_n - (P_3 \cup K_2)$, by (3), $r^*(P_3, K_n - 2P_3) \leq 2(n - 3)^2 + 1$.

**Theorem 3.6.** For $n \geq 5$,

$$r^*(P_3, K_n - P_4) = r^*(P_3, K_n - 2K_2) = 2(n - 3)^2 + 2.$$  

**Proof.** Note that $K_{n-2} \subseteq K_n - P_4 \subseteq K_n - 2K_2$. Since $\beta(K_n - P_4) = \beta(K_n - 2K_2) = 2$, Theorem 2.4 implies $r(P_3, K_n - P_4) = r(P_3, K_n - 2K_2) = 2n - 5$. The Ramsey number $r(P_3, K_{n-2}) = 2n - 5$ too. For the lower bound, we consider all graphs $F$ with $v(F) = 2n - 5$ and $e(F) = 2(n - 3)^2 + 1$. However, if $F$ is a $(P_3, K_{n-2})$-arrowing graph, then Lemma 2.3 implies $\delta(F) \geq 2n - 7$. The only graph satisfies the above conditions is $F = K_{2n-5} - (|M| - 1)K_2$, with $M$ a maximal matching in $K_{2n-5}$. Take a red-blue coloring $\phi$ of all edges in $F$ such that $G_F \cong P_{2n-6} \cup K_1$. According to Lemma 2.2, $K_{2n-5} - M$ is a $(P_3, K_{n-2})$-arrowing graph and according to Observation 3.1, $F$ is also a $(P_3, K_{n-2})$-arrowing graph. Suppose $V'$ is the set of vertices that induces a $K_{n-2}$ in $\overline{G}_F$. Since $G_F$ is $P_{2n-6} \cup K_1$, there must be a leaf $v$ of $P_{2n-6}$ such that $v \notin V'$ which is adjacent to exactly one vertex $v' \in V'$ in $G_F$. It means we cannot extend the subgraph $K_{n-2}$ to have a subgraph $K_n - P_4$ in $\overline{G}_F$. Thus $F$ is not a $(P_3, K_n - P_4)$-arrowing graph and $r^*(P_3, K_n - P_4) \geq 2(n - 3)^2 + 2$. Since $K_n - P_4 \subseteq K_n - 2K_2$, by (3) we have $r^*(P_3, K_n - 2K_2) \geq 2(n - 3)^2 + 2$.

For the upper bound, let $F = K_{2n-5} - (|M| - 2)K_2$ with $M$ a maximal matching in $K_{2n-5}$. Note that $e(F) = 2(n - 3)^2 + 2$. We will show that $F$ is a $(P_3, K_n - 2K_2)$-arrowing graph. According to Lemma 2.2, $K_{2n-5} - M$ is a $(P_3, K_{n-2})$-arrowing graph and according to Observation 3.1, $F$ is also a $(P_3, K_{n-2})$-arrowing graph. Consider any red-blue coloring $\phi$ of all edges in $F$ not containing a red $P_3$. The graph $G_F$ will consists of even cycles and paths with at least two path of even order. Suppose $V'$ is the set of vertices that induced $K_{n-2}$ in the $\overline{G}_F$. Since there are two path of even order in $G_F$, there must be at least two vertices $u, v \notin V'$ adjacent to exactly a vertex $v' \in V'$ in $G_F$. It means we can extend the subgraph $K_{n-2}$ to have a subgraph $K_n - 2K_2$ in $\overline{G}_F$. As a consequence, $r^*(P_3, K_n - 2K_2) \leq 2(n - 3)^2 + 2$. Since $K_n - P_4 \subseteq K_n - 2K_2$, by (3) we have $r^*(P_3, K_n - P_4) \leq 2(n - 3)^2 + 2$.  

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