Electronic Journal of Graph Theory and Applications **14** (1) (2026), 153–178

Electronic Journal of
Graph Theory and Applications

Odd order C_4-face-magic projective grid graphs

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Abstract

For a graph $G = (V, E)$ embedded in the projective plane, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic projective graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels around C_n is a constant S. We consider the $m \times n$ grid graph, denoted by $\mathcal{P}_{m,n}$, embedded in the projective plane in the natural way.

Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. It is known that the C_4-face-magic value of a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ is either $2mn + 1$, $2mn + 2$, or $2mn + 3$. The characterization of C_4-face-magic labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$ is known. In this paper, we determine a category of C_4-face-magic labelings on $\mathcal{P}_{m,n}$ for which the C_4-face-magic value is either $2mn + 1$ or $2mn + 3$. It is conjectured that these are the only C_4-face-magic labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 1$ or $2mn + 3$.

Keywords: C_4-face-magic graphs, polyomino, projective grid graphs
2010 Mathematics Subject Classification : 05C78
DOI: 10.5614/ejgta.2026.14.1.10

1. Introduction

Kotzig and Rosa introduced graph labelings [14] in 1970. Graph labelings have a number of interesting applications such as radar pulse code designs, communication network models, X-ray

Received: 28 June 2021, Revised: 13 September 2025, Accepted: 20 January 2026.

crystallography, and graph decomposition problems. We direct readers to J. A. Gallian's dynamic survey on graph labelings [11] for further inquiry.

Readers should consult Chartrand, Lesniak, and Zhang [5] for concepts and notation not explicitly defined in this paper. We investigate C_4-face-magic labelings on the projective grid graph in this paper. Let $G = (V, E)$ be a planar (toroidal, Klein bottle, projective) graph, and let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a *C_n-face-magic planar (toroidal, Klein bottle, projective)* graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labels around C_n is a constant S. We call the constant S a *C_n-face-magic value* of G. More generally, C_n-face-magic planar graph labelings are a special case of (a, b, c)-magic labelings introduced by Lih [15]. For $a, b, c \in \{0, 1\}$, an (a, b, c)-magic labeling is a bijective assignment of the elements of the set $\{1, 2, ..., a|V| + b|E| + c|F|\}$ to the vertices, edges, and faces of $G = (V, E, F)$ that assigns a labels to each vertex, b labels to each edge, and c labels to each face such that the sum of labels for each face (including vertices, edges, and the face itself) is constant. For assorted values of a, b and c, Baca and others [1, 2, 3, 12, 13, 15] have analyzed the problem for various classes of graphs. Wang [16] showed that the toroidal grid graphs $C_m \times C_n$ are antimagic for all integers $m, n \geqslant 3$. Recall that an antimagic labeling is a bijective assignment of the set $\{1, 2, \ldots, |E|\}$ onto the edges of G such that, for every vertex, the sum of the labels on edges incident to the vertex is unique. Butt et al. [4] investigated face antimagic labelings on toroidal and Klein bottle grid graphs; they found bijective assignments of the elements from the set $\{1, 2, \ldots, |V| + |E| + |F|\}$ onto $V \cup E \cup F$ such that the induced face labels, formed by the sum of the labels on all vertices and edges incident to the face and the label on the face itself, produce arithmetic sequences with various common differences.

Curran, Low and Locke [6, 7] examined C_4-face-magic toroidal labelings on an $m \times n$ toroidal grid graph. They showed that $C_m \times C_n$ admits a C_4-face-magic toroidal labeling if and only if either $m = 2$, or $n = 2$, or both m and n are even. Curran, Low and Locke [8] also investigated C_4-face-magic Klein bottle labelings on an $m \times n$ Klein bottle grid graph. They showed that an $m \times n$ Klein bottle grid graph admits a C_4-face-magic Klein bottle labeling if and only if n is even.

Curran [9] showed that an $m \times n$ projective grid graph admits a C_4-face-magic projective labeling if and only if either both m and n are even, or both m and n are odd. On one hand, when m and n are even, the C_4-face-magic value of a C_4-face-magic labeling on an $m \times n$ projective grid graph is $2mn + 2$. On the other hand, when m and n are odd, the C_4-face-magic value of a C_4-face-magic labeling on an $m \times n$ projective grid graph is either $2mn+1$, $2mn+2$, or $2mn+3$. When m and n are odd, the C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$ were characterized in [9]. In this paper, we determine a category of the C_4-face-magic labelings on $\mathcal{P}_{m,n}$, for m and n odd, having C_4-face-magic value $2mn + 1$ or $2mn + 3$. We conjecture that these are the only C_4-face-magic labelings on $\mathcal{P}_{m,n}$, for m and n odd, having C_4-face-magic value $2mn + 1$ or $2mn + 3$.

2. Preliminaries

In this section we introduce definitions and known results about C_4-face-magic projective labelings on an $m \times n$ projective grid graph.

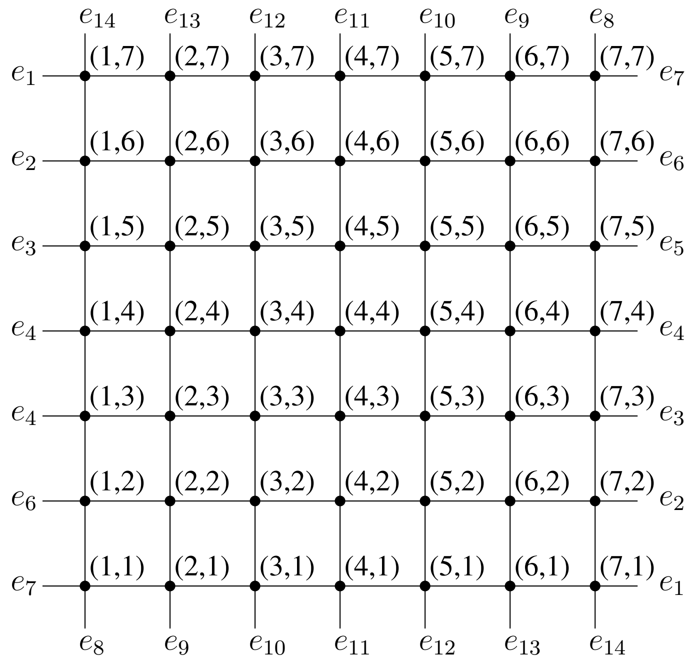

Figure 1. 7×7 projective grid graph $\mathcal{P}_{7,7}$.

Definition 2.1. *Suppose m and n are integers such that $m, n \geqslant 2$. The $m \times n$ projective grid graph, denoted by $\mathcal{P}_{m,n}$, is the graph with vertex set given by*

$$V(\mathcal{P}_{m,n}) = \{(i,j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\},$$

and with edge set consisting of the following edges:

- *there is an edge from (i,j) to $(i+1,j)$, for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n$,*
- *there is an edge from (m,j) to $(1,n+1-j)$, for $1 \leqslant j \leqslant n$,*
- *there is an edge from (i,j) to $(i,j+1)$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$, and*
- *there is an edge from (i,n) to $(m+1-i,1)$, for $1 \leqslant i \leqslant m$.*

The graph $\mathcal{P}_{m,n}$ has a natural embedding in the projective plane. Since there are double edges on the vertex sets $\{(1,1),(m,n)\}$ and $\{(m,1),(1,n)\}$, $\mathcal{P}_{m,n}$ is a multigraph.

Example 2.1. The 7×7 projective grid graph $\mathcal{P}_{7,7}$ is illustrated in Figure 1. Due to the orientation of the vertices in $\mathcal{P}_{m,n}$, we refer to the vertices $\{(i,j) : 1 \leqslant j \leqslant n\}$ as column i of $V(\mathcal{P}_{m,n})$ and $\{(i,j) : 1 \leqslant i \leqslant m\}$ as row j of $V(\mathcal{P}_{m,n})$.

The following result determines when $\mathcal{P}_{m,n}$ admits a C_4-face-magic projective labeling.

Lemma 2.1 ([9], Lemma 2.4)**.** *Suppose m and n are integers such that $m, n \geqslant 2$. Then $\mathcal{P}_{m,n}$ admits a C_4-face-magic projective labeling if and only if m and n have the same parity.*

Furthermore, one can determine the possible C_4-face-magic values of a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. The next lemma determines the C_4-face-magic value of a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ when m and n are even integers.

Lemma 2.2 ([9], Lemma 2.5)**.** *Suppose $m \geqslant 2$ and $n \geqslant 2$ are even integers. Let $\{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Then $S = 2mn + 2$.*

In fact, when odd integers m and n, one can determine the digon face sum values on the digon vertex sets $\{(1,1),(m,n)\}$ and $\{(m1),(1,n)\}$ of $\mathcal{P}_{m,n}$.

Lemma 2.3 ([9], Lemma 2.6)**.** *Suppose $m \geqslant 3$ and $n \geqslant 3$ are odd integers. Let $\{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Let $D_1 = x_{1,1} + x_{m,n}$ and $D_2 = x_{m,1} + x_{1,n}$ be the face sums of the two digons constructed from the pair of vertices at opposite corners of $\mathcal{P}_{m,n}$. Then either*

1. $S = 2mn + 1$ *and* $D_1 = D_2 = \frac{3}{2}mn + \frac{1}{2}$,
2. $S = 2mn + 2$ *and* $D_1 = D_2 = mn + 1$, *or*
3. $S = 2mn + 3$ *and* $D_1 = D_2 = \frac{1}{2}mn + \frac{3}{2}$.

Example 2.2. Figure 2 illustrates a C_4-face-magic projective labeling on the 7×7 projective grid graph $\mathcal{P}_{7,7}$ such that the C_4-face-magic value is 101 and the digon face sum is $D_1 = D_2 = 26$.

For odd integers $m \geqslant 3$ and $n \geqslant 3$, the C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+2$ were characterized in [9]. The statement of this characterization involves several technical definitions. So, we refer the reader to [9] for the details of this characterization. However, it is relatively easy to count the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$. We need the following definition in order to state this result.

Definition 2.2. *Suppose there exists a positive integer k such that one of the two following conditions hold.*

1. *There are factorizations of $m = m_1 m_2 \dots m_k$ and $n = n_1 n_2 \dots n_k$, where $m_i > 1$ and $n_i > 1$ for all $1 \leqslant i \leqslant k$.*
2. *There are factorizations of $m = m'_1 m'_2 \dots m'_k m'_{k+1}$ and $n = n'_1 n'_2 \dots n'_k$, where $m'_i > 1$ for all $1 \leqslant i \leqslant k+1$, and $n'_i > 1$ for all $1 \leqslant i \leqslant k$.*

We say that $(m_1, n_1, m_2, n_2, \dots, m_k, n_k)$ is an (m,n)-projective factorization sequence of length $2k$. Also, we say $(m'_1, n'_1, m'_2, n'_2, \dots, m'_k, n'_k, m'_{k+1})$ is an (m,n)-projective factorization sequence of length $2k+1$. For convenience, we let $n'_{k+1} = 1$ and refer to $(m'_1, n'_1, m'_2, n'_2, \dots, m'_{k+1}, n'_{k+1})$ as an (m,n)-projective factorization sequence of length $2k+1$. In addition, we say that $(m_1, n_1, m_2, n_2, \dots, m_k, n_k)$ and $(m'_1, n'_1, m'_2, n'_2, \dots, m'_{k+1}, n'_{k+1})$ are (m,n)-projective factorization sequences.

Furthermore, we let $\tau(m,n)$ denote the number of distinct (m,n)-projective factorization sequences.

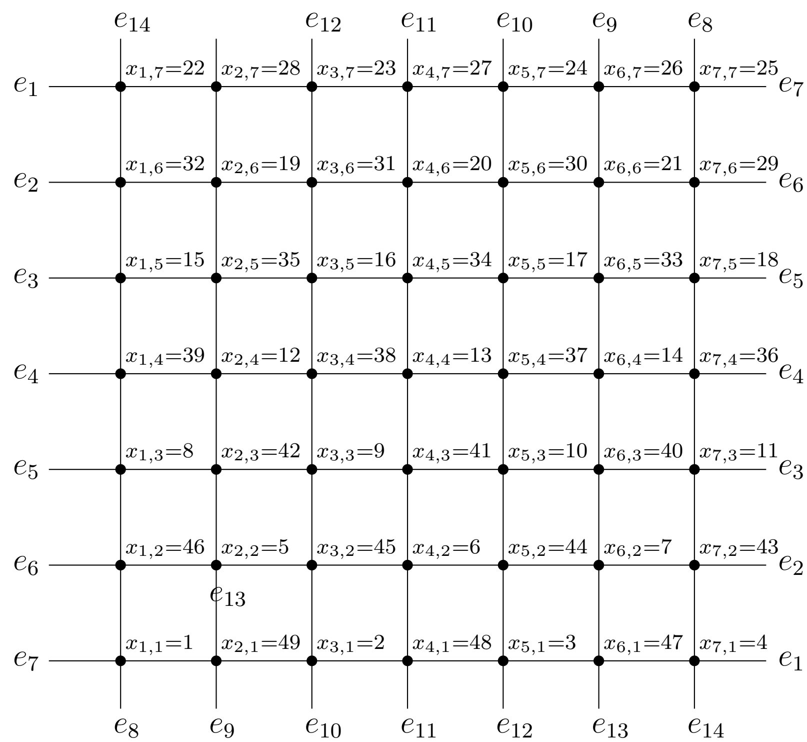

Figure 2. C_4-face-magic projective labeling on $\mathcal{P}_{7,7}$ having C_4-face-magic value 101.

The next result provides the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$ for distinct odd integers m and n.

Theorem 2.1 ([9], Theorem 3.40)**.** *Let $m \geqslant 3$ and $n \geqslant 3$ be distinct odd integers. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn + 2$ (up to symmetries on the projective plane) is*

$$\big(\tau(m,n) + \tau(n,m)\big)\, 2^{m/2+n/2-3}\, (\tfrac{m-1}{2})!(\tfrac{n-1}{2})!.$$

The number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ having C_4-face-magic value $2m^2 + 2$ for an odd integer m is stated below.

Theorem 2.2 ([9], Theorem 3.41)**.** *Let $m \geqslant 3$ be an odd integer. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ having C_4-face-magic value $2m^2 + 2$ (up to symmetries on the projective plane) is*

$$\tau(m,m)\, 2^{m-3} \big((\tfrac{m-1}{2})!\big)^2.$$

We now direct our attention to C_4-face-magic labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn + 1$ or $2mn + 3$.

Definition 2.3. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. We define a labeling Y on $\mathcal{P}_{m,n}$ given by*

$$y_{i,j} = mn + 1 - x_{i,j} \quad \textit{for all } (i,j) \in V(\mathcal{P}_{m,n}).$$

We say that Y is the order plus one complement of X labeling on $\mathcal{P}_{m,n}$. We call the transformation $\mathcal{C}(X) = Y$ the order plus one complement labeling transformation.

Proposition 2.1. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value S. Then the order plus one complement of X labeling $Y = \mathcal{C}(X)$ on $\mathcal{P}_{m,n}$ is a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value $4mn + 4 - S$.*

Proof. Let (i_1, j_1), (i_2, j_2), (i_3, j_3), and (i_4, j_4) be the vertices of any C_4-face on $\mathcal{P}_{m,n}$. Then

$$\sum_{k=1}^{4} x_{i_k,j_k} = S.$$

Thus, the order plus one complement of X labeling $Y = \mathcal{C}(X)$ satisfies

$$\sum_{k=1}^{4} y_{i_k,j_k} = \sum_{k=1}^{4} (mn + 1 - x_{i_k,j_k}) = 4mn + 4 - S.$$

Since $h : \{1, 2, \ldots, mn\} \to \{1, 2, \ldots, mn\}$ defined by $h(x) = mn + 1 - x$ is a bijection, so is $f : V(P_m \times P_n) \to \{1, 2, \ldots, mn\}$ defined by $f(i,j) = h(x_{i,j})$. □

Remark 2.1. We observe that $\mathcal{C}(\mathcal{C}(X)) = X$ for all C_4-face-magic projective labelings X on $\mathcal{P}_{m,n}$. Thus, $\mathcal{C}$ is a one-to-one correspondence between C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value S and C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $4mn + 4 - S$.

Hence, for odd integers $m \geqslant 3$ and $n \geqslant 3$, $\mathcal{C}$ is a one-to-one correspondence between C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn + 1$ and those with C_4-face-magic value $2mn + 3$.

3. Bicentrally Balanced C_4-face-magic Labelings on $\mathcal{P}_{m,n}$

Notation 3.1. Throughout this section, we assume that both $m \geqslant 3$ and $n \geqslant 3$ are odd integers. We write $m = 2m_0 + 1$ and $n = 2n_0 + 1$ for positive integers m_0 and n_0. For any positive integer N, we let $N^+ = N + 1$. In particular, we have $m_0^+ = m_0 + 1$ and $n_0^+ = n_0 + 1$.

Notation 3.2. We refer to the vertex $(\frac{1}{2}(m+1), \frac{1}{2}(n+1)) = (m_0^+, n_0^+)$ as the *center* of the projective grid graph $\mathcal{P}_{m,n}$. The graph automorphisms of $\mathcal{P}_{m,n}$ that are induced by a homeomorphism of the projective plane are described in relation to the center of $\mathcal{P}_{m,n}$. We let R_θ denote the rotation by θ degrees in the counter-clockwise direction about the center. The symmetry H (V) is the reflection about the center row (column). Since the corner vertices of $\mathcal{P}_{m,n}$ are the only vertices incident to a double edge, a symmetry of $\mathcal{P}_{m,n}$ sends each corner vertex to another corner vertex. Thus, for distinct integers m and n, the set of symmetries on $\mathcal{P}_{m,n}$ is $\{R_0, R_{180}, H, V\}$. We let D_+ (D_-) denote the reflection about the diagonal with positive (negative) slope passing through the center. When $m = n$, the set of symmetries on $\mathcal{P}_{m,m}$ is $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D_+, D_-\}$.

Definition 3.1. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face value $S = 2mn + 3$. For all $(i,j) \in V(\mathcal{P}_{m,n})$, let*

$$S(i,j) = \tfrac{1}{2}mn + \tfrac{3}{2} \text{ if } i+j \text{ is even, and}$$
$$S(i,j) = \tfrac{3}{2}mn + \tfrac{3}{2} \text{ if } i+j \text{ is odd.}$$

We say that X is bicentrally balanced *if, for all $(i,j) \in V(\mathcal{P}_{m,n})$,*

$$x_{i,j} + x_{m+1-i,n+1-j} = S(i,j).$$

Remark 3.1. Suppose $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ is a C_4-face-magic labeling on $\mathcal{P}_{m,n}$ that is bicentrally balanced. Then the C_4-face-magic value S of X is

$$S = x_{1,1} + x_{m,n} + x_{2,1} + x_{m-1,n} = S(1,1) + S(2,1) = 2mn + 3.$$

We observe that the C_4-face-magic projective labeling on $\mathcal{P}_{5,5}$ in Figure 2 is bicentrally balanced.

Lemma 3.1. *Suppose $m \geqslant 3$ and $n \geqslant 3$ are odd integers. Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value $S = 2mn + 3$. Then X is bicentrally balanced. Furthermore, we have $x_{m_0^+, n_0^+} = \frac{1}{2}S(m_0^+, n_0^+)$. Thus, $x_{m_0^+, n_0^+} = \frac{1}{4}mn + \frac{3}{4}$ if $m_0^+ + n_0^+$ is even, or $x_{m_0^+, n_0^+} = \frac{3}{4}mn + \frac{3}{4}$ if $m_0^+ + n_0^+$ is odd.*

Proof. By Lemma 2.3, the digons formed by the vertex sets $\{(1,1),(m,n)\}$ and $\{(m,1),(1,n)\}$ have face values

$$D_1 = x_{1,1} + x_{m,n} = \tfrac{1}{2}mn + \tfrac{3}{2} \text{ and}$$
$$D_2 = x_{m,1} + x_{1,n} = \tfrac{1}{2}mn + \tfrac{3}{2}, \text{ respectively.}$$

Suppose that for some integer $1 \leqslant i < m$,

$$x_{i,1} + x_{m+1-i,n} = S(i,1).$$

See Definition 3.1 for the definition of $S(i,j)$. Since

$$x_{i,1} + x_{i+1,1} + x_{m+1-i,n} + x_{m-i,n} = S,$$

we have

$$x_{i+1,1} + x_{m-i,n} = S - S(i,1) = S(i+1,1).$$

Hence,

$$x_{i,1} + x_{m+1-i,n} = S(i,1)$$

for all $1 \leqslant i \leqslant m$. A similar argument shows that

$$x_{1,j} + x_{m,n+1-j} = S(1,j)$$

for all $1 \leqslant j \leqslant n$.

We use induction to show that $x_{i,j} + x_{m+1-i,n+1-j} = S(i,j)$ for all $(i,j) \in V\mathcal{P}_{m,n})$. Suppose there exist integers $1 < i < m$ and $1 < j < n$ such that

1. for all $1 \leqslant i' < i$ and $1 \leqslant j' \leqslant n$, $x_{i',j'} + x_{m+1-i',n+1-j'} = S(i',j')$, and
2. for all $1 \leqslant j' < j$, $x_{i,j'} + x_{m+1-i,n+1-j'} = S(i,j')$.

We need to show that $x_{i,j} + x_{m+1-i,n+1-j} = S(i,j)$. When we add the two C_4-face-values

$$x_{i-1,j-1} + x_{i-1,j} + x_{i,j-1} + x_{i,j} = S \text{ and}$$
$$x_{m+2-i,n+2-j} + x_{m+2-i,n+1-j} + x_{m+1-i,n+2-j} + x_{m+1-i,n+1-j} = S,$$

we obtain

$$\begin{aligned}&(x_{i-1,j-1} + x_{m+2-i,n+2-j}) + (x_{i-1,j} + x_{m+2-i,n+1-j})\\ &+(x_{i,j-1} + x_{m+1-i,n+2-j}) + (x_{i,j} + x_{m+1-i,n+1-j}) = 2S.\end{aligned}$$

Since

$$\begin{aligned}x_{i-1,j-1} + x_{m+2-i,n+2-j} &= S(i-1,j-1),\\ x_{i-1,j} + x_{m+2-i,n+1-j} &= S(i-1,j), \text{ and}\\ x_{i,j-1} + x_{m+1-i,n+2-j} &= S(i,j-1),\end{aligned}$$

we have

$$S(i-1,j-1) + S(i-1,j) + S(i,j-1) + (x_{i,j} + x_{m+1-i,n+1-j}) = 2S.$$

Thus

$$x_{i,j} + x_{m+1-i,n+1-j} = S(i,j).$$

Since

$$2x_{m_0^+,n_0^+} = x_{m_0^+,n_0^+} + x_{m+1-m_0^+,n+1-n_0^+} = S(m_0^+,n_0^+),$$

we have

$$x_{m_0^+,n_0^+} = \tfrac{1}{2}S(m_0^+,n_0^+).$$

Thus, $x_{m_0^+,n_0^+} = \frac{1}{4}mn + \frac{3}{4}$ if $m_0^+ + n_0^+$ is even, or $x_{m_0^+,n_0^+} = \frac{3}{4}mn + \frac{3}{4}$ if $m_0^+ + n_0^+$ is odd. □

3.1. Structure of a Bicentrally Balanced C_4-face-magic Labeling

Lemma 3.2. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. For $1 \leqslant j \leqslant n_0$, let $a_j = x_{1,j} + x_{1,j+1}$. Then,*

1. *for all $1 \leqslant i \leqslant m_0$ where i is odd, and $1 \leqslant j \leqslant n_0$, we have*

$$x_{i,j} + x_{i,j+1} = a_j, \qquad x_{i,n+1-j} + x_{i,n-j} = S - a_j,$$
$$x_{m+1-i,j} + x_{m+1-i,j+1} = a_j, \textit{and} \qquad x_{m+1-i,n+1-j} + x_{m+1-i,n-j} = S - a_j, \quad \textit{and}$$

2. *for all $1 \leqslant i \leqslant m_0$ where i is even, and $1 \leqslant j \leqslant n_0$, we have*

$$x_{i,j} + x_{i,j+1} = S - a_j, \qquad x_{i,n+1-j} + x_{i,n-j} = a_j,$$
$$x_{m+1-i,j} + x_{m+1-i,j+1} = S - a_j, \textit{and} \qquad x_{m+1-i,n+1-j} + x_{m+1-i,n-j} = a_j.$$

Proof. When we equate the two C_4-face sums

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S \text{ and}$$
$$x_{i+1,j} + x_{i+1,j+1} + x_{i+2,j} + x_{i+2,j+1} = S,$$

we obtain

$$x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}. \tag{1}$$

By (1), for all $1 \leqslant i \leqslant m_0$ where i is odd, and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,j} + x_{i,j+1} = a_j \text{ and } x_{m+1-i,j} + x_{m+1-i,j+1} = a_j.$$

Since

$$a_j + x_{2,j} + x_{2,j+1} = x_{1,j} + x_{1,j+1} + x_{2,j} + x_{2,j+1} = S,$$

we have

$$x_{2,j} + x_{2,j+1} = S - a_j$$

for all $1 \leqslant j \leqslant n_0$. Also by (1), for all $1 \leqslant i \leqslant m_0$ where i is even, and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,j} + x_{i,j+1} = S - a_j \text{ and } x_{m+1-i,j} + x_{m+1-i,j+1} = S - a_j.$$

Since

$$a_j + x_{1,n+1-j} + x_{1,n-j} = x_{m,j} + x_{m,j+1} + x_{1,n+1-j} + x_{1,n-j} = S,$$

we have

$$x_{1,n+1-j} + x_{1,n-j} = S - a_j$$

for all $1 \leqslant j \leqslant n_0$. Thus, by (1), for all $1 \leqslant i \leqslant m_0$ where i is odd, and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,n+1-j} + x_{i,n-j} = S - a_j \text{ and } x_{m+1-i,n+1-j} + x_{m+1-i,n-j} = S - a_j.$$

Since

$$(S - a_j) + x_{2,n+1-j} + x_{2,n-j} = x_{1,n+1-j} + x_{1,n-j} + x_{2,n+1-j} + x_{2,n-j} = S,$$

we have

$$x_{2,n+1-j} + x_{2,n-j} = a_j$$

for all $1 \leqslant j \leqslant n_0$. Hence, by (1), for all $1 \leqslant i \leqslant m_0$ where i is even, and $1 \leqslant j \leqslant n_0$, we have

$$x_{i,n+1-j} + x_{i,n-j} = a_j \text{ and } x_{m+1-i,n+1-j} + x_{m+1-i,n-j} = a_j.$$

□

3.2. Row and Column Permutations on a Bicentrally Balanced Labeling

Definition 3.2. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Let η be a permutation on the set $\{1, 2, \ldots, m_0\}$ such that $\eta(i) \equiv i \pmod 2$ for all $1 \leqslant i \leqslant m_0$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n$, we have*

$$\begin{aligned} z_{i,j} &= x_{\eta(i),j}, \\ z_{m_0^+,j} &= x_{m_0^+,j}, \text{ and} \\ z_{m+1-i,j} &= x_{m+1-\eta(i),j}. \end{aligned}$$

We let $\mathcal{E}_\eta$ denote the labeling operation given by $\mathcal{E}_\eta(X) = Z$.

Lemma 3.3. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$, and let η be a permutation on the set $\{1, 2, \ldots, m_0\}$ such that $\eta(i) \equiv i \pmod 2$ for all $1 \leqslant i \leqslant m_0$. Let $\mathcal{E}_\eta$ be the labeling operation defined in Definition 3.2. Then the labeling $Z = \mathcal{E}_\eta(X)$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.*

Proof. We first verify that Z is bicentrally balanced. Suppose that $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n$. Since $\eta(i) - i$ is even, we have

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{\eta(i),j} + x_{m+1-\eta(i),n+1-j} = S(i,j).$$

Furthermore, we have

$$z_{m_0^+,j} + z_{m+1-m_0^+,n+1-j} = x_{m_0^+,j} + x_{m+1-m_0^+,n+1-j} = S(m_0^+, j).$$

Next, we show that Z is a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. For all $1 \leqslant i < m$ and $1 \leqslant j < n$, by Lemma 3.2, we have

$$z_{i,j} + z_{i,j+1} = x_{i,j} + x_{i,j+1} \text{ and } z_{i+1,j} + z_{i+1,j+1} = x_{i+1,j} + x_{i+1,j+1}.$$

Thus

$$z_{i,j} + z_{i,j+1} + z_{i+1,j} + z_{i+1,j+1} = x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S.$$

Since Z is bicentrally balanced, for $1 \leqslant i < m$, we have

$$z_{i,n} + z_{m+1-i,1} + z_{i+1,n} + z_{m-i,1} = S(i,n) + S(i+1,n) = S.$$

Also, since Z is bicentrally balanced, for $1 \leqslant j < n$, we have

$$z_{m,j} + z_{1,n+1-j} + z_{m,j+1} + z_{1,n-j} = S(m,j) + S(m,j+1) = S.$$

$\square$

Definition 3.3. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Let κ be a permutation on the set $\{1,2,\ldots,n_0\}$ such that $\kappa(j) \equiv j \pmod 2$ for all $1 \leqslant j \leqslant n_0$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_0$, we have*

$$z_{i,j} = x_{i,\kappa(j)},$$
$$z_{i,n_0^+} = x_{i,n_0^+}, \text{ and}$$
$$z_{i,n+1-j} = x_{i,n+1-\kappa(j)}.$$

We let $\mathcal{E}_\kappa$ denote the labeling operation given by $\mathcal{E}_\kappa(X) = Z$.

Lemma 3.4. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$, and let κ be a permutation on the set $\{1,2,\ldots,n_0\}$ such that $\kappa(j) \equiv j \pmod 2$ for all $1 \leqslant j \leqslant n_0$. Let $\mathcal{E}_\kappa$ be the labeling operation defined in Definition 3.3. Then the labeling $Z = \mathcal{E}_\kappa(X)$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.*

The proof of Lemma 3.4 is similar to the proof of Lemma 3.3.

Definition 3.4. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Let $\alpha : \{1,2,\ldots,m_0\} \to \{0,1\}$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n$, we have*

$$z_{i,j} = x_{(1-\alpha(i))i+\alpha(i)(m+1-i),j}, \quad \text{and}$$
$$z_{m+1-i,j} = x_{\alpha(i)i+(1-\alpha(i))(m+1-i),j}.$$

We let $\mathcal{E}_\alpha$ denote the labeling operation given by $\mathcal{E}_\alpha(X) = Z$. The labeling operation $\mathcal{E}_\alpha$ has the effect of keeping the labelings on the vertices of columns i and $m+1-i$ the same if $\alpha(i) = 0$ and swapping the labelings on the vertices of column i with those of column $m+1-i$ if $\alpha(i) = 1$.

Lemma 3.5. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$, and let $\alpha : \{1,2,\ldots,m_0\} \to \{0,1\}$. Let $\mathcal{E}_\alpha$ be the labeling operation defined in Definition 3.4. Then the labeling $Z = \mathcal{E}_\alpha(X)$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.*

Proof. First, we show that Z is bicentrally balanced. Suppose $\alpha(i) = 0$. Then

$$z_{i,j} = x_{i,j} \quad \text{and} \quad z_{m+1-i,j} = x_{m+1-i,j}.$$

Thus

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{i,j} + x_{m+1-i,n+1-j} = S(i,j).$$

Suppose $\alpha(i) = 1$. Then

$$z_{i,j} = x_{m+1-i,j} \quad \text{and} \quad z_{m+1-i,j} = x_{i,j}.$$

Thus

$$z_{i,j} + z_{m+1-i,n+1-j} = x_{m+1-i,j} + x_{i,n+1-j} = S(m+1-i,j) = S(i,j)$$

since $m+1-i \equiv i \pmod 2$. The proof that Z is a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn+3$ is similar to that in the proof of Lemma 3.3. $\square$

Definition 3.5. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Let $\delta : \{1,2,\ldots,n_0\} \to \{0,1\}$. We define a labeling on $\mathcal{P}_{m,n}$, $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$, such that for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_0$, we have*

$$z_{i,j} = x_{i,(1-\delta(j))j+\delta(j)(n+1-j)}, \quad \textit{and}$$
$$z_{i,n+1-j} = x_{i,\delta(j)j+(1-\delta(j))(n+1-j)}.$$

We let $\mathcal{E}_\delta$ denote the labeling operation given by $\mathcal{E}_\delta(X) = Z$. The labeling operation $\mathcal{E}_\delta$ has the effect of keeping the labelings on the vertices of rows j and $n+1-j$ the same if $\delta(j) = 0$ and swapping the labelings on the vertices of row j with those of row $n+1-j$ if $\delta(j) = 1$.

Lemma 3.6. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$, and let $\delta : \{1,2,\ldots,n_0\} \to \{0,1\}$. Let $\mathcal{E}_\delta$ be the labeling operation defined in Definition 3.5. Then the labeling $Z = \mathcal{E}_\delta(X)$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.*

The proof of Lemma 3.6 is similar to the proof of Lemma 3.5.

Definition 3.6. *We call each of the labeling operations $\mathcal{E}_\eta$ in Definition 3.2, $\mathcal{E}_\kappa$ in Definition 3.3, $\mathcal{E}_\alpha$ in Definition 3.4, and $\mathcal{E}_\delta$ in Definition 3.5 an elementary projective labeling operation.*

Definition 3.7. *We say that two bicentrally balanced C_4-face-magic labelings on $\mathcal{P}_{m,n}$ are projective labeling equivalent if one labeling can be obtained from the other by applying a sequence of elementary projective labeling operations to it.*

3.3. Standard Bicentrally Balanced Labeling

Given a bicentrally balanced C_4-face-magic projective labeling X on $\mathcal{P}_{m,n}$, the next theorem identifies a canonical bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ that is projective labeling equivalent to X.

Theorem 3.1. *Let $X = \{x_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ be a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Then there is a unique bicentrally balanced C_4-face-magic projective labeling $Z = \{z_{i,j} : (i,j) \in V(\mathcal{P}_{m,n})\}$ on $\mathcal{P}_{m,n}$ that is projective labeling equivalent to X such that*

- $z_{i,n_0^+} < z_{i+2,n_0^+}$ *for all $1 \leqslant i \leqslant m-2$ and $i + n_0^+$ is even,*
- $z_{i,n_0^+} > z_{i+2,n_0^+}$ *for all $1 \leqslant i \leqslant m-2$ and $i + n_0^+$ is odd,*
- $z_{m_0^+,j} < z_{m_0^+,j+2}$ *for all $1 \leqslant j \leqslant n-2$ and $m_0^+ + j$ is even, and*
- $z_{m_0^+,j} > z_{m_0^+,j+2}$ *for all $1 \leqslant j \leqslant n-2$ and $m_0^+ + j$ is odd.*

Proof. By Lemma 3.1, we have $x_{m_0^+,n_0^+} = \frac{1}{2}S(m_0^+, n_0^+)$. It is easy to check that this value remains the same regardless of the elementary projective labeling operation that we apply to X. Since X is bicentrally balanced, for all $1 \leqslant i \leqslant m_0$, we have

$$x_{i,n_0^+} + x_{m+1-i,n_0^+} = S(i, n_0^+).$$

Thus, either $x_{i,n_0^+} < \frac{1}{2}S(i, n_0^+)$ or $x_{m+1-i,n_0^+} < \frac{1}{2}S(i, n_0^+)$. We define a function $\alpha : \{1, 2, \ldots, m_0\} \to \{0, 1\}$ as follows. For each $1 \leqslant i \leqslant m_0$, let

$$\alpha(i) = \begin{cases} 0, & \text{if } i + n_0^+ \text{ is even and } x_{i,n_0^+} < \frac{1}{2}S(i, n_0^+), \\ 1, & \text{if } i + n_0^+ \text{ is even and } x_{i,n_0^+} > \frac{1}{2}S(i, n_0^+), \\ 0, & \text{if } i + n_0^+ \text{ is odd and } x_{i,n_0^+} > \frac{1}{2}S(i, n_0^+), \text{ and} \\ 1, & \text{if } i + n_0^+ \text{ is odd and } x_{i,n_0^+} < \frac{1}{2}S(i, n_0^+). \end{cases}$$

By Lemma 3.5, $\mathcal{E}_\alpha(X)$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Replace X with $\mathcal{E}_\alpha(X)$. Then X satisfies, for all $1 \leqslant i \leqslant m_0$,

$$\begin{gathered} x_{i,n_0^+} < \tfrac{1}{2}S(i, n_0^+) \text{ and } x_{m+1-i,n_0^+} > \tfrac{1}{2}S(i, n_0^+) \text{ if } i + n_0^+ \text{ is even, and} \\ x_{i,n_0^+} > \tfrac{1}{2}S(i, n_0^+) \text{ and } x_{m+1-i,n_0^+} < \tfrac{1}{2}S(i, n_0^+) \text{ if } i + n_0^+ \text{ is odd.} \end{gathered}$$

Choose a permutation η of $\{1, 2, \ldots, m_0\}$ with $\eta(i) \equiv i \pmod 2$ for all $1 \leqslant i \leqslant m_0$ such that for all $1 \leqslant i \leqslant m_0 - 2$,

$$\begin{aligned} x_{\eta(i),n_0^+} &< x_{\eta(i+2),n_0^+} \quad \text{if } i + n_0^+ \text{ is even, and} \\ x_{\eta(i),n_0^+} &> x_{\eta(i+2),n_0^+} \quad \text{if } i + n_0^+ \text{ is odd.} \end{aligned}$$

By Lemma 3.3, $\mathcal{E}_\eta(X)$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Replace X with $\mathcal{E}_\eta(X)$. Then X satisfies, for all $1 \leqslant i \leqslant m - 2$,

$$\begin{aligned} x_{i,n_0^+} &< x_{i+2,n_0^+} \quad \text{if } i + n_0^+ \text{ is even, and} \\ x_{i,n_0^+} &> x_{i+2,n_0^+} \quad \text{if } i + n_0^+ \text{ is odd.} \end{aligned}$$

A similar argument allows us to choose a function $\delta : \{1, 2, \ldots, n_0\} \to \{0, 1\}$ and a permutation κ on $\{1, 2, \ldots, n_0\}$ with $\kappa(j) \equiv j \pmod 2$ for all $1 \leqslant j \leqslant n_0$ such that $Z = \mathcal{E}_\kappa(\mathcal{E}_\delta(X))$ is a bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ and

$$\begin{aligned} z_{i,n_0^+} &< z_{i+2,n_0^+} \quad \text{for all } 1 \leqslant i \leqslant m-2 \text{ and } i + n_0^+ \text{ is even,} \\ z_{i,n_0^+} &> z_{i+2,n_0^+} \quad \text{for all } 1 \leqslant i \leqslant m-2 \text{ and } i + n_0^+ \text{ is odd,} \\ z_{m_0^+,j} &< z_{m_0^+,j+2} \quad \text{for all } 1 \leqslant j \leqslant n-2 \text{ and } m_0^+ + j \text{ is even, and} \\ z_{m_0^+,j} &> z_{m_0^+,j+2} \quad \text{for all } 1 \leqslant j \leqslant n-2 \text{ and } m_0^+ + j \text{ is odd.} \end{aligned}$$

□

Definition 3.8. *We refer to the bicentrally balanced C_4-face-magic projective labeling Z in Theorem 3.1 as the standard projective labeling associated with X. We say that Z is a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.*

As a result of Theorem 3.1, we only need to find the standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.

Example 3.1. Table 1 illustrates a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{9,9}$. For convenience, we display the 9×9 projective grid graph as a 9×9 checkerboard where each square cell represents a vertex and square cells that share an edge are adjacent.

Table 1. A standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{9,9}$ with C_4-face-magic value 165.

31	51	32	47	36	46	40	42	41
53	30	52	34	48	35	44	39	43
28	54	29	50	33	49	37	45	38
65	18	64	22	60	23	56	27	55
16	66	17	62	21	61	25	57	26
68	15	67	19	63	20	59	24	58
4	78	5	74	9	73	13	69	14
80	3	79	7	75	8	71	12	70
1	81	2	77	6	76	10	72	11

3.4. Partial Bicentrally Balanced Labeling

We need to introduce labelings on subgrids of $\mathcal{P}_{m,n}$ in order to determine a category of standard bicentrally balanced C_4-face-magic labelings on $\mathcal{P}_{m,n}$.

Definition 3.9. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers, and let $M \geqslant m$ and $N \geqslant n$ be odd integers. Let $\mathrm{Grid}(m,n) = \{(i,j) : 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}$ be the $m \times n$ subgrid of $\mathcal{P}_{M,N}$. Let $P_m \times P_n$ represent the $m \times n$ planar grid subgraph of $\mathcal{P}_{M,N}$ on $\mathrm{Grid}(m,n)$. Let $X = \{x_{i,j} : (i,j) \in \mathrm{Grid}(m,n)\}$ be a labeling on $P_m \times P_n$ We say X is a partial bicentrally balanced C_4-face-magic labeling on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$ if*

1. *the sum of the labels on the vertices of each C_4-face of $P_m \times P_n$ is $2MN + 3$,*
2. $\{x_{i,j} : (i,j) \in \mathrm{Grid}(m,n)$ *and $i + j$ is even*$\} = \{1, 2, \ldots, \frac{1}{2}mn + \frac{1}{2}\}$,
3. $\{x_{i,j} : (i,j) \in \mathrm{Grid}(m,n)$ *and $i+j$ is odd*$\} = \{MN - \frac{1}{2}mn + \frac{3}{2}, MN - \frac{1}{2}mn + \frac{5}{2}, \ldots, MN\}$,
4. $x_{i,j} + x_{m+1-i,n+1-j} = \frac{1}{2}mn + \frac{3}{2}$ *if $(i,j) \in \mathrm{Grid}(m,n)$ and $i + j$ is even,*
5. $x_{i,j} + x_{m+1-i,n+1-j} = 2MN - \frac{1}{2}mn + \frac{3}{2}$ *if $(i,j) \in \mathrm{Grid}(m,n)$ and $i + j$ is odd, and*

6. X *satisfies*

$$x_{i,n_0^+} < x_{i+2,n_0^+} \text{ for all } 1 \leqslant i \leqslant m-2 \text{ and } i+n_0^+ \text{ is even,}$$
$$x_{i,n_0^+} > x_{i+2,n_0^+} \text{ for all } 1 \leqslant i \leqslant m-2 \text{ and } i+n_0^+ \text{ is odd,}$$
$$x_{m_0^+,j} < x_{m_0^+,j+2} \text{ for all } 1 \leqslant j \leqslant n-2 \text{ and } m_0^+ + j \text{ is even, and}$$
$$x_{m_0^+,j} > x_{m_0^+,j+2} \text{ for all } 1 \leqslant j \leqslant n-2 \text{ and } m_0^+ + j \text{ is odd.}$$

3.5. Partial Alternating Lexicographic Labeling

We introduce two partial bicentrally balanced C_4-face-magic labelings on an $m \times n$ subgrid of $\mathcal{P}_{M,N}$.

Definition 3.10. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers, and let $M \geqslant m$ and $N \geqslant n$ be odd integers. The partial horizontal alternating lexicographic labeling on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$, denoted by $\mathrm{HALL}_{M,N}(m,n)$, is the labeling $\mathrm{HALL}_{M,N}(m,n) = \{x_{i,j} : (i,j) \in \mathrm{Grid}(m,n)\}$ given by*

- $x_{2i-1,2j-1} = m(j-1)+i$, *for all* $1 \leqslant i \leqslant m_0^+$ *and* $1 \leqslant j \leqslant n_0^+$,
- $x_{2i,2j} = m(j-1)+m_0^+ + i$, *for all* $1 \leqslant i \leqslant m_0$ *and* $1 \leqslant j \leqslant n_0$,
- $x_{2i,2j-1} = MN + m(1-j)+1-i$, *for all* $1 \leqslant i \leqslant m_0$ *and* $1 \leqslant j \leqslant n_0^+$, *and*
- $x_{2i-1,2j} = MN - mj + m_0^+ + 1 - i$, *for all* $1 \leqslant i \leqslant m_0^+$ *and* $1 \leqslant j \leqslant n_0$.

Similarly, the partial vertical alternating lexicographic labeling on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$, denoted by $\mathrm{VALL}_{M,N}(m,n)$, is the labeling $\mathrm{VALL}_{M,N}(m,n) = \{y_{i,j} : (i,j) \in \mathrm{Grid}(m,n)\}$ given by

- $y_{2i-1,2j-1} = n(i-1)+j$, *for all* $1 \leqslant i \leqslant m_0^+$ *and* $1 \leqslant j \leqslant n_0^+$,
- $y_{2i,2j} = n(i-1)+n_0^+ + j$, *for all* $1 \leqslant i \leqslant m_0$ *and* $1 \leqslant j \leqslant n_0$,
- $y_{2i,2j-1} = MN - ni + n_0^+ + 1 - j$, *for all* $1 \leqslant i \leqslant m_0$ *and* $1 \leqslant j \leqslant n_0^+$, *and*
- $y_{2i-1,2j} = MN + n(1-i)+1-j$, *for all* $1 \leqslant i \leqslant m_0^+$ *and* $1 \leqslant j \leqslant n_0$.

Example 3.2. The labeling in Figure 3 is the partial vertical alternating lexicographic labeling on the 5×5 subgrid of $\mathcal{P}_{15,5}$ denoted by $\mathrm{VALL}_{15,5}(5,5)$.

Theorem 3.2. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers, and let $M \geqslant m$ and $N \geqslant n$ be odd integers. The horizontal alternating lexicographic labeling $\mathrm{HALL}_{M,N}(m,n)$ and the vertical alternating lexicographic labeling $\mathrm{VALL}_{M,N}(m,n)$ are partial bicentrally balanced C_4-face-magic labelings on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$.*

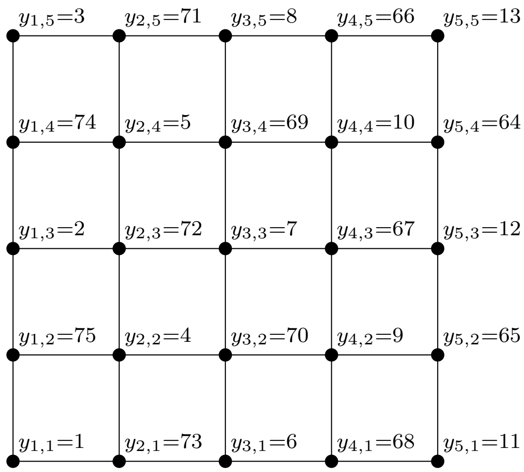

Figure 3. The partial bicentrally balanced C_4-face-magic projective labeling $\mathrm{VALL}_{15,5}(5,5)$.

Proof. We show that $\mathrm{VALL}_{M,N}(m,n)$ is a partial bicentrally balanced C_4-face-magic labelings on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$. The proof that $\mathrm{HALL}_{M,N}(m,n)$ is a partial bicentrally balanced C_4-face-magic labelings on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$ is similar.

We observe that for the vertices (i,j) where $i+j$ even, we assign the labels $1, 2, \ldots, \frac{1}{2}mn + \frac{1}{2}$ in lexicographic order; however, for the vertices (i,j) where $i+j$ odd, we assign the labels $MN - (\frac{1}{2}mn - \frac{3}{2}), MN - (\frac{1}{2}mn - \frac{5}{2}), \ldots, MN$ in reverse lexicographic order.

We have $y_{2i-1,2j-1} + y_{2i-1,2j} = MN + 1$ for $1 \leqslant i \leqslant m_0 + 1$ and $1 \leqslant j \leqslant n_0$. Also, we have $y_{2i,2j-1} + y_{2i,2j} = MN + 2$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$. Thus, for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n_0$, we have

$$y_{i,2j-1} + y_{i,2j} + y_{i+1,2j-1} + y_{i+1,2j} = 2MN + 3.$$

Next, we have $y_{2i-1,2j} + y_{2i-1,2j+1} = MN + 2$ for $1 \leqslant i \leqslant m_0 + 1$ and $1 \leqslant j \leqslant n_0$. Also, we have $y_{2i,2j} + y_{2i,2j+1} = MN + 1$ for $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$. Thus, for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n_0$, we have

$$y_{i,2j} + y_{i,2j+1} + y_{i+1,2j} + y_{i+1,2j+1} = 2MN + 3.$$

We observe that, for $1 \leqslant j \leqslant n_0 + 1$, $y_{1,2j-1} + y_{m,n+2-2j} = \frac{1}{2}mn + \frac{3}{2}$, and, for $1 \leqslant j \leqslant n_0$, $y_{1,2j} + y_{m,n+1-2j} = 2MN - \frac{1}{2}mn + \frac{3}{2}$. Thus, for $1 \leqslant j \leqslant n-1$, we have

$$y_{1,j} + y_{m,n+1-j} + y_{1,j+1} + y_{m,n-j} = 2MN + 3.$$

Similarly, for $1 \leqslant i \leqslant m_0 + 1$, $y_{2i-1,1} + y_{m+2-2i,n} = \frac{1}{2}mn + \frac{3}{2}$, and, for $1 \leqslant i \leqslant m_0$, $y_{2i,1} + y_{m+1-2i,n} = 2MN - \frac{1}{2}mn + \frac{3}{2}$. Thus, for $1 \leqslant i \leqslant m-1$, we have

$$y_{i,1} + y_{m+1-i,n} + y_{i+1,1} + y_{m-i,n} = 2MN + 3.$$

For all $1 \leqslant i \leqslant m_0^+$ and $1 \leqslant j \leqslant n_0^+$,

$$y_{2i-1,2j-1} + y_{m+2-2i,n+2-2j} = \tfrac{1}{2}mn + \tfrac{3}{2}.$$

Also, for all $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0$,

$$y_{2i,2j} + y_{m+1-2i,n+1-2j} = \tfrac{1}{2}mn + \tfrac{3}{2}.$$

We observe that, for all $1 \leqslant i \leqslant m_0^+$ and $1 \leqslant j \leqslant n_0$,

$$y_{2i-1,2j} + y_{m+2-2i,n+1-2j} = 2MN - \tfrac{1}{2}mn + \tfrac{3}{2}.$$

Similarly, for all $1 \leqslant i \leqslant m_0$ and $1 \leqslant j \leqslant n_0^+$,

$$y_{2i,2j-1} + y_{m+1-2i,n+2-2j} = 2MN - \tfrac{1}{2}mn + \tfrac{3}{2}.$$

Suppose $n \equiv 1 \pmod 4$. Then n_0^+ is odd, and we let n_0' be the positive integer such that $n_0^+ = 2n_0' - 1$. Thus

$$\begin{aligned} y_{2i-1,2n_0'-1} &= n(i-1) + n_0' \ \text{ for all } 1 \leqslant i \leqslant m_0^+, \text{ and} \\ y_{2i,2n_0'-1} &= MN - ni + n_0' \ \text{ for all } 1 \leqslant i \leqslant m_0. \end{aligned}$$

Hence,

$$y_{i,n_0^+} < y_{i+2,n_0^+} \ \text{ for all } 1 \leqslant i \leqslant m-2 \text{ and } i + n_0^+ \text{ is even, and} \tag{2}$$

$$y_{i,n_0^+} > y_{i+2,n_0^+} \ \text{ for all } 1 \leqslant i \leqslant m-2 \text{ and } i + n_0^+ \text{ is odd.} \tag{3}$$

A similar argument shows that (2) and (3) hold when $n \equiv 3 \pmod 4$. Also, a similar argument shows that

$$\begin{aligned} y_{m_0^+,j} &< y_{m_0^+,j+2} \ \text{ for all } 1 \leqslant j \leqslant n-2 \text{ and } m_0^+ + j \text{ is even, and} \\ y_{m_0^+,j} &> y_{m_0^+,j+2} \ \text{ for all } 1 \leqslant j \leqslant n-2 \text{ and } m_0^+ + j \text{ is odd.} \end{aligned}$$

□

3.6. Alternating Connected Sums

Definition 3.11. *Let $r \geqslant 3$ be an odd integer and let r_0 be the positive integer such that $r = 2r_0+1$. Let X be a partial bicentrally balanced C_4-face-magic labeling on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$. Suppose $r \leqslant M/m$. The r-horizontal alternating connected sum of X, denoted by $\mathrm{HASC}^r(X)$, is the labeling $Y = \{y_{i,j} : (i,j) \in \mathrm{Grid}(mr,n)\}$ on $\mathrm{Grid}(mr,n)$ given by, for all $(i,j) \in \mathrm{Grid}(m,n)$ and for all integers $0 \leqslant k \leqslant r_0$,*

- $y_{(2k-1)m+i,j} = MN - x_{i,j} + (k - \frac{1}{2})(mn) + \frac{3}{2}$ *if* $1 \leqslant k \leqslant r_0$ *and* $i + j$ *is odd,*
- $y_{(2k-1)m+i,j} = MN - x_{i,j} - (k - \frac{1}{2})(mn) + \frac{3}{2}$ *if* $1 \leqslant k \leqslant r_0$ *and* $i + j$ *is even,*
- $y_{(2k)m+i,j} = x_{i,j} - k(mn)$ *if* $0 \leqslant k \leqslant r_0$ *and* $i + j$ *is odd, and*
- $y_{(2k)m+i,j} = x_{i,j} + k(mn)$ *if* $0 \leqslant k \leqslant r_0$ *and* $i + j$ *is even.*

Suppose $r \leqslant N/n$. The r-vertical alternating connected sum of X, denoted by $\mathrm{VASC}^r(X)$, *is the labeling $Y = \{y_{i,j} : (i,j) \in \mathrm{Grid}(m, nr)\}$ on $\mathrm{Grid}(m, nr)$ given by, for all $(i,j) \in \mathrm{Grid}(m,n)$ and for all integers $0 \leqslant k \leqslant r_0$,*

- $y_{i,(2k-1)n+j} = MN - x_{i,j} + (k - \frac{1}{2})(mn) + \frac{3}{2}$ *if $1 \leqslant k \leqslant r_0$ and $i + j$ is odd,*
- $y_{i,(2k-1)n+j} = MN - x_{i,j} - (k - \frac{1}{2})(mn) + \frac{3}{2}$ *if $1 \leqslant k \leqslant r_0$ and $i + j$ is even,*
- $y_{i,(2k)n+j} = x_{i,j} - k(mn)$ *if $0 \leqslant k \leqslant r_0$ and $i + j$ is odd, and*
- $y_{i,(2k)n+j} = x_{i,j} + k(mn)$ *if $0 \leqslant k \leqslant r_0$ and $i + j$ is even.*

Example 3.3. The 3-horizontal alternating connected sum of $\mathrm{HALL}_{15,5}(5,5)$ is given in Table 2. For convenience, we display the 15×5 projective grid graph as a 15×5 checkerboard.

Table 2. A 3-horizontal alternating connected sum of $\mathrm{HALL}_{15,5}(5,5)$ on $\mathcal{P}_{15,5}$ with C_4-face-magic value 153.

11	65	12	64	13	53	24	52	25	51	36	40	37	39	38
68	9	67	10	66	21	55	22	54	23	43	34	42	35	41
6	70	7	69	8	58	19	57	20	56	31	45	32	44	33
73	4	72	5	71	16	60	17	59	18	48	29	47	30	46
1	75	2	74	3	63	14	62	15	61	26	50	27	49	28

Theorem 3.3. *Suppose X is a partial bicentrally balanced C_4-face-magic labeling on the $m \times n$ subgrid of $\mathcal{P}_{M,N}$. If r is an odd positive integer such that $mr \leqslant M$, then the r-horizontal alternating connected sum of X,* $\mathrm{HASC}^r(X)$, *is a partial bicentrally balanced C_4-face-magic labeling on the $mr \times n$ subgrid of $\mathcal{P}_{M,N}$. Similarly, if r is an odd positive integer such that $nr \leqslant N$, then the r-vertical alternating connected sum of X,* $\mathrm{VASC}^r(X)$, *is a partial bicentrally balanced C_4-face-magic labeling on the $m \times nr$ subgrid of $\mathcal{P}_{M,N}$.*

Proof. We show that $\mathrm{HASC}^r(X)$, is a partial bicentrally balanced C_4-face-magic labeling on the $mr \times n$ subgrid of $\mathcal{P}_{M,N}$. The proof that $\mathrm{VASC}^r(X)$, is a partial bicentrally balanced C_4-face-magic labeling on the $m \times nr$ subgrid of $\mathcal{P}_{M,N}$ is similar.

First, when $1 \leqslant k \leqslant r_0$, we have

$$y_{(2k-1)m+2i-1,j} + y_{(2k-1)m+2i-1,j+1} = 2MN + 3 - (x_{2i-1,j} + x_{2i-1,j+1}) \text{ and}$$
$$y_{(2k-1)m+2i,j} + y_{(2k-1)m+2i,j+1} = 2MN + 3 - (x_{2i,j} + x_{2i,j+1}).$$

Thus, for all $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n-1$, and $1 \leqslant k \leqslant r_0$, we have

$$y_{(2k-1)m+i,j} + y_{(2k-1)m+i,j+1} + y_{(2k-1)m+i+1,j} + y_{(2k-1)m+i+1,j+1} = 2MN + 3.$$

Second, when $0 \leqslant k \leqslant r_0$, we have

$$y_{(2k)m+2i-1,j} + y_{(2k)m+2i-1,j+1} = x_{2i-1,j} + x_{2i-1,j+1} \text{ and}$$
$$y_{(2k)m+2i,j} + y_{(2k)m+2i,j+1} = x_{2i,j} + x_{2i,j+1}.$$

Thus, for all $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n-1$, and $0 \leqslant k \leqslant r_0$, we have

$$y_{(2k)m+i,j} + y_{(2k)m+i,j+1} + y_{(2k)m+i+1,j} + y_{(2k)m+i+1,j+1} = 2MN + 3.$$

When we set the C_4-face sums below equal,

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = x_{i+1,j} + x_{i+1,j+1} + x_{i+2,j} + x_{i+2,j+1},$$

we obtain

$$x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}.$$

Thus, for all $1 \leqslant j \leqslant n-1$, we have

$$x_{1,j} + x_{1,j+1} = x_{m,j} + x_{m,j+1}. \tag{4}$$

Third, for all $1 \leqslant j \leqslant n-1$ and $1 \leqslant k \leqslant r_0$, we have

$$y_{(2k)m,j} + y_{(2k)m,j+1} = 2MN + 3 - (x_{m,j} + x_{m,j+1}).$$

Similarly, we have

$$y_{(2k)m+1,j} + y_{(2k)m+1,j+1} = x_{1,j} + x_{1,j+1}.$$

By (4), we have

$$y_{(2k)m,j} + y_{(2k)m,j+1} + y_{(2k)m+1,j} + y_{(2k)m+1,j+1} = 2MN + 3.$$

A similar argument shows that, for all $1 \leqslant j \leqslant n-1$ and $1 \leqslant k \leqslant r_0$, we have

$$y_{(2k-1)m,j} + y_{(2k-1)m,j+1} + y_{(2k-1)m+1,j} + y_{(2k-1)m+1,j+1} = 2MN + 3.$$

From the construction of Y, we have

$$\begin{aligned}\{y_{i,j} : (i,j) \in \mathrm{Grid}(mr,n) \text{ and } i+j \text{ is even}\} &= \{1, 2, \ldots, \tfrac{1}{2}mnr + \tfrac{1}{2}\}, \text{ and}\\ \{y_{i,j} : (i,j) \in \mathrm{Grid}(mr,n) \text{ and } i+j \text{ is odd}\} &= \{MN - \tfrac{1}{2}mnr + \tfrac{3}{2},\\ &\qquad MN - \tfrac{1}{2}mnr + \tfrac{5}{2}, \ldots, MN\}.\end{aligned}$$

If $i+j$ is even, we have

$$\begin{aligned}y_{(2k-1)m+i,j} + y_{(2(r_0-k+1)-1)m+(m+1-i),n+1-j} &= 2MN - \tfrac{1}{2}mnr + \tfrac{3}{2}, \text{ and}\\ y_{(2k)m+i,j} + y_{2(r_0-k)m+(m+1-i),n+1-j} &= \tfrac{1}{2}mnr + \tfrac{3}{2}.\end{aligned}$$

If $i+j$ is odd, we have

$$\begin{aligned}y_{(2k-1)m+i,j} + y_{(2(r_0-k+1)-1)m+(m+1-i),n+1-j} &= \tfrac{1}{2}mnr + \tfrac{3}{2}, \text{ and}\\ y_{(2k)m+i,j} + y_{2(r_0-k)m+(m+1-i),n+1-j} &= 2MN - \tfrac{1}{2}mnr + \tfrac{3}{2}.\end{aligned}$$

Thus, for all $1 \leqslant i \leqslant mr$ and $1 \leqslant j \leqslant n$,

$$y_{i,j} + y_{mr+1-i,n+1-j} = \tfrac{1}{2}mnr + \tfrac{3}{2}, \ \text{ if } i+j \text{ is even, and}$$
$$y_{i,j} + y_{mr+1-i,n+1-j} = 2MN - \tfrac{1}{2}mnr + \tfrac{3}{2}, \ \text{ if } i+j \text{ is odd.}$$

We have, for all $1 \leqslant i \leqslant m$ and $0 \leqslant k \leqslant r_0$,

$$y_{(2k-1)m+i,n_0^+} = MN - x_{i,n_0^+} + (k - \tfrac{1}{2})(mn) + \tfrac{3}{2} \ \text{ if } i + n_0^+ \text{ is odd and}$$
$$y_{(2k)m+i,n_0^+} = x_{i,n_0^+} + k(mn) \ \text{ if } i + n_0^+ \text{ is even.}$$

Since, for all $1 \leqslant i \leqslant m-2$,

$$x_{i,n_0^+} < x_{i+2,n_0^+} \ \text{ if } i + n_0^+ \text{ is even and}$$
$$x_{i,n_0^+} > x_{i+2,n_0^+} \ \text{ if } i + n_0^+ \text{ is odd,}$$

we have

$$y_{(2k-1)m+i,n_0^+} < y_{(2k-1)m+i+2,n_0^+} \ \text{ if } (2k-1)m + i + n_0^+ \text{ is even and}$$
$$y_{(2k)m+i,n_0^+} < y_{(2k)m+i+2,n_0^+} \ \text{ if } (2k)m + i + n_0^+ \text{ is even.}$$

In addition, we have

$$(k-1)(mn) + \tfrac{1}{2}mn + \tfrac{3}{2} \leqslant y_{(2k-1)m+i,n_0^+} \leqslant k(mn)$$
$$\text{if } (2k-1)m + i + n_0^+ \text{ is even and}$$
$$k(mn) + 1 \leqslant y_{(2k)m+i,n_0^+} \leqslant k(mn) + \tfrac{1}{2}mn + \tfrac{1}{2}$$
$$\text{if } (2k)m + i + n_0^+ \text{ is even.}$$

Hence, for all $1 \leqslant i \leqslant mr-2$,

$$y_{i,n_0^+} < y_{i+2,n_0^+} \ \text{ if } i + n_0^+ \text{ is even.}$$

A similar argument shows that, for all $1 \leqslant i \leqslant mr-2$,

$$y_{i,n_0^+} > y_{i+2,n_0^+} \ \text{ if } i + n_0^+ \text{ is odd.}$$

Let M_0^+ be the positive integer such that $mr = 2M_0^+ - 1$. Then $M_0^+ = mr_0 + m_0^+$. Suppose r_0 is odd. Let r_0' be the positive integer such that $r_0 = 2r_0' - 1$. For all $1 \leqslant j \leqslant n$, we have

$$y_{(2r_0'-1)m+m_0^+,j} = MN - x_{m_0^+,j} + (r_0' - \tfrac{1}{2})(mn) + \tfrac{3}{2} \ \text{ if } m_0^+ + j \text{ is odd and}$$
$$y_{(2r_0'-1)m+m_0^+,j} = MN - x_{m_0^+,j} - (r_0' - \tfrac{1}{2})(mn) + \tfrac{3}{2} \ \text{ if } m_0^+ + j \text{ is even.}$$

Since, for all $1 \leqslant j \leqslant n-2$,

$$x_{m_0^+,j} < x_{m_0^+,j+2} \ \text{ if } m_0^+ + j \text{ is even and}$$
$$x_{m_0^+,j} > x_{m_0^+,j+2} \ \text{ if } m_0^+ + j \text{ is odd,}$$

we have

$$y_{(2r'_0-1)m+m_0^+,j} < y_{(2r'_0-1)m+m_0^+,j+2} \text{ if } (2r'_0-1)m+m_0^++j \text{ is even and}$$
$$y_{(2r'_0-1)m+m_0^+,j} > y_{(2r'_0-1)m+m_0^+,j+2} \text{ if } (2r'_0-1)m+m_0^++j \text{ is odd.}$$

Thus, for all $1 \leqslant j \leqslant n-2$, we have

$$y_{M_0^+,j} < y_{M_0^+,j+2} \text{ if } M_0^++j \text{ is even and} \tag{5}$$
$$y_{M_0^+,j} > y_{M_0^+,j+2} \text{ if } M_0^++j \text{ is odd.} \tag{6}$$

A similar argument shows that (5) and (6) hold when r_0 is even. □

3.7. Labelings Associated with a Projective Factorization Sequence

Definition 3.12. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers.*

1. *Let $F = (m_i, n_i : 1 \leqslant i \leqslant k)$ be an (m,n)-projective factorization sequence. See Definition 2.2. Let $X_1 = \mathrm{HALL}(m_1, n_1)$. For $2 \leqslant i \leqslant k$, let $Y_i = \mathrm{HACS}^{m_i}(X_{i-1})$ and $X_i = \mathrm{VACS}^{n_i}(Y_i)$. The horizontal bicentrally balanced labeling associated with F is denoted by $\mathrm{HBBL}(F) = X_k$.*
2. *Let $F' = (n'_i, m'_i : 1 \leqslant i \leqslant k)$ be an (n,m)-projective factorization sequence. Let $X'_1 = \mathrm{VALL}(m'_1, n'_1)$. For $2 \leqslant i \leqslant k$, let $Y'_i = \mathrm{VACS}^{n'_i}(X'_{i-1})$ and $X'_i = \mathrm{HACS}^{m'_i}(Y'_i)$. The vertical bicentrally balanced labeling associated with F' is denoted by $\mathrm{VBBL}(F') = X'_k$.*

Theorem 3.4. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. Suppose X is constructed in one of the following two ways.*

1. *Suppose $X = \mathrm{HBBL}(F)$ for some (m,n)-projective factorization sequence $F = (m_1, n_1, m_2, n_2, \ldots, m_k, n_k)$.*
2. *Suppose $X = \mathrm{VBBL}(F')$ for some (n,m)-projective factorization sequence $F' = (n'_1, m'_1, n'_2, m'_2, \ldots, n'_k, m'_k)$.*

Then X is a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.

Furthermore, distinct (m,n)-projective factorization sequences F_1 and F_2 give rise to distinct standard bicentrally balanced C_4-face-magic projective labelings $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ on $\mathcal{P}_{m,n}$. Similarly, distinct (n,m)-projective factorization sequences F'_1 and F'_2 give rise to distinct standard bicentrally balanced C_4-face-magic projective labelings $\mathrm{VBBL}(F'_1)$ and $\mathrm{VBBL}(F'_2)$ on $\mathcal{P}_{m,n}$.

Proof. We first show that $\mathrm{HBBL}(F)$ is a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. By Theorem 3.2, $X_1 = \mathrm{HALL}_{m,n}(m_1, n_1)$ is a partial bicentrally balanced C_4-face-magic labeling on the $m_1 \times n_1$ subgrid of $\mathcal{P}_{m,n}$. For $1 \leqslant i \leqslant k$, let

$$M_i = m_1 m_2 \cdots m_i \text{ and}$$
$$N_i = n_1 n_2 \cdots n_i.$$

For some integer $2 \leqslant i \leqslant k$, suppose X_{i-1} is a partial bicentrally balanced C_4-face-magic labeling on the $M_{i-1} \times N_{i-1}$ subgrid of $\mathcal{P}_{m,n}$. By Theorem 3.3, $Y_i = \mathrm{HACS}^{m_i}(X_{i-1})$ is a partial bicentrally balanced C_4-face-magic labeling on the $M_i \times N_{i-1}$ subgrid of $\mathcal{P}_{m,n}$ and $X_i = \mathrm{VACS}^{n_i}(Y_i)$ is a partial bicentrally balanced C_4-face-magic labeling on the $M_i \times N_i$ subgrid of $\mathcal{P}_{m,n}$ Thus $X_k = \mathrm{HBBL}(F)$ is a partial bicentrally balanced C_4-face-magic labeling on the $m \times n$ subgrid of $\mathcal{P}_{m,n}$. Hence, $\mathrm{HBBL}(F)$ is a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.

A similar argument shows that, for an (n, m)-projective factorization sequence F', $\mathrm{VBBL}(F')$ is a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$.

Let $F_i = (m_{i,j}, n_{i,j} : 1 \leqslant j \leqslant k_i)$, for $i = 1, 2$, be distinct (m, n)-projective factorization sequences. We need to show that $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ are distinct standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$. Suppose $m_{1,j'}$ and $m_{2,j'}$ are the first entries in F_1 and F_2, respectively, such that $m_{1,j'} \neq m_{2,j'}$.

First, assume $j' = 1$. Without loss of generality, we may assume $m_{1,1} < m_{2,1}$. Then the labels on $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ are the same on the $m_{1,1} \times 1$ subgrid of $\mathcal{P}_{m,n}$. Let $Z = \{y_{i,1} : 1 \leqslant i \leqslant m_{1,1}\}$ be the common labels of $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ on the $m_{1,1} \times 1$ subgrid of $\mathcal{P}_{m,n}$. Let z be the smallest positive integer such that $z \in \mathrm{Label}_Z(\mathrm{Grid}(m_{1,1}, 1))$ and $z + 1 \notin \mathrm{Label}_Z(\mathrm{Grid}(m_{1,1}, 1))$. Let $m' = m_{1,1}$. Then $z = y_{m',1} = \frac{1}{2}(m' + 1)$. In $\mathrm{HBBL}(F_1)$ we have $y_{2,2} = z + 1$, and in $\mathrm{HBBL}(F_2)$ we have $y_{m'+2,1} = z + 1$. Thus $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ are distinct standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.

Suppose $j' > 1$. Without loss of generality, we may assume $m_{1,j'} < m_{2,j'}$. Let $m' = m_{1,j'}M_{j'-1}$ and $n' = N_{j'-1}$. Let $Z = \{y_{i,j} : (i, j) \in \mathrm{Grid}(m', n')\}$ be the common labels of $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ on the $m' \times n'$ subgrid of $\mathcal{P}_{m,n}$. Let z be the smallest positive integer such that $z \in \mathrm{Label}_Z(\mathrm{Grid}(m', n'))$ and $z + 1 \notin \mathrm{Label}_Z(\mathrm{Grid}(m', n'))$. Then $z = y_{m',n'} = \frac{1}{2}(m'n' + 1)$. In $\mathrm{HBBL}(F_1)$ we have $y_{2,n'+1} = z + 1$, and in $\mathrm{HBBL}(F_2)$ we have $y_{m'+2,1} = z + 1$. Thus $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ are distinct standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.

Suppose $n_{1,j'}$ and $n_{2,j'}$ are the first entries in F_1 and F_2, respectively, such that $n_{1,j'} \neq n_{2,j'}$. A similar argument to the one above shows that $\mathrm{HBBL}(F_1)$ and $\mathrm{HBBL}(F_2)$ are distinct standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.

Similarly, we can show that distinct (n, m)-projective factorization sequences F_1' and F_2' give rise to distinct standard bicentrally balanced C_4-face-magic projective labelings $\mathrm{VBBL}(F_1')$ and $\mathrm{VBBL}(F_2')$ on $\mathcal{P}_{m,n}$. □

4. Enumerating Bicentrally Balanced Labelings

We will enumerate the minimum number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.

Notation 4.1. Let $m \geqslant 3$ be an odd integer. We define the function β given by

$$\beta(m) = \begin{cases} \left((\frac{m-1}{4})!\right)^2, & \text{if } m \equiv 1 \pmod 4, \\ (\frac{m-3}{4})!(\frac{m+1}{4})!, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

The following theorem gives us a lower bound on the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+1$ (or $2mn+3$) for distinct odd integers m and n.

Theorem 4.1. *Let $m \geqslant 3$ and $n \geqslant 3$ be distinct odd integers. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ having C_4-face-magic value $2mn+1$ (or $2mn+3$) (up to symmetries on the projective plane) is at least*

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\beta(m)\beta(n),$$

where $\tau(m,n)$ is the number of distinct (m,n)-projective factorization sequences. See Definition 2.2.

Proof. Let X be a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Let η be a permutation on $\{1,2,\ldots,m_0\}$ such that $\eta(i) \equiv i \pmod 2$ for all $1 \leqslant i \leqslant m_0$ and $\mathcal{E}_\eta(X)$ be the labeling given in Definition 3.2. By Lemma 3.3, there are $\beta(m)$ distinct bicentrally balanced C_4-face-magic projective labelings of type $\mathcal{E}_\eta(X)$ on $\mathcal{P}_{m,n}$. Let $\alpha : \{1,2,\ldots,m_0\} \to \{0,1\}$ and $\mathcal{E}_\alpha(X)$ be the labeling given in Definition 3.4. By Lemma 3.5, there are 2^{m_0} distinct bicentrally balanced C_4-face-magic projective labelings of type $\mathcal{E}_\alpha(X)$ on $\mathcal{P}_{m,n}$. Similarly, by Lemma 3.4, there are $\beta(n)$ distinct bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ associated with the elementary projective labeling operation given in Definition 3.3. Also, by Lemma 3.6, there are 2^{n_0} distinct bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ associated with the elementary projective labeling operation given in Definition 3.5. Thus, there are $\beta(m)2^{m_0}\beta(n)2^{n_0}$ distinct bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ that are projective labeling equivalent to X. For each of these labelings, there are four labelings that are projective labeling equivalent by one of the symmetries R_0, R_{180}, H, or V of the projective plane. Hence, there are $\frac{1}{4}\beta(m)2^{m_0}\beta(n)2^{n_0} = 2^{m/2+n/2-3}\beta(m)\beta(n)$ distinct bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ that are projective labeling equivalent to X up to symmetries of the projective plane. By Remark 3.1 and Lemma 3.1, a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ is bicentrally balanced if and only if its C_4-face-magic value is $2mn+3$. Therefore, there are $2^{m/2+n/2-3}\beta(m)\beta(n)$ distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn+3$ that are projective labeling equivalent to X up to symmetries of the projective plane.

By Theorem 3.4, each (m,n)-projective factorization sequence F and each (n,m)-projective factorization sequence F' are associated with unique standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ given by $\mathrm{HBBL}(F)$ and $\mathrm{VBBL}(F')$, respectively. Therefore, there are at least

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\beta(m)\beta(n)$$

distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) that have C_4-face-magic value $2mn+3$.

By Remark 2.1, there are at least

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\beta(m)\beta(n)$$

distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) that have C_4-face-magic value $2mn+1$. □

The next proposition gives us a lower bound on the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ having C_4-face-magic value $2m^2+1$ (or $2m^2+3$).

Theorem 4.2. *Let $m \geqslant 3$ be an odd integer. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ having C_4-face-magic value $2m^2+1$ (or $2m^2+3$) (up to symmetries on the projective plane) is at least*

$$\tau(m,m)2^{m-3}\big(\beta(m)\big)^2.$$

The proof of Theorem 4.2 is similar to that of Theorem 4.1.

Theorem 4.3. *Let $m \geqslant 3$ and $n \geqslant 3$ be distinct odd integers. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) is at least*

$$\big(\tau(m,n)+\tau(n,m)\big)2^{m/2+n/2-3}\big((\tfrac{m-1}{2})!(\tfrac{n-1}{2})!+2\beta(m)\beta(n)\big).$$

Proof. By Lemma 2.3, the C_4-face-magic value of a C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$ is either $2mn+1$, $2mn+2$, or $2mn+3$. By Theorem 2.1, there are exactly

$$(\tau(m,n)+\tau(n,m))2^{m/2+n/2-3}(\tfrac{m-1}{2})!(\tfrac{n-1}{2})!$$

distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) with C_4-face-magic value $2mn+2$. By Theorem 4.1, there are at least

$$(\tau(m,n)+\tau(n,m))2^{m/2+n/2-3}\beta(m)\beta(n)$$

distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ (up to symmetries on the projective plane) with C_4-face-magic value $2mn+1$ (or $2mn+3$). The result follows. □

Theorem 4.4. *Let $m \geqslant 3$ be an odd integer. Then the number of distinct C_4-face-magic projective labelings on $\mathcal{P}_{m,m}$ (up to symmetries on the projective plane) is at least*

$$\tau(m,m)2^{m-3}\Big(\big((\tfrac{m-1}{2})!\big)^2+2\big(\beta(m)\big)^2\Big).$$

The proof Theorem 4.4 is similar to that of Theorem 4.3.

5. Open Problems

We conjecture that the labelings given in Theorem 3.4 are the only standard bicentrally balanced C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.

Conjecture 1. *Let $m \geqslant 3$ and $n \geqslant 3$ be odd integers. Suppose X is a standard bicentrally balanced C_4-face-magic projective labeling on $\mathcal{P}_{m,n}$. Then either*

1. *there exists an (m,n)-projective factorization sequence F such that $X = HBBL(F)$ or*
2. *there exists an (n,m)-projective factorization sequence F' such that $X = VBBL(F')$.*

Suppose $m \geqslant 3$ and $n \geqslant 3$ are odd integers. The characterization of C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ with C_4-face-magic value $2mn + 2$ is given in [9]. It is natural to ask if there is a characterization of C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$ when $m \geqslant 2$ and $n \geqslant 2$ are even integers. By Lemma 2.2, the C_4-face-magic value of such a labeling is $2mn + 2$.

Problem 1. *Let $m \geqslant 2$ and $n \geqslant 2$ be even integers. Find a characterization of the C_4-face-magic projective labelings on $\mathcal{P}_{m,n}$.*

Curran and Locke [10] have characterized the C_4-face-magic projective labelings on the 4×4 projective grid graph $\mathcal{P}_{4,4}$. They show that there are 144 C_4-face-magic projective labelings on $\mathcal{P}_{4,4}$ up to symmetries on the projective plane.

References

[1] M. Baca, On magic labelings of Mobius ladders. *J. Franklin Inst.* **326** (1989), 885–888. doi.org/10.1016/0016-0032(89)90010-0

[2] M. Baca, On magic labelings of honeycomb. *Discrete Math.* **105** (1992), 305–311. doi.org/10.1016/0012-365X(92)90153-7

[3] M. Baca, On magic labelings of grid graphs. *Ars Combin*. **33** (1992), 295–299.

[4] S.I. Butt, M. Numan, S. Ali, and A. Semaničová-Feňovčíková, Face antimagic labelings of toroidal and Klein bottle grid graphs. *AKCE Int. J. Graphs Comb.* **17** (2020), 109–117. doi.org/10.1016/j.akcej.2018.09.005

[5] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, 6th ed., CRC Press, Boca Raton, (2016).

[6] S.J. Curran and R.M. Low, C_4-face-magic torus labelings on $C_4 \times C_4$, *Cong. Numer.* **233** (2019), 79–94.

[7] S.J. Curran, R.M. Low, and S.C. Locke, C_4-face-magic toroidal labelings on $C_m \times C_n$, *Art Discrete Appl. Math.* **4** (2021), #P1.04, 33 pp. doi.org/10.26493/2590-9770.1368.f37

[8] S.J. Curran, R M. Low and S.C. Locke, Equatorially balanced C_4-face-magic labelings on Klein bottle grid graphs, arXiv:2206.02028 [math.CO]. doi.org/10.48550/arXiv.2206.02028

[9] S.J. Curran, C_4-face-magic labelings on odd order projective grid graphs, *Australas. J. Combin.* **83(3)** (2022), 361–396. https://ajc.maths.uq.edu.au/pdf/83/ajc_v83_p361.pdf

[10] S.J. Curran and S.C. Locke, C_4-face-magic labelings on even order projective grid graphs. In: F. Hoffman, S. Holliday, Z. Rosen, F. Shahrokhi, J., Wierman (eds) *Combinatorics, Graph Theory and Computing*. SEICCGTC 2021. Springer Proceedings in Mathematics & Statistics, vol 448. Springer, Cham. (2024), 177–201. doi.org/10.1007/978-3-031-52969-6_17

[11] J.A. Gallian, A dynamic survey of graph labeling. *Electron. J. Combin.* **24** (2021), #DS6. doi.org/10.37236/27

[12] A. Kasif, H. Muhammad, A. Ali, and M. Miller, Magic labeling of type (a, b, c) of families of wheels. *Mathematics in Computer Science*. **7** (2013), 315–319. doi.org/10.1007/s11786-013-0162-9

[13] K. Kathiresan and S. Gokulakrishnan, On magic labelings of type $(1, 1, 1)$ for the special classes of plane graphs. *Util. Math.* **63** (2003), 25–32.

[14] A. Kotzig, and A. Rosa, Magic valuations of finite graphs. *Canad. Math. Bull*. **13** (1970), 451–461. doi.org/10.4153/CMB-1970-084-1

[15] K.W. Lih, On magic and consecutive labelings of plane graphs. *Util. Math.* **24** (1983), 165–197.

[16] T.M. Wang, Toroidal grids are anti-magic. In: Wang L. (eds) *Computing and Combinatorics.* COCOON 2005. Lecture Notes in Computer Science, vol 3595. Springer, Berlin, Heidelberg, (2005), 671–679. doi.org/10.1007/11533719_68