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On Odd Sum Graphs

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Abstract: An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u)+f(v)$, for all $uv \in E(G)$, is bijective and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper, we have studied the odd sum property for the graphs paths P_p, cycles C_p, $C_p \odot K_1$, the ladder $P_2 \times P_p$, $P_m \odot nK_1$, the balloon graph $P_n(C_p)$, quadrilateral snake $Q_n, [P_m; C_n]$, $(P_m; Q_3), T_p^{(n)}, H_n \odot mK_1$, bistar graph and cyclic ladder $P_2 \times C_p$.

Key Words: Labeling, odd sum labeling, odd sum graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

Path on p vertices is denoted by P_p and a cycle on p vertices is denoted by C_p whose length is p. If m number of pendant vertices are attached at each vertex of G, then the resultant graph obtained from G is the graph $G \odot mK_1$. When $m = 1, G \odot K_1$ is the corona of G. The bistar graph $B_{m,n}$ is the graph obtained from K_2 by identifying the central vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of K_2 respectively. The graph $P_2 \times P_p$ is the ladder and $P_2 \times C_p$ is the cyclic ladder. The balloon of a graph G, $P_n(G)$ is the graph obtained from G by identifying an end vertex of P_n at a vertex of G. Let v be a fixed vertex of G. The graph $[P_m; G]$ is obtained from m copies of G and the path $P_m : u_1u_2 \ldots u_m$ by identifying u_i with the vertex v of the i^{th} copy of G, for $1 \le i \le m$. The graph $(P_m; G)$ is obtained from m copies of G and the path $P_m : u_1u_2 \ldots u_m$ by joining u_i with the vertex v of the i^{th} copy of G by means of an edge, for $1 \le i \le m$ [7]. The cube graph Q_3 is $P_2 \times C_4$. A quadrilateral snake is obtained from a path by

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identifying each edge of the path with an edge of the cycle C_4. The graph $T_p^{(n)}$ is a tree formed from n copies of path on p vertices by joining an edge uu' between every pair of consecutive paths where u is a vertex in i^{th} copy of the path and u' is the corresponding vertex in the $(i+1)^{th}$ copy of the path.

In [2], an odd edge labeling of a graph is defined as follows: A labeling $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ is called an odd edge labeling of G if for the edge labeling f^+ on $E(G)$ defined by $f^+(uv) = f(u) + f(v)$ for any edge $uv \in E(G)$, for a connected graph G, the edge labeling is not necessarily injective. In [5], the concept of pair sum labeling was introduced. An injective function $f : V(G) \to \{\pm1, \pm2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e : E(G) \to \mathbb{Z} - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm \frac{k_q}{2}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\} \cup \{\frac{k_{q+1}}{2}\}$ according as q is even or odd. A graph with a pair sum labeling defined on it is called a pair sum graph. In [6], the concept of mean labeling was introduced. An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is said to be a mean labeling if the induced edge labeling f^* defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is injective and $f^*(E(G)) = \{1, 2, \ldots, q\}$. A graph G is said to be odd mean if there exists an injective function f from $V(G)$ to $\{0, 1, 2, 3, \ldots, 2q-1\}$ such that the induced map f^* from $E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

is a bijection [6].

Motivated by these, we introduce a new concept called odd sum labeling. An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u)+f(v)$, for all $uv \in E(G)$, is bijective and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper, we have studied the odd sum property for the graphs paths P_p, cycles C_p, $C_p \odot K_1$, the ladder $P_2 \times P_p$, $P_m \odot nK_1$, the balloon graph $P_n(C_p)$, quadrilateral snake Q_n, $[P_m; C_n]$, $(P_m; Q_3)$, $T_p^{(n)}$, $H_n \odot mK_1$, bistar graph and cyclic ladder $P_2 \times C_p$.

§2. Main Results

Observation 2.1 *Every graph having an odd cycle is not an odd sum graph.*

Proof If a graph has a cycle of odd length, then at least one edge uv on the cycle such that $f(u)$ and $f(v)$ are of same suit and hence its induced edge label $f^*(uv)$ is even. □

Proposition 2.2 *Every path $P_p, p \geq 2$ is an odd sum graph.*

Proof Let $v_1, v_2, \ldots, v_p$ be the vertices of the path P_p. The labeling $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is defined as $f(v_i) = i - 1$ for $1 \le i \le p$ and the induced edge label is $f^*(v_i v_{i+1}) = 2i - 1$, for $1 \le i \le p - 1$. Then f is an odd sum labeling and hence P_p is an odd sum graph. □

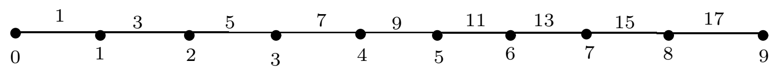

Figure 1: An odd sum labeling of P_{10}.

Proposition 2.3 *Cycle C_p is an odd sum graph only when $p \equiv 0 (mod\ 4)$.*

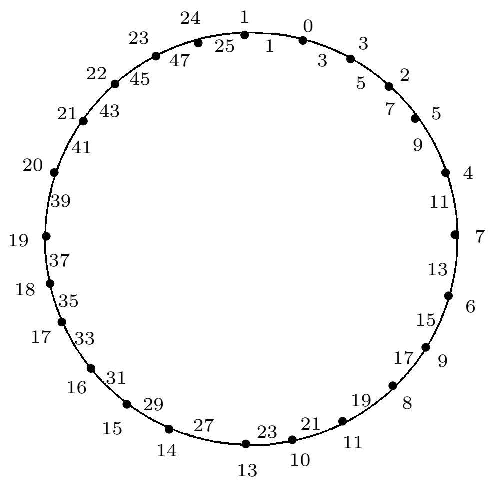

Figure 2: An odd sum labeling of C_{24}.

Proof By Observation 2.1, C_p is not an odd sum graph when p is odd. Suppose $p = 2m, m \ge 2$ and C_p admits an odd sum labeling. Then $\sum\limits_{uv \in E(G)} f^*(uv) = \sum\limits_{uv \in E(G)} (f(u) + f(v))$. This implies that $1 + 3 + \cdots + (4m - 1) = 2(0 + 1 + 2 + \cdots + 2m) - 2i$ where i is not a vertex label of C_p. From this we have, $i = m$. If m is odd, then the number of even values is in excess of 2 that of the number of odd values and they are to be assigned as vertex labels in C_p. Thus if C_p admits an odd sum labeling, then m should be even and hence p is a multiple of 4.

Suppose $p = 4m, m \ge 1$. Let $v_1, v_2, \ldots, v_p$ be the vertices of the cycle C_p. The labeling $f : V(G) \to \{0, 1, 2, \ldots, 4m\}$ is defined as follows.

$$f(v_i) = \begin{cases} i, & 1 \le i \le 2m - 1 \text{ and } i \text{ is odd,} \\ i - 2, & 1 \le i \le 2m \text{ and } i \text{ is even,} \\ i, & 2m + 1 \le i \le 4m. \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(v_i v_{i+1}) = \begin{cases} 2i - 1, & 1 \le i \le 2m, \\ 2i + 1, & 2m + 1 \le i \le 4m - 1 \text{ and} \end{cases}$$

$$f^*(v_{4m} v_1) = 4m + 1.$$

Hence f is an odd sum labeling of C_p only when $p \equiv 0(mod\ 4)$. □

Proposition 2.4 *For each even integer $p \geq 4$, $C_p \odot K_1$ is an odd sum graph.*

Proof In $C_p \odot K_1$, let $v_1, v_2, \ldots, v_p$ be the vertices on the cycle and let u_i be the pendant vertex of v_i at each i, $1 \leq i \leq p$.

Case 1 $p = 4m$, for $m \geq 1$.

The labeling $f : V(C_p \odot K_1) \to \{0, 1, 2, \ldots, 8m\}$ is defined as follows.

$$f(v_i) = \begin{cases} 2i-2, & 1 \leq i \leq 2m-1 \text{ and } i \text{ is odd,} \\ 2i, & 2m+1 \leq i \leq 4m-1 \text{ and } i \text{ is odd,} \\ 2i-1, & 2 \leq i \leq 4m \text{ and } i \text{ is even and} \end{cases}$$

$$f(u_i) = \begin{cases} 2i-1, & 1 \leq i \leq 4m-1 \text{ and } i \text{ is odd,} \\ 2i-2, & 2 \leq i \leq 2m \text{ and } i \text{ is even,} \\ 2i, & 2m+2 \leq i \leq 4m \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(v_i v_{i+1}) = \begin{cases} 4i-1, & 1 \leq i \leq 2m-1, \\ 4i+1, & 2m \leq i \leq 4m-1, \end{cases}$$

$$f^*(v_{4m} v_1) = 8m-1 \textit{ and}$$

$$f^*(u_i v_i) = \begin{cases} 4i-3, & 1 \leq i \leq 2m, \\ 4i-1, & 2m+1 \leq i \leq 4m. \end{cases}$$

Thus f is an odd sum labeling of $C_p \odot K_1$. Hence $C_p \odot K_1$ is an odd sum graph when $p = 4m$.

Case 2 $p = 4m+2$, for $m \geq 1$.

The labeling $f : V(C_p \odot K_1) \to \{0, 1, 2, \ldots, 8m+4\}$ is defined as follows.

$$f(v_i) = \begin{cases} 2i-2, & 1 \leq i \leq 2m+1 \text{ and } i \text{ is odd,} \\ 2i, & 2m+3 \leq i \leq 4m+1 \text{ and } i \text{ is odd,} \\ 2i-1, & 2 \leq i \leq 2m \text{ and } i \text{ is even,} \\ 2i+1, & i = 2m+2, \\ 2i-1, & 2m+4 \leq i \leq 4m+2 \text{ and } i \text{ is even and} \end{cases}$$

$$f(u_i) = \begin{cases} 2i-1, & 1 \leq i \leq 2m+1 \text{ and } i \text{ is odd,} \\ 2i-3, & i = 2m+3, \\ 2i-1, & 2m+5 \leq i \leq 4m+1 \text{ and } i \text{ is odd,} \\ 2i-2, & 2 \leq i \leq 2m+2 \text{ and } i \text{ is even,} \\ 2i, & 2m+4 \leq i \leq 4m+2 \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(v_iv_{i+1}) = \begin{cases} 4i-1, & 1 \le i \le 2m, \\ 4i+1, & i = 2m+1, \\ 4i+3, & i = 2m+2, \\ 4i+1, & 2m+3 \le i \le 4m+1, \end{cases}$$

$$f^*(v_{4m+2}v_1) = 8m+3 \textit{ and}$$

$$f^*(u_iv_i) = \begin{cases} 4i-3, & 1 \le i \le 2m+1, \\ 4i-1, & i = 2m+2, \\ 4i-3, & i = 2m+3 \\ 4i-1, & 2m+4 \le i \le 4m+2. \end{cases}$$

Thus f is an odd sum labeling of $C_p \odot K_1$. Hence $C_p \odot K_1$ is an odd sum graph. □

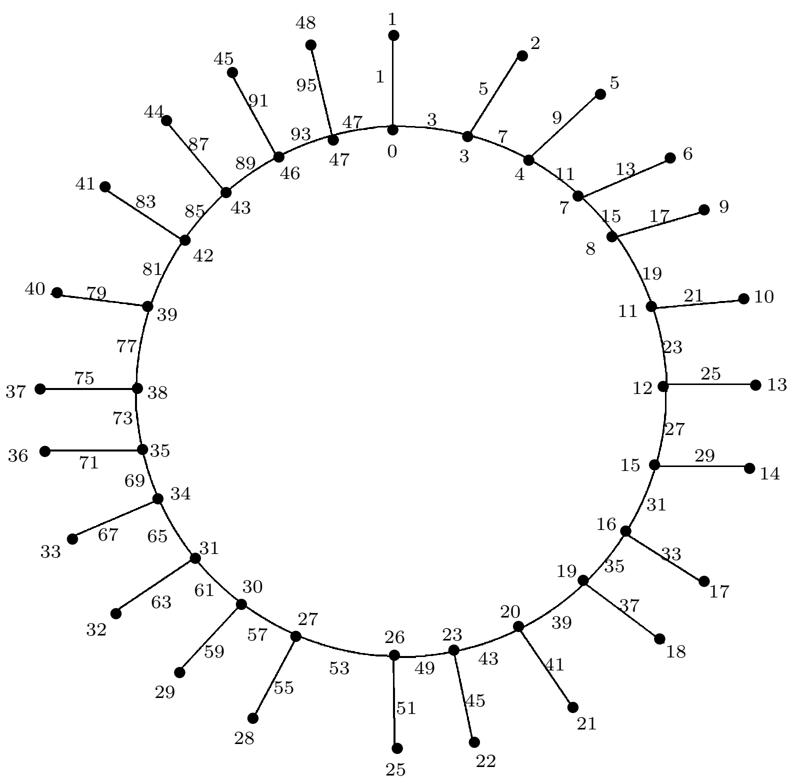

Figure 3: An odd sum labeling of $C_{24} \odot K_1$.

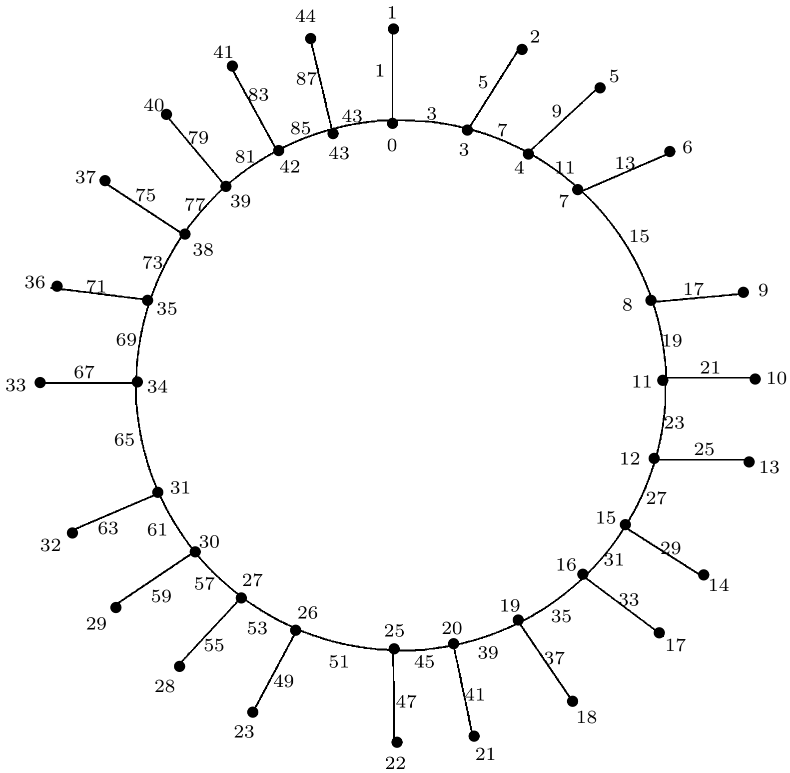

Figure 4: An odd sum labeling of $C_{22} \odot K_1$.

Proposition 2.5 *For every positive integer $p \geq 2$, the ladder $P_2 \times P_p$ is an odd sum graph.*

Proof Let $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_p$ be the vertices of the two copies of P_p. The labeling $f : V(P_2 \times P_p) \to \{0, 1, 2, \ldots, 3p - 2\}$ is defined as follows.

$$f(u_i) = 3i - 3, for\ 1 \leq i \leq p\ and$$
$$f(v_i) = 3i - 2, for\ 1 \leq i \leq p.$$

The induced edge labels are obtained as follows.

$$f^*(u_i u_{i+1}) = 6i - 3,\ \ for\ 1 \leq i \leq p - 1,$$
$$f^*(v_i v_{i+1}) = 6i - 1,\ \ for\ 1 \leq i \leq p - 1\ and$$
$$f^*(u_i v_i) = 6i - 5, for\ 1 \leq i \leq p.$$

Thus f is an odd sum labeling of $P_2 \times P_p$. Hence $P_2 \times P_p$ is an odd sum graph. □

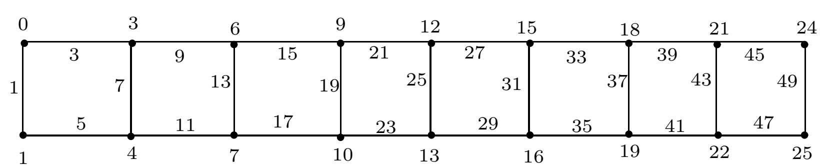

Figure 5: An odd sum labeling of $P_2 \times P_9$.

Proposition 2.6 *The graph $P_m \odot nK_1$ is an odd sum graph if either m is an even positive integer and n is any positive integer or m is an odd positive integer and $n = 1, 2$.*

Proof In $P_m \odot nK_1$, let $u_1, u_2, \ldots, u_m$ be the vertices on the path and $\{u_{i,j} : 1 \leq j \leq n\}$ be the pendant vertices attached at $u_i, 1 \leq i \leq m$.

Case 1 m is even.

The labeling $f : V(P_m \odot nK_1) \to \{0, 1, 2, \ldots, m(n+1) - 1\}$ is defined as follows.
For $1 \leq i \leq m$,

$$f(u_i) = \begin{cases} (n+1)(i-1), & i \text{ is odd,} \\ (n+1)i - 1, & i \text{ is even.} \end{cases}$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$f(u_{i,j}) = \begin{cases} (n+1)(i-1) + 2j - 1, & i \text{ is odd,} \\ (n+1)(i-2) + 2j, & i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(u_i u_{i,j}) = 2(n+1)(i-1) + 2j - 1, \ \textit{for } 1 \leq i \leq m \textit{ and } 1 \leq j \leq n \textit{ and}$$
$$f^*(u_i u_{i+1}) = 2(n+1)i - 1, \ \textit{for } 1 \leq i \leq m - 1.$$

Thus f is an odd sum labeling of $P_m \odot nK_1$.

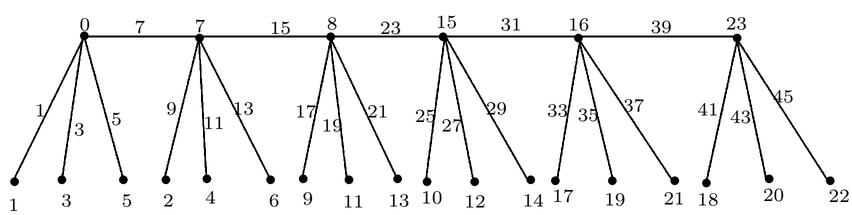

Figure 6: An odd sum labeling of $P_6 \odot 3K_1$.

Case 2 m is odd.

If $P_m \odot nK_1$ has an odd sum labeling f when m is odd, then f is a bijection from $V(P_m \odot nK_1)$ to the set $\{0, 1, 2, \ldots, m(n+1) - 1\}$. Since the number of even integers in this set is either equal to or one excess to the number of odd integers in this set, n should be less than or equal to 2.

In case of m is odd and $n = 1, 2$, the labeling $f : V(P_m \odot nK_1) \to \{0, 1, 2, \ldots, m(n+1) - 1\}$ is defined as follows.

For $1 \leq i \leq m$,

$$f(u_i) = \begin{cases} (n+1)(i-1) + 1, & i \text{ is odd,} \\ (n+1)i - 2, & i \text{ is even.} \end{cases}$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$f(u_{i,j}) = \begin{cases} (n+1)(i-1)+2(j-1), & i \text{ is odd,} \\ (n+1)(i-2)+2j+1, & i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows.

For $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$f^*(u_i u_{i,j}) = 2(n+1)(i-1)+2j-1.$$

For $1 \leq i \leq m-1$,

$$f^*(u_i u_{i+1}) = 2(n+1)i-1.$$

Thus f is an odd sum labeling of $P_m \odot nK_1$. □

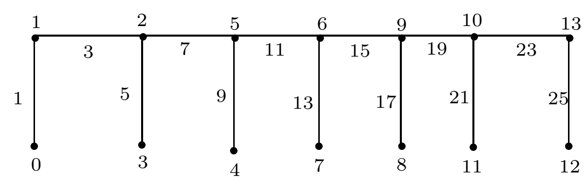

Figure 7: An odd sum labeling of $P_7 \odot K_1$.

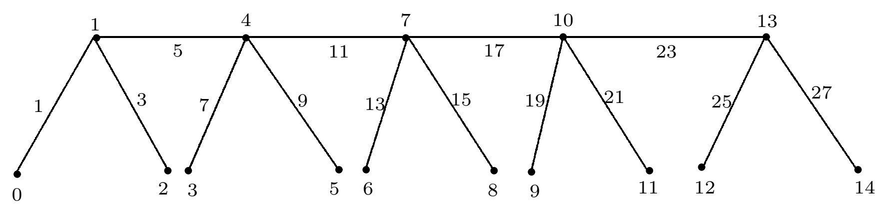

Figure 8: An odd sum labeling of $P_5 \odot 2K_1$.

Proposition 2.7 *The graph $P_n(C_p)$ is an odd sum graph if either $p \equiv 0(mod\ 4)$ or $p \equiv 2(mod\ 4)$ and $n \not\equiv 1(mod\ 3)$.*

Proof Let $u_1, u_2, \ldots, u_p$ be the vertices of C_p and $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n and u_p be identified with v_1 in $P_n(C_p)$.

Case 1 $p \equiv 0(mod\ 4)$.

Let $p = 4m, m \geq 1$. The labeling $f : V(P_n(C_p)) \to \{0, 1, 2, \ldots, 4m+n-1\}$ is defined as follows.

$$f(u_i) = \begin{cases} i, & 1 \leq i \leq 4m \text{ and } i \text{ is odd,} \\ i-2, & 1 \leq i \leq 2m \text{ and } i \text{ is even,} \\ i, & 2m+1 \leq i \leq 4m \text{ and } i \text{ is even,} \end{cases}$$

$f(v_i) = 4m+i-1, 2 \leq i \leq n.$

The induced edge labels are obtained as follows.

$$f^*(u_iu_{i+1}) = \begin{cases} 2i-1, & 1 \le i \le 2m, \\ 2i+1, & 2m+1 \le i \le 4m-1. \end{cases}$$

$$f^*(u_1u_{4m}) = 4m+1 \text{ and}$$
$$f^*(v_iv_{i+1}) = 8m+2i-1, \quad 1 \le i \le n-1.$$

Thus f is an odd sum labeling of $P_n(C_p)$.

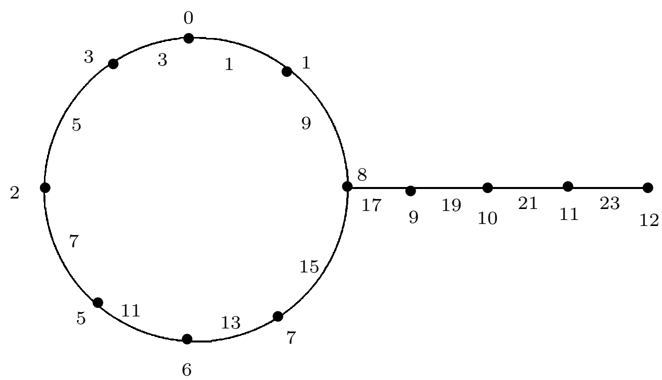

Figure 9: An odd sum labeling of $P_5(C_8)$.

Case 2 $p \equiv 2(mod\ 4)$.

Let $p = 4m+2, m \ge 1$.

Subcase 2.1 $n \equiv 0(mod\ 3)$.

The labeling $f : V(P_n(C_p)) \to \{0, 1, 2, \ldots, 4m+n+1\}$ is defined as follows.

$$f(u_1) = 4m+3,$$
$$f(u_i) = \begin{cases} i-2, & 1 \le i \le 2m+3, \\ i, & 2m+4 \le i \le 4m+2 \text{ and } i \text{ is even}, \\ i-2, & 2m+4 \le i \le 4m+2 \text{ and } i \text{ is odd and} \end{cases}$$

$$f(v_i) = \begin{cases} 4m+i+1, & 1 \le i \le n-3 \text{ and } i \equiv 1(mod\ 3), \\ 4m+i-1, & 1 \le i \le n-1 \text{ and } i \equiv 2(mod\ 3), \\ 4m+i+3, & 1 \le i \le n-1 \text{ and } i \equiv 0(mod\ 3), \\ 4m+n+1, & i = n-2, \\ 4m+n-1, & i = n. \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(u_iu_{i+1}) = \begin{cases} 4m+3, & i=1, \\ 2i-3, & 2 \leq i \leq 2m+2, \\ 2i-1, & 2m+3 \leq i \leq 4m+1. \end{cases}$$

$f^*(u_{4m+2}u_1) = 8m+5$ *and*

$$f^*(v_iv_{i+1}) = \begin{cases} 8m+2i+3, & 2 \leq i \leq n-4 \text{ and } i \equiv 2(mod\ 3), \\ 8m+2i+5, & 2 \leq i \leq n-4 \text{ and } i \equiv 0(mod\ 3), \\ 8m+2i+1, & 2 \leq i \leq n-4 \text{ and } i \equiv 1(mod\ 3), \\ 8m+4n-2i-5, & n-3 \leq i \leq n-1. \end{cases}$$

Thus f is an odd sum labeling of $P_n(C_p)$.

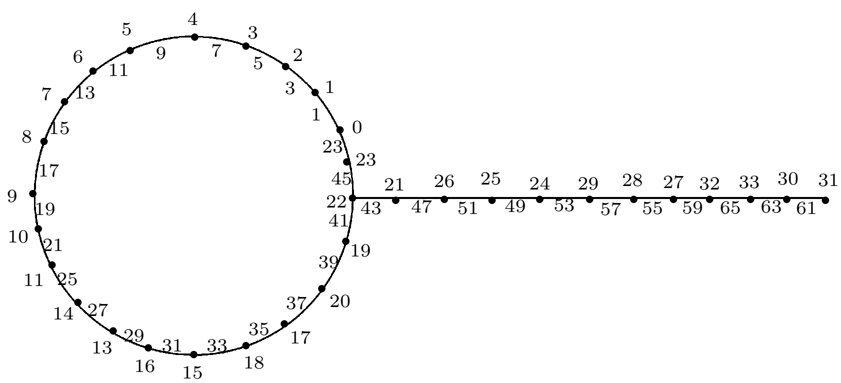

Figure 10: An odd sum labeling of $P_{12}(C_{22})$.

Subcase 2.2 $n \equiv 2(mod\ 3)$.

The labeling $f : V(P_n(C_p)) \to \{0, 1, 2, \ldots, 4m+n+1\}$ is defined as follows.

$$f(u_1) = 4m+3,$$

$$f(u_i) = \begin{cases} i-2, & 1 \leq i \leq 2m+3, \\ i, & 2m+4 \leq i \leq 4m+2 \text{ and } i \text{ is even}, \\ i-2, & 2m+4 \leq i \leq 4m+2 \text{ and } i \text{ is odd}, \end{cases}$$

$$and\ f(v_i) = \begin{cases} 4m+i+1, & 1 \leq i \leq n \text{ and } i \equiv 1(mod\ 3), \\ 4m+i-1, & 1 \leq i \leq n \text{ and } i \equiv 2(mod\ 3), \\ 4m+i+3, & 1 \leq i \leq n \text{ and } i \equiv 0(mod\ 3). \end{cases}$$

The induced edge labels are obtained as follows.

$$f^*(u_iu_{i+1}) = \begin{cases} 4m+3, & i=1, \\ 2i-3, & 2 \le i \le 2m+2, \\ 2i-1, & 2m+3 \le i \le 4m+1, \end{cases}$$

$f^*(u_{4m+2}u_1) = 8m+5$ *and*

$$f^*(v_iv_{i+1}) = \begin{cases} 8m+2i+1, & 1 \le i \le n \text{ and } i \equiv 1(mod\ 3), \\ 8m+2i+3, & 1 \le i \le n \text{ and } i \equiv 2(mod\ 3), \\ 8m+2i+5, & 1 \le i \le n \text{ and } i \equiv 0(mod\ 3). \end{cases}$$

Thus f is an odd sum labeling of $P_n(C_p)$. Hence $P_n(C_p)$ is an odd sum graph. □

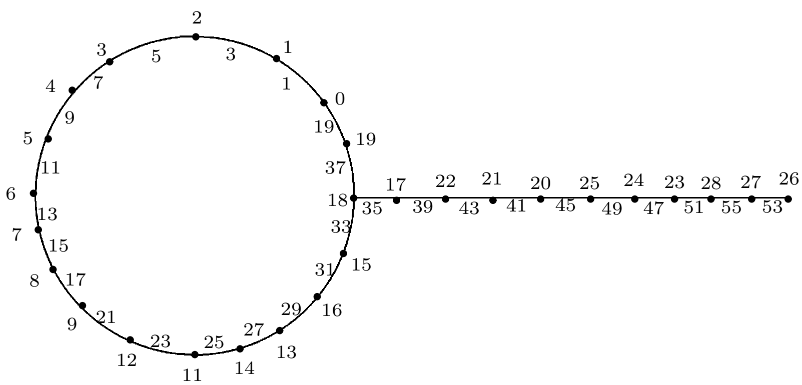

Figure 11: An odd sum labeling of $P_{11}(C_{18})$.

Proposition 2.8 *$[P_m; C_n]$ is an odd sum graph for $n \equiv 0(mod\ 4)$ and any $m \ge 2$.*

Proof In $[P_m; C_n]$, let $v_1, v_2, \ldots, v_m$ be the vertices on the path P_m, $v_{i,1}, v_{i,2}, \ldots, v_{i,n}$ be the vertices of the i^{th} cycle C_n, for $1 \le i \le m$ and each vertex $v_{i,1}$ of the i^{th} cycle C_n is identified with the vertex v_i of the path P_m, $1 \le i \le m$.

Suppose $n = 4t, t \ge 1$. The labeling $f : V([P_m; C_n]) \to \{0, 1, 2, 3, \ldots, m(n+1)-1\}$ is defined as follows.
For $1 \le i \le m$,

$$f(v_{i,j}) = \begin{cases} (n+1)(i-1)+j-1, & 1 \le j \le 2t,\ i \text{ and } j \text{ are odd}, \\ (n+1)(i-1)+j+1, & 2t+1 \le j \le 4t,\ i \text{ and } j \text{ are odd}, \\ (n+1)(i-1)+j-1, & 1 \le j \le 4t,\ i \text{ is odd and } j \text{ are even}, \\ (n+1)i-j, & 1 \le j \le 2t,\ i \text{ is even and } j \text{ is odd}, \\ (n+1)i-j-2, & 2t+1 \le j \le 4t,\ i \text{ is even and } j \text{ is odd}, \\ (n+1)i-j, & 1 \le j \le 4t,\ i \text{ is even and } j \text{ is even}. \end{cases}$$

For $1 \le i \le m$, the induced edge label is obtained as follows.

$$f^*(v_{i,j}v_{i,j+1}) = \begin{cases} 2(n+1)(i-1)+2j-1, & 1 \le j \le 2t-1 \text{ and } i \text{ is odd,} \\ 2(n+1)(i-1)+2j+1, & 2t \le j \le 4t-1 \text{ and } i \text{ is odd,} \\ 2(n+1)(i-1)+9, & j = 1 \text{ and } i \text{ is even,} \\ 2(n+1)(i-1)+2j-3, & 2 \le j \le 2t+1 \text{ and } i \text{ is even,} \\ 2(n+1)(i-1)+2j-1, & 2t+2 \le j \le 4t-1 \text{ and } i \text{ is even} \end{cases}$$

$$\textit{and } f^*(v_{i,4t}v_{i,1}) = \begin{cases} 2(n+1)(i-1)+4t-1, & i \text{ is odd,} \\ 2(n+1)(i-1)+8t-1, & i \text{ is even.} \end{cases}$$

Thus f is an odd sum labeling of $[P_m; C_n]$. Hence $[P_m; C_n]$ is an odd sum graph. □

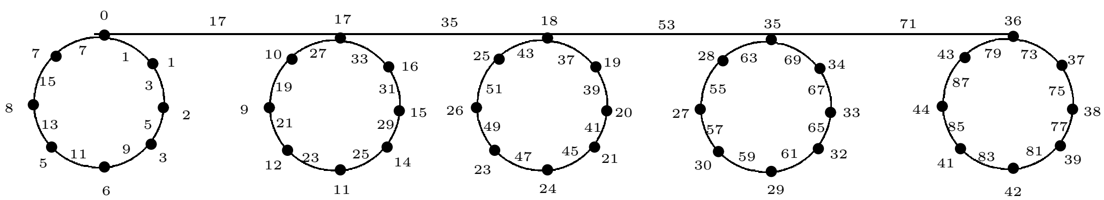

Figure 12: An odd sum labeling of $[P_5; C_8]$.

Proposition 2.9 *Quadrilateral snake Q_n is an odd sum graph for $n \ge 1$.*

Proof The vertex set and edge set of the Quadrilateral snake Q_n are $V(Q_n) = \{u_i, v_j, w_j : 1 \le i \le n+1, 1 \le j \le n\}$ and $E(Q_n) = \{u_iv_i, v_iw_i, u_iu_{i+1}, u_{i+1}w_i : 1 \le i \le n\}$ respectively. The labeling $f : V(Q_n) \to \{0, 1, 2, \ldots, 4n\}$ is defined as follows.

$$f(u_i) = \begin{cases} 4i-4, & 1 \le i \le n+1 \text{ and } i \text{ is odd,} \\ 4i-5, & 1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_i) = \begin{cases} 4i-3, & 1 \le i \le n \text{ and } i \text{ is odd,} \\ 4i-2, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$\textit{and } f(w_i) = \begin{cases} 4i, & 1 \le i \le n \text{ and } i \text{ is odd,} \\ 4i-1, & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows

$$f^*(u_iu_{i+1}) = 8i-5, \quad 1 \le i \le n,$$
$$f^*(u_iv_i) = 8i-7, \quad 1 \le i \le n,$$
$$f^*(v_iw_i) = 8i-3, \quad 1 \le i \le n,$$
$$f^*(w_iu_{i+1}) = 8i-1, \quad 1 \le i \le n.$$

Thus f is an odd sum labeling of Q_n. Hence the Quadrilateral snake Q_n is an odd sum graph for $n \geq 1$. □

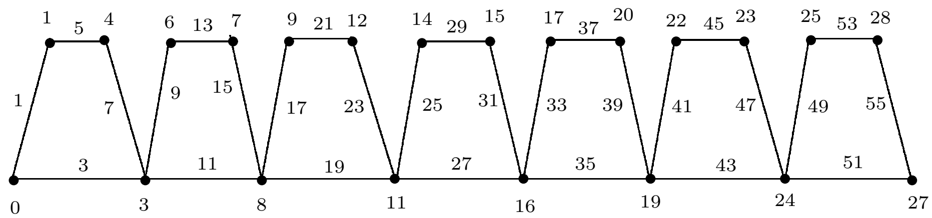

Figure 13: An odd sum labeling of Q_7.

Proposition 2.10 *$(P_m; Q_3)$ is an odd sum graph for any positive integer $m \geq 1$.*

Proof Let $v_{i,j}$, $1 \leq j \leq 8$ be the vertices in the i^{th} copy of Q_3, $1 \leq i \leq m$ and $u_1, u_2, \ldots, u_m$ be the vertices on the path P_m. $\{u_iu_{i+1} : 1 \leq i \leq m-1\} \cup \{u_iv_{i,1} : 1 \leq i \leq m\} \cup \{v_{i,1}v_{i,2}, v_{i,1}v_{i,4}, v_{i,1}v_{i,6}, v_{i,2}v_{i,3}, v_{i,2}v_{i,7}, v_{i,3}v_{i,4}, v_{i,3}v_{i,8}, v_{i,4}v_{i,5}, v_{i,5}v_{i,6}, v_{i,5}v_{i,8}, v_{i,6}v_{i,7}, v_{i,7}v_{i,8} : 1 \leq i \leq m\}$ be the edge set of $(P_m; Q_3)$.

The labeling $f : V[(P_m; Q_3)] \to \{0, 1, 2, \ldots, 14m-1\}$ is defined as follows:

For $1 \leq i \leq m$,

$$f(u_i) = \begin{cases} 14(i-1), & i \text{ is odd}, \\ 14i-1, & i \text{ is even}. \end{cases}$$

For $1 \leq i \leq m$ and i is odd,

$$f(v_{i,j}) = \begin{cases} 14i-13, & j = 1, \\ 14i-12+j, & 2 \leq j \leq 3, \\ 14i-12, & j = 4, \\ 14i-5, & j = 5, \\ 14i-8+j, & 6 \leq j \leq 7, \\ 14i-4, & j = 8. \end{cases}$$

For $1 \leq i \leq m$ and i is even,

$$f(v_{i,j}) = \begin{cases} 14i-2, & j = 1, \\ 14i-j-3, & 2 \leq j \leq 3, \\ 14i-3, & j = 4, \\ 14i-10, & j = 5, \\ 14i-j-7, & 6 \leq j \leq 7, \\ 14i-11, & j = 8. \end{cases}$$

The induced edge label of $(P_m; Q_3)$ is obtained as follows:

For $1 \le i \le m-1$,

$$f^*(u_iu_{i+1}) = 28i - 1.$$

For $1 \le i \le m$,

$$f^*(u_iv_{i,1}) = \begin{cases} 28i-27, & i \text{ is odd,} \\ 28i-3, & i \text{ is even.} \end{cases}$$

For $1 \le i \le m$ and i is odd	For $1 \le i \le m$ and i is even
$f^*(v_{i,1}v_{i,2}) = 28i - 23$	$f^*(v_{i,1}v_{i,2}) = 28i - 7$
$f^*(v_{i,1}v_{i,4}) = 28i - 25,$	$f^*(v_{i,1}v_{i,4}) = 28i - 5$
$f^*(v_{i,1}v_{i,6}) = 28i - 15$	$f^*(v_{i,1}v_{i,6}) = 28i - 15$
$f^*(v_{i,2}v_{i,3}) = 28i - 19$	$f^*(v_{i,2}v_{i,3}) = 28i - 11$
$f^*(v_{i,2}v_{i,7}) = 28i - 11$	$f^*(v_{i,2}v_{i,7}) = 28i - 19$
$f^*(v_{i,3}v_{i,4}) = 28i - 21$	$f^*(v_{i,3}v_{i,4}) = 28i - 9$
$f^*(v_{i,3}v_{i,8}) = 28i - 13$	$f^*(v_{i,3}v_{i,8}) = 28i - 17$
$f^*(v_{i,4}v_{i,5}) = 28i - 17$	$f^*(v_{i,4}v_{i,5}) = 28i - 13$
$f^*(v_{i,5}v_{i,6}) = 28i - 7$	$f^*(v_{i,5}v_{i,6}) = 28i - 23$
$f^*(v_{i,5}v_{i,8}) = 28i - 9$	$f^*(v_{i,5}v_{i,8}) = 28i - 21,$
$f^*(v_{i,6}v_{i,7}) = 28i - 3$	$f^*(v_{i,6}v_{i,7}) = 28i - 27$
$f^*(v_{i,7}v_{i,8}) = 28i - 5$	$f^*(v_{i,7}v_{i,8}) = 28i - 25$

Thus f is an odd sum labeling of $(P_m; Q_3)$. Hence $(P_m; Q_3)$ is an odd sum graph. □

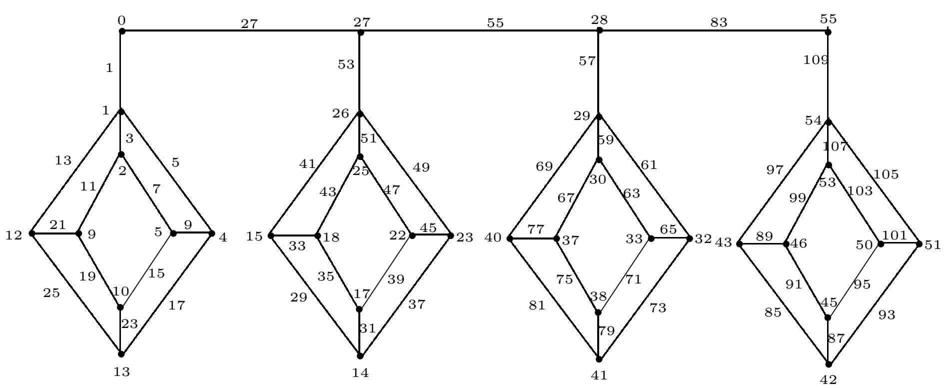

Figure 14: An odd sum labeling of $(P_4; Q_3)$.

Proposition 2.11 *For all positive integers p and n, the graph $T_p^{(n)}$ is an odd sum graph.*

Proof Let $v_i^{(j)}$, $1 \le i \le p$ be the vertices of the j^{th} copy of the path on p vertices, $1 \le j \le n$. The graph $T_p^{(n)}$ is formed by adding an edge $v_i^{(j)}v_i^{(j+1)}$ between j^{th} and $(j+1)^{th}$ copy of the path at some i, $1 \le i \le p$., The labeling $f : V(G) \to \{0, 1, 2, \ldots, np-1\}$ is defined as follows:

For $1 \le j \le n$ and $1 \le i \le p$,

$$f(v_i^{(j)}) = \begin{cases} p(j-1)+i-1, & j \text{ is odd,} \\ pj-i, & j \text{ is even.} \end{cases}$$

The induced edge labeling is obtained as follows:

For $1 \le j \le n$ and $1 \le i \le p-1$,

$$f^*(v_i^{(j)}v_{i+1}^{(j)}) = \begin{cases} 2p(j-1)+2i-1, & j \text{ is odd,} \\ 2pj-2i-1, & j \text{ is even} \quad \text{and} \end{cases}$$

$$f^*(v_i^{(j)}v_i^{(j+1)}) = 2pj-1.$$

Thus f is an odd sum labeling of the graph $T_p^{(n)}$. Hence $T_p^{(n)}$ is an odd sum graph. □

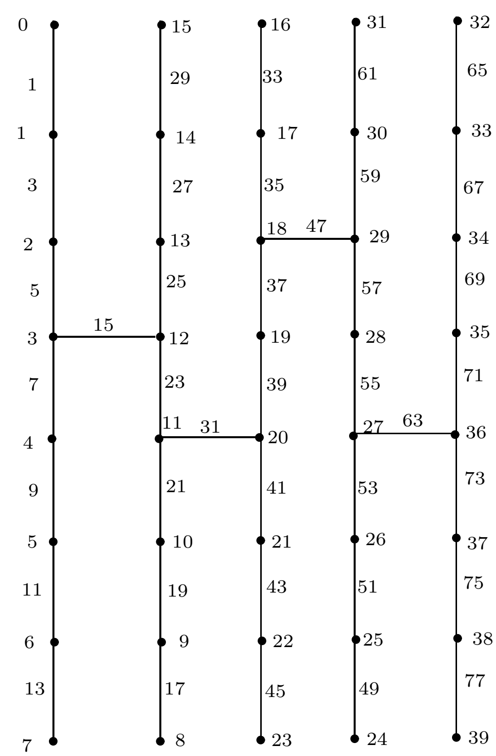

Figure 15: An odd sum labeling of $T_8^{(5)}$.

Proposition 2.12 *The graph $H_n \odot mK_1$ is an odd sum graph for all positive integers m and n.*

Proof Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the path of length $n-1$. Let $x_{i,k}$ and $y_{i,k}$, $1 \le k \le m$, be the pendant vertices at u_i and v_i respectively, for $1 \le i \le n$. Define $f : V(H_n \odot mK_1) \to \{0, 1, 2, \ldots, 2n(m+1)-1\}$ as follows:

For $1 \leq i \leq n$,

$$f(u_i) = \begin{cases} i + m(i-1), & i \text{ is odd,} \\ i(m+1) - 2, & i \text{ is even} \quad \text{and} \end{cases}$$

$$f(v_i) = \begin{cases} f(u_i) + n(m+1) + m - 2, & i \text{ is odd and } n \text{ is odd,} \\ f(u_i) + n(m+1) - m + 2, & i \text{ is even and } n \text{ is odd,} \\ f(u_i) + n(m+1), & n \text{ is even.} \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq k \leq m$,

$$f(x_{i,k}) = \begin{cases} (m+1)(i-1) + 2k - 2, & i \text{ is odd,} \\ (m+1)(i-2) + 2k + 1, & i \text{ is even} \quad \text{and} \end{cases}$$

$$f(y_{i,k}) = \begin{cases} f(x_{i,k}) + n(m+1) - m + 2, & i \text{ is odd and } n \text{ is odd,} \\ f(x_{i,k}) + n(m+1) + m - 2, & i \text{ is even and } n \text{ is odd,} \\ f(x_{i,k}) + n(m+1), & n \text{ is even.} \end{cases}$$

The induced edge labels are obtained as follows:

For $1 \leq i \leq n-1$,

$$f^*(u_i u_{i+1}) = 2i(m+1) - 1 \text{ and}$$
$$f^*(v_i v_{i+1}) = f^*(u_i u_{i+1}) + 2n(m+1).$$

For $1 \leq i \leq n$ and $1 \leq k \leq m$,

$$f^*(u_i x_{i,k}) = 2(m+1)(i-1) + 2k - 1 \text{ and}$$
$$f^*(v_i y_{i,k}) = f^*(u_i x_{i,k}) + 2n(m+1).$$

When n is odd,

$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = 2n(m+1) - 1.$$

When n is even,

$$f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = 2n(m+1) - 1.$$

Thus f is an odd sum labeling of $H_n \odot mK_1$. Hence $H_n \odot mK_1$ is an odd sum graph for all positive integers m and n. □

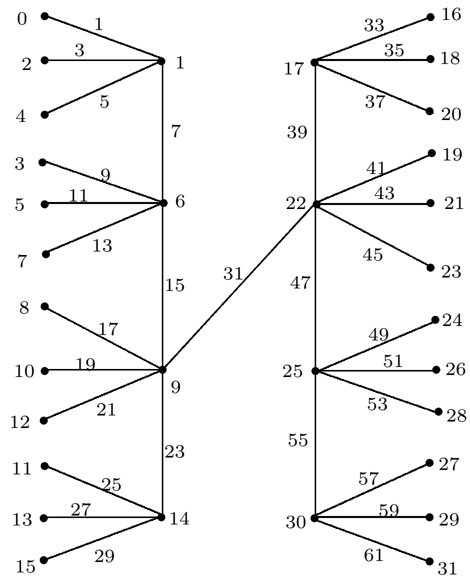

Figure 16: An odd sum labeling of $H_4 \odot 3K_1$.

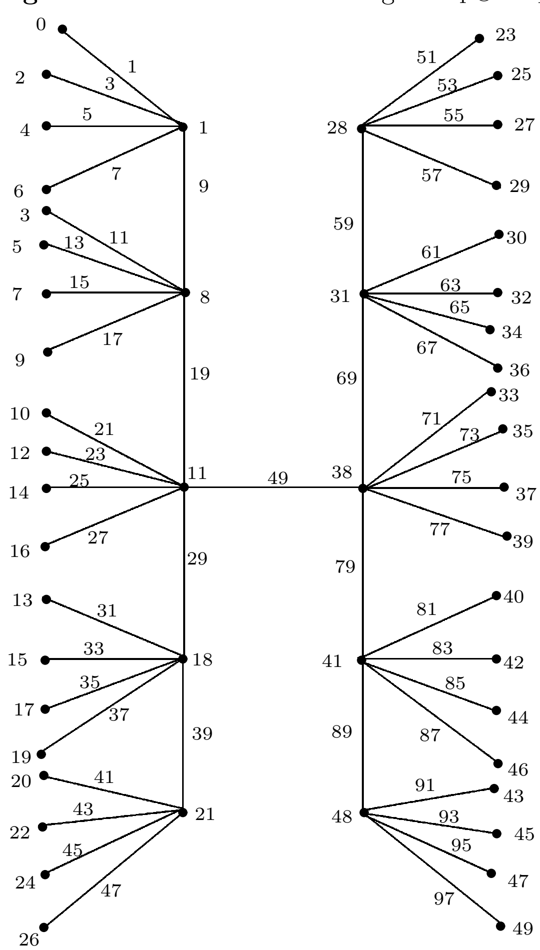

Figure 17: An odd sum labeling of $H_5 \odot 4K_1$.

Corollary 2.13 *For any positive integer m, the bistar graph $B(m,m)$ is an odd sum graph.*

Proof By taking $n = 1$ in Proposition 2.12, the result follows. □

Proposition 2.14 *For any even integer $p \geq 4$, the cyclic ladder $P_2 \times C_p$ is an odd sum graph.*

Proof Let $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_p$ be the vertices of the inner and outer cycle which are joined by the edges $\{u_i v_i : 1 \leq i \leq p\}$.

Case 1 $p = 4m, m \geq 2$.

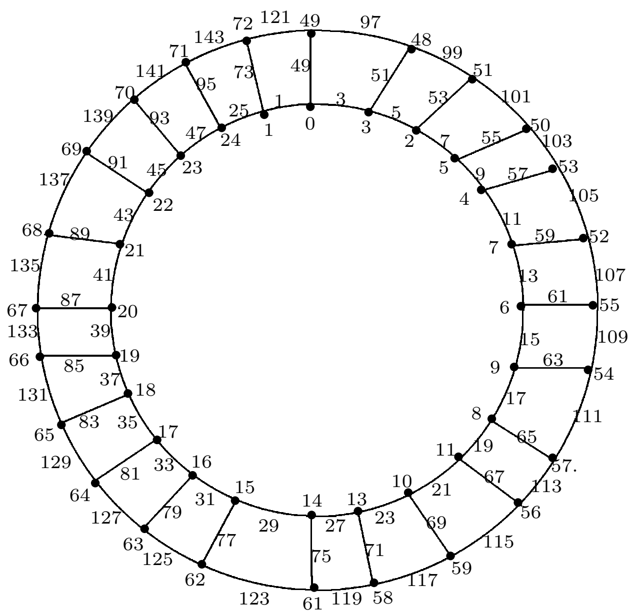

Figure 18: An odd sum labeling of $P_2 \times C_{24}$.

The labeling $f : V(P_2 \times C_p) \rightarrow \{0, 1, 2, \ldots, 12m\}$ is defined as follows:

$$f(u_i) = \begin{cases} i-1, & 1 \leq i \leq 2m-1 \text{ and } i \text{ is odd}, \\ i+1, & 2 \leq i \leq 4m-2 \text{ and } i \text{ is even}, \\ i+1, & 2m+1 \leq i \leq 4m-1 \text{ and } i \text{ is odd}, \end{cases}$$

$f(u_{4m}) = 1$ *and*

$$f(v_i) = \begin{cases} 8k+i, & 1 \leq i \leq 4m-1 \text{ and } i \text{ is odd}, \\ 8k+i-2, & 2 \leq i \leq 2m \text{ and } i \text{ is even}, \\ 8k+i, & 2m+2 \leq i \leq 4m \text{ and } i \text{ is even}. \end{cases}$$

The induced edge labeling is obtained as follows.

$$f^*(u_iu_{i+1}) = \begin{cases} 2i+1, & 1 \le i \le 2m-1, \\ 2i+3, & 2m \le i \le 4m-2, \\ i+2, & i = 4m-1, \end{cases}$$

$$f^*(u_1u_{4m}) = 1,$$

$$f^*(v_iv_{i+1}) = \begin{cases} 16m+2i-1, & 1 \le i \le 2m \\ 16m+2i+1, & 2m+1 \le i \le 4m-1, \end{cases}$$

$$f^*(v_1v_{4m}) = 20m+1,$$

$$f^*(u_iv_i) = \begin{cases} 8m+2i-1, & 1 \le i \le 2m, \\ 8m+2i+1, & 2m+1 \le i \le 4m-1 \text{ and} \end{cases}$$

$$f^*(u_{4m}v_{4m}) = 12m+1.$$

Thus f is an odd sum labeling of $P_2 \times C_p$. Hence $P_2 \times C_p$ is an odd sum graph when $p = 4m$.

Case 2 $p = 4m+2, m \ge 1$.

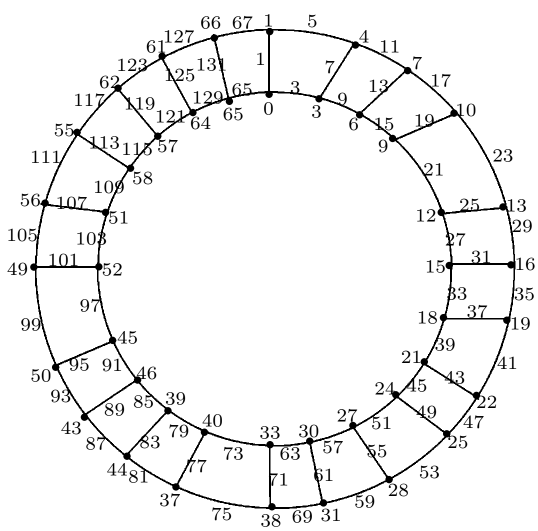

Figure 19: An odd sum labeling of $P_2 \times C_{22}$.

The labeling $f : V(P_2 \times C_p) \to \{0, 1, 2, \ldots, 12m\}$ is defined as follows:

$$f(u_i) = \begin{cases} 3i-3, & 1 \le i \le 2m+2, \\ 3i+1, & 2m+3 \le i \le 4m+1 \text{ and } i \text{ is odd}, \end{cases}$$

$$f(u_i)=\begin{cases}3i-3, & 2m+4\le i\le 4m \text{ and } i \text{ is even},\\ 3i-1, & i=4m+2 \;\text{ and}\end{cases}$$

$$f(v_i)=\begin{cases}3i-2, & 1\le i\le 2m+1,\\ 3i+2, & 2m+2\le i\le 4m \text{ and } i \text{ is even},\\ 3i-2, & 2m+3\le i\le 4m+1 \text{ and } i \text{ is odd},\\ 3i, & i=4m+2.\end{cases}$$

The induced edge labels are given as

$$f^*(u_iu_{i+1})=\begin{cases}6i-3, & 1\le i\le 2m+1,\\ 6i+1, & 2m+2\le i\le 4m,\\ 6i+3, & i=4m+1,\end{cases}$$

$$f^*(u_1u_{4m+2})=12m+5,$$

$$f^*(v_iv_{i+1})=\begin{cases}6i-1, & 1\le i\le 2m,\\ 6i+3, & 2m+1\le i\le 4m,\\ 6i+1, & i=4m+1,\end{cases}$$

$$f^*(v_1v_{4m+2})=12m+7 \textit{ and}$$

$$f^*(u_iv_i)=\begin{cases}6i-5, & 1\le i\le 2m+1,\\ 6i-1, & 2m+2\le i\le 4m+2.\end{cases}$$

Thus f is an odd sum labeling of $P_2\times C_p$. Whence $P_2\times C_p$ is an odd sum graph if $p=4m+2$. □

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