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On the restricted size Ramsey number for P_3 versus dense connected graphs

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Abstract

Let F, G and H be simple graphs. A graph F is said a $(G, H)-$arrowing graph if in any red-blue coloring of edges of F we can find a red G or a blue H. The size Ramsey number of G and H, $\hat{r}(G, H)$, is the minimum size of F. If the order of F equals to the Ramsey number of G and H, $r(G, H)$, then the minimum size of F is called the restricted size Ramsey number of G and H, $r^*(G, H)$. The Ramsey number of G and H, $r(G, H)$, is the minimum order of F. In this paper, we study the restricted size number involving a P_3. The value of $r^*(P_3, K_n)$ has been given by Faudree and Sheehan. Here, we examine $r^*(P_3, H)$ where H is dense connected graph.

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1. Introduction

Let G be a graph with the vertex and edge set $V(G)$ and $E(G)$, respectively. We denote the order of G by $v(G)$ and and the size of G by $e(G)$. A $\delta(G)$ (resp. $\Delta(G)$) denotes the minimum (resp. maximum) degree of vertices in G. If G is a graph and H is a subgraph of G, then graph

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$G - H$ has $V(G-H) = V(G)$ and $E(G-H) = E(G) \setminus E(H)$. Further terminologies in graphs can be found in [3].

A graph F is a $(G, H)-$*arrowing graph* if in any red-blue coloring of the edges of F we can find a red G or a blue H. Let F be $(G, H)-$arrowing graph. The *Ramsey number* of G and H, $r(G, H)$, is the smallest order of F and the *size Ramsey number* of G and H, $\hat{r}(G, H)$, is the smallest size of F. The *restricted size Ramsey number* of G and H, $r^*(G, H)$ is the smallest size of a F when its order equals the Ramsey number $r(G, H)$.

The size Ramsey number for a pair of graphs was introduced by Erdős *et al.* in 1978 [4], while the restricted size Ramsey number for a pair of graphs is a direct consequence of the concept of Ramsey and size Ramsey number in graphs. Some previous results on the (restricted) size Ramsey number of graphs was given in [1, 5] and the previous results on the restricted size Ramsey number involving a P_3 can be found in [9, 10, 11, 12, 13].

In 1972, Chvátal and Harary [2] introduced the off-diagonal Ramsey number, where the pair of graphs involved are from different classes. One of their results is the Ramsey number for P_3 and any graph without isolated vertices. In 1983, Faudree and Sheehan [7] investigated the size and the restricted Ramsey numbers involving stars. One of their results is the size and the restricted size Ramsey number for P_3 and K_n and they found that these both values are the same, namely, $\hat{r}(P_3, K_n) = r^*(P_3, K_n)$.

Furthermore, it was known that the lower and upper bounds of the size and the restricted size Ramsey number for any pair of graph G and H as follows.

$$e(G) + e(H) - 1 \leq \hat{r}(G, H) \leq r^*(G, H) \leq \binom{r(G, H)}{2}. \tag{1}$$

The first inequality was given by Harary and Miller [8].

In our previous work in [10], we have characterized all graphs H such that $r^*(P_3, H)$ attains the upper and lower bounds of (1). In this paper, we continue the investigation on the restricted size Ramsey number involving a P_3. We give $r^*(P_3, H)$ with H a dense graph. Since H is dense, we can obtain it by removing some edge froms a complete graphs.

2. Preliminaries

The size and the restricted size Ramsey numbers for a path P_3 and a complete graph K_n was given by Faudree and Sheehan [7], as stated in Theorem 2.1. From the proof of Theorem 2.1 [7], we have Lemma 2.2 and Lemma 2.3.

Theorem 2.1. *[7] For a positive integer $n \geq 2$,*

$$\hat{r}(P_3, K_n) = r^*(P_3, K_n) = 2(n-1)^2.$$

Lemma 2.2. *[7] For a positive integer $n \geq 2$, $F = K_{2n-1} - M$ is a $(P_3, K_n)-$ arrowing graph, with M is a maximal matching in K_{2n-1}.*

Lemma 2.3. *[7] For a positive integer $n \geq 2$, let F be a graph with $v(F) = 2n-1$. If F is a $(P_3, K_n)-$arrowing graph, then $\delta(F) \geq 2n-3$.*

The Ramsey number for P_3 and any graph H without isolated vertices was given by Chvátal and Harary [2], as stated in Theorem 2.4. This result gives the order of $(P_3, H)-$arrowing graph to find $r^*(P_3, H)$.

Theorem 2.4. *[2] For any graph H with no isolates,*

$$r(P_3, H) = \begin{cases} v(H), & \overline{H}\ has\ 1-factor, \\ 2v(H) - 2\beta(\overline{H}) - 1, & otherwise, \end{cases}$$

with $\beta(\overline{H})$ the maximum number of independent edges in the complement of H.

Let H be a connected graph with $v(H) = n$. From Theorem 2.4 we have $r(P_3, H) = n$ if $\beta(\overline{H}) = \lfloor \frac{n}{2} \rfloor$ and $r(P_3, H) > n$ otherwise. In [10], we showed that $r^*(P_3, H)$ is less than the upper bound of (1) for all H with $r(P_3, H) > n$. Here, we find the exact value of $r^*(P_3, H)$ for some H with $r(P_3, H) > n$.

The following monotonicity property is clear from the definition of the (restricted) size Ramsey number of graphs. If $F_1' \subseteq F_1$ and $F_2' \subseteq F_2$, then

$$\hat{r}(F_1', F_2') \leq \hat{r}(F_1, F_2). \tag{2}$$

and

$$r^*(F_1', F_2') \leq r^*(F_1, F_2). \tag{3}$$

Note that Chvátal and Harary [2] also gave the same monotonicity property for Ramsey number of graphs.

3. Main Results

First, we investigate the restricted size Ramsey number $r^*(P_3, H)$ for H a connected graph obtained by deleting some edges incident to a vertex from a complete graph. The results are given in Theorem 3.2 and 3.3.

Second, we investigate the restricted size Ramsey number $r^*(P_3, H)$ for H a connected graph obtained by deleting some edges incident to two vertices from a complete graph. The results are given in Theorem 3.4, 3.5, and 3.6.

To prove those theorems, we adopt the idea from Faudree and Sheehan in [7] by using a graph G_F which is defined as follows. Let F be a $(G, H)-$arrowing graph with all edges are colored by red and blue. A G_F is a graph with $V(G_F) = V(F)$ and $E(G_F)$ consists of red edges in F and edges in $\overline{F}$. Notice that $\overline{G_F}$ is the blue subgraph of F.

Additionally, we will also use Observation 3.1.

Observation 3.1. *Let F, F', G, and H be graphs. If F is a $(G, H)-$arrowing graph and $F \subseteq F'$, then F' is a $(G, H)-$arrowing graph.*

Proof. Suppose to the contrary F' is not a $(G, F)-$arrowing graph. It means there is a red-blue coloring ϕ of all edges in F' not containing a red H or a blue G. However, since $F \subseteq F'$, ϕ is also a red-blue coloring of all edges in F not containing a red H or a blue G. Thus, F is not a $(G, F)-$arrowing graph. A contradiction. □

Theorem 3.2. *Let n and t be integers with $2 \leq t \leq n-2$. For $n \geq 3$,*

$$r^*(P_3, K_n - K_{1,t}) = 2(n-2)^2.$$

Proof. Note that $K_{n-1} \subsetneq K_n - K_{1,t}$ for any t. Since $\beta(\overline{K_n - K_{1,t}}) = 1$, Theorem 2.4 implies the Ramsey number $r(P_3, K_n - K_{1,t}) = 2n-3$ for $2 \leq t \leq n-2$. However, the Ramsey number $r(P_3, K_{n-1}) = 2n-3$ too. From Theorem 2.1 we have $r^*(P_3, K_{n-1}) = 2(n-2)^2$. Since $K_{n-1} \subseteq K_n - K_{1,t}$, by (3) we get $r^*(P_3, K_n - K_{1,t}) \geq 2(n-2)^2$ for all t in $2 \leq t \leq n-2$. This completes the proof for the lower bound.

For the upper bound, let $F = K_{2n-3} - M$ with M a maximal matching in K_{2n-3}. Note that $e(F) = 2(n-2)^2$. Consider any red-blue coloring ϕ of all edges in F not containing a red P_3. The graph G_F will consist of even cycles and paths. It means $\Delta(G_F) \leq 2$. Furthermore, according to Lemma 2.2, F is a $(P_3, K_{n-1})-$arrowing graph. Since $\Delta(G_F) \leq 2$, we can extend K_{n-1} to $K_n - K_{1,t}$ for $2 \leq t \leq n-2$ in $\overline{G}_F$. Thus, F is a $(P_3, K_n - K_{1,t})-$arrowing graph and $r^*(P_3, K_n - K_{1,t}) \leq 2(n-2)^2$ for $2 \leq t \leq n-2$. □

The restricted size Ramsey number $r^*(P_3, H)$ for $H = K_n - K_{1,1}$ is given in Theorem 3.3. We use K_2 in terms of $K_{1,1}$.

Theorem 3.3. *For $n \geq 4$,*

$$r^*(P_3, K_n - K_2) = 2(n-2)^2 + 1.$$

Proof. Note that $K_{n-1} \subsetneq K_n - K_2$. Since $\beta(\overline{K_n - K_2}) = 1$, Theorem 2.4 implies $r(P_3, K_n - K_2) = 2n-3$. Note again that the Ramsey number $r(P_3, K_{n-1}) = 2n-3$ too. For the lower bound, we consider all graphs F with $v(F) = 2n-3$ and $e(F) = 2(n-2)^2$. However, if F is a $(P_3, K_{n-1})-$arrowing graph, Lemma 2.3 implies $\delta(F) \geq 2n-5$. The only graph satisfies the above conditions is $F = K_{2n-3} - M$, with M a maximal matching in K_{2n-3}. Take a red-blue coloring of all edges in F not containing a red P_3 such that $G_F \cong P_{2n-3}$. According to Lemma 2.2, F is a $(P_3, K_{n-1})-$arrowing graph. But, each vertex which does not belong to subgraph K_{n-1} in $\overline{G}_F$ is adjacent to exactly two vertices that induced K_{n-1} in $\overline{G}_F$. It means we cannot extend K_{n-1} to $K_n - K_2$ in $\overline{G}_F$. Thus, F is not a $(P_3, K_n - P_2)-$arrowing graph and $r^*(P_3, K_n - P_2) \geq 2(n-2)^2 + 1$.

For the upper bound, let $F = K_{2n-3} - (|M| - 1)K_2$ with M a maximal matching in K_{2n-3}. Note that $e(F) = 2(n-2)^2 + 1$. We will show that F is a $(P_3, K_n - K_2)-$arrowing graph. According to Lemma 2.2, $K_{2n-3} - M$ is a $(P_3, K_{n-1})-$arrowing graph. Since $K_{2n-3} - M \subseteq F$, Observation 3.1 implies F is also a $(P_3, K_{n-1})-$arrowing graph. Consider any red-blue coloring ϕ of all edges in F not containing a red P_3. The graph G_F will consists of even cycles and paths with at least one path of even order. Suppose V' is the set of vertices that induces a K_{n-1} in $\overline{G}_F$. Since there is a path of even order in G_F, there must be at least one vertex $v \in V \backslash V'$ that is adjacent to exactly one vertex $v' \in V'$. It means we can extend K_{n-1} to have a $K_n - K_2$ in $\overline{G}_F$. Thus, F is a $(P_3, K_n - P_2)-$arrowing graph and $r^*(P_3, K_n - K_2) \leq 2(n-2)^2 + 1$. □

The next result is $r^*(P_3, H)$ with H a connected graph obtained by deleting some edges incident to two vertices in K_n. First, we consider collection of graphs obtained by deleting edges in $K_{1,s} \cup K_{1,t}$ for $2 \leq s \leq n-2$ and $3 \leq t \leq n-2$, as given in Theorem 3.4. When $1 \leq s, t \leq 2$, the graph $K_{1,s} \cup K_{1,t}$ is one of $2P_3$, $P_3 \cup K_2$, P_4, $2K_2$, P_3, or K_2. The values of $r^*(P_3, H)$ with H either $K_n - P_3$ or $K_n - K_2$ already include in Theorems 3.2 and 3.3. Theorems 3.5 and 3.6 give $r^*(P_3, H)$ for H a graph obtained by deleting edges in $2P_3$, $P_3 \cup K_2$, P_4, or $2K_2$ from K_n.

Theorem 3.4. *Let n, s, t be integers with $2 \le s \le n-2$ and $3 \le t \le n-2$. For $n \ge 5$,*

$$r^*(P_3, K_n - (K_{1,s} \cup K_{1,t})) = 2(n-3)^2.$$

Proof. Note that $K_{n-2} \subseteq K_n - (K_{1,s} \cup K_{1,t})$ for $2 \le s \le n-2$ and $3 \le t \le n-2$. Since $\beta(\overline{K_n - (K_{1,s} \cup K_{1,t})}) = 2$, Theorem 2.4 implies $r(P_3, K_n - (K_{1,s} \cup K_{1,t})) = 2n-5$. Note that the Ramsey number $r(P_3, K_{n-2}) = 2n-5$ too. From Theorem 2.1 we have $r^*(P_3, K_{n-2}) = 2(n-3)^2$. Since $K_{n-2} \subseteq K_n - (K_{1,s} \cup K_{1,t})$, by (3) we get $r^*(P_3, K_n - (K_{1,s} \cup K_{1,t})) \ge 2(n-3)^2$ for $2 \le s \le n-2$ and $3 \le t \le n-2$. This completes the proof for the lower bound.

For the upper bound, let $F = K_{2n-5} - M$ with M a maximal matching in K_{2n-5}. Note that $e(F) = 2(n-3)^2$. Lemma 2.2 implies F is a $(P_3, K_{n-2})-$arrowing graph. Consider any red-blue coloring of all edges of F not containing a red P_3. The graph G_F will consist of even cycles and paths. It means $\Delta(G_F) \le 2$. As a consequence, we can extend the subgraph K_{n-2} to have a subgraph $K_n - (K_{1,s} \cup K_{1,t})$ for $2 \le s \le n-2$ and $3 \le t \le n-2$ in $\overline{G}_F$. Therefore, F is a $(P_3, K_n - (K_{1,s} \cup K_{1,t}))-$arrowing graph and $r^*(P_3, K_n - (K_{1,s} \cup K_{1,t})) \le 2(n-3)^2$ for $2 \le s \le n-2$ and $3 \le t \le n-2$. □

Theorem 3.5. *For $n \ge 5$,*

$$r^*(P_3, K_n - 2P_3) = r^*(P_3, K_n - (P_3 \cup K_2)) = 2(n-3)^2 + 1.$$

Proof. Note that $K_{n-2} \subseteq K_n - 2P_3 \subseteq K_n - (P_3 \cup K_2)$. Since $\beta(\overline{K_n - 2P_3}) = \beta(\overline{K_n - (P_3 \cup K_2)}) = 2$, Theorem 2.4 implies $r(P_3, K_n - 2P_3) = r(P_3, K_n - (P_3 \cup K_2)) = 2n-5$. Note that the Ramsey number $r(P_3, K_{n-2}) = 2n-5$ too. For the lower bound, we consider all graphs F with $v(F) = 2n-5$ and $e(F) = 2(n-3)^2$. However, if F is a $(P_3, K_{n-2})-$arrowing graph, then Lemma 2.3 implies $\delta(F) \ge 2n-7$. The only graph satisfies the above conditions is $F = K_{2n-5} - M$, with M a maximal matching in K_{2n-5}. Take a red-blue coloring ϕ of all edges in F such that $G_F \cong P_{2n-5}$. According to Lemma 2.2, F is a $(P_3, K_{n-2})-$arrowing graph. Suppose V' is the set of vertices that induces K_{n-2} in $\overline{G}_F$. Since $G_F \cong P_{2n-5}$, each vertex $v \in V \backslash V'$ is adjacent to exactly two vertices that belong to V' in G_F. It means we cannot extend the subgraph K_{n-2} to have a subgraph $K_n - 2P_3$ in $\overline{G}_F$. As a consequence, $r^*(P_3, K_n - 2P_3) \ge 2(n-3)^2 + 1$. Since $K_n - 2P_3 \subseteq K_n - (P_3 \cup K_2)$, by (3), $r^*(P_3, K_n - (P_3 \cup K_2)) \ge 2(n-3)^2 + 1$.

For the upper bound, let $F = K_{2n-5} - (|M| - 1)K_2$ with M a maximal matching in K_{2n-5}. Note that $e(F) = 2(n-3)^2 + 1$. We will show that F is a $(P_3, K_n - (P_3 \cup K_2))-$ arrowing graph. According to Lemma 2.2, $K_{2n-3} - M$ is a $(P_3, K_{n-2})-$arrowing graph. Since $K_{2n-5} - M \subseteq F$, Observation 3.1 implies F is also a $(P_3, K_{n-2})-$arrowing graph. Consider any red-blue coloring ϕ of all edges in F not containing a red P_3. The graph G_F will consists of even cycles and paths with at least one path of even order. Suppose V' is the set of vertices that induces a K_{n-2} in $\overline{G}_F$. Since there is a path of even order in G_F, there must be at least one vertex $v \in V \backslash V'$ that adjacent to exactly one vertex $v' \in V'$ in G_F. It means we can extend the subgraph K_{n-2} to have a subgraph $K_n - 2K_2$ in $\overline{G}_F$. As a consequence, F is a $(P_3, K_n - (P_3 \cup K_2))-$ arrowing graph and $r^*(P_3, K_n - (P_3 \cup K_2)) \leq 2(n-3)^2 + 1$. Since $K_n - 2P_3 \subseteq K_n - (P_3 \cup K_2)$, by (3), $r^*(P_3, K_n - 2P_3) \leq 2(n-3)^2 + 1$. □

Theorem 3.6. *For $n \geq 5$,*

$$r^*(P_3, K_n - P_4) = r^*(P_3, K_n - 2K_2) = 2(n-3)^2 + 2.$$

Proof. Note that $K_{n-2} \subseteq K_n - P_4 \subseteq K_n - 2K_2$. Since $\beta(\overline{K_n - P_4}) = \beta(\overline{K_n - 2K_2}) = 2$, Theorem 2.4 implies $r(P_3, K_n - P_4) = r(P_3, K_n - 2K_2) = 2n - 5$. The Ramsey number $r(P_3, K_{n-2}) = 2n-5$ too. For the lower bound, we consider all graphs F with $v(F) = 2n-5$ and $e(F) = 2(n-3)^2 + 1$. However, if F is a $(P_3, K_{n-2})-$arrowing graph, then Lemma 2.3 implies $\delta(F) \geq 2n - 7$. The only graph satisfies the above conditions is $F = K_{2n-5} - (|M| - 1)K_2$, with M a maximal matching in K_{2n-5}. Take a red-blue coloring ϕ of all edges in F such that $G_F \cong P_{2n-6} \cup K_1$. According to Lemma 2.2, $K_{2n-5} - M$ is a $(P_3, K_{n-2})-$arrowing graph and according to Observation 3.1, F is also a $(P_3, K_{n-2})-$arrowing graph. Suppose V' is the set of vertices that induces a K_{n-2} in $\overline{G}_F$. Since G_F is $P_{2n-6} \cup K_1$, there must be a leaf v of P_{2n-6} such that $v \notin V'$ which is adjacent to exactly one vertex $v' \in V'$ in G_F. It means we cannot extend the subgraph K_{n-2} to have a subgraph $K_n - P_4$ in $\overline{G}_F$. Thus F is not a $(P_3, K_n - P_4)-$arrowing graph and $r^*(P_3, K_n - P_4) \geq 2(n-3)^2 + 2$. Since $K_n - P_4 \subseteq K_n - 2K_2$, by (3) we have $r^*(P_3, K_n - 2K_2) \geq 2(n-3)^2 + 2$.

For the upper bound, let $F = K_{2n-5} - (|M| - 2)K_2$ with M a maximal matching in K_{2n-5}. Note that $e(F) = 2(n-3)^2 + 2$. We will show that F is a $(P_3, K_n - 2K_2)-$arrowing graph. According to Lemma 2.2, $K_{2n-5} - M$ is a $(P_3, K_{n-2})-$arrowing graph and according to Observation 3.1, F is also a $(P_3, K_{n-2})-$arrowing graph. Consider any red-blue coloring ϕ of all edges in F not containing a red P_3. The graph G_F will consists of even cycles and paths with at least two path of even order. Suppose V' is the set of vertices that induced K_{n-2} in the $\overline{G}_F$. Since there are two path of even order in G_F, there must be at least two vertices $u, v \notin V'$ adjacent to exactly a vertex $v' \in V'$ in G_F. It means we can extend the subgraph K_{n-2} to have a subgraph $K_n - 2K_2$ in $\overline{G}_F$. As a consequence, $r^*(P_3, K_n - 2K_2) \leq 2(n-3)^2 + 2$. Since $K_n - P_4 \subseteq K_n - 2K_2$, by (3) we have $r^*(P_3, K_n - P_4) \leq 2(n-3)^2 + 2$. □

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