Electronic Journal of Graph Theory and Applications **6** (2) (2018), 317–325

Electronic Journal of Graph Theory and Applications

A note on Fibonacci and Lucas number of domination in path

Leomarich F. Casinillo

Department of Mathematics and Physics
Visayas State University
Visca, Baybay City, Leyte, Philippines

leomarich_casinillo@yahoo.com

Abstract

Let $G = (V(G), E(G))$ be a path of order $n \geq 1$. Let $f_m(G)$ be a path with $m \geq 0$ independent dominating vertices which follows a Fibonacci string of binary numbers where 1 is the dominating vertex. A set $F(G)$ contains all possible $f_m(G)$, $m \geq 0$, having the cardinality of the Fibonacci number F_{n+2}. Let $F_d(G)$ be a set of $f_m(G)$ where $m = i(G)$ and $F_d^{max}(G)$ be a set of paths with maximum independent dominating vertices. Let $l_m(G)$ be a path with $m \geq 0$ independent dominating vertices which follows a Lucas string of binary numbers where 1 is the dominating vertex. A set $L(G)$ contains all possible $l_m(G)$, $m \geq 0$, having the cardinality of the Lucas number L_n. Let $L_d(G)$ be a set of $l_m(G)$ where $m = i(G)$ and $L_d^{max}(G)$ be a set of paths with maximum independent dominating vertices. This paper determines the number of possible elements in the sets $F_d(G)$, $L_d(G)$, $F_d^{max}(G)$ and $L_d^{max}(G)$ by constructing a combinatorial formula. Furthermore, we examine some properties of $F(G)$ and $L(G)$ and give some important results.

Keywords: Fibonacci numbers, Lucas numbers, path, independent domination number
Mathematics Subject Classification : 05C69
DOI: 10.5614/ejgta.2018.6.2.11

Received: 22 December 2016, Revised: 25 May 2018, Accepted: 2 September 2018.

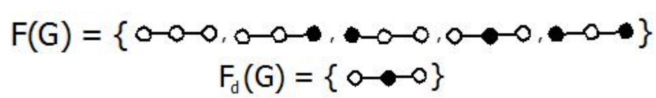
1. Introduction

A *Fibonacci number* F_n and a *Lucas number* L_n can be obtained by the following equations $F_n = F_{n-2} + F_{n-1}$, for $n \in \{3, 4, 5, \cdots\}$, where $F_1 = F_2 = 1$ and $L_n = L_{n-2} + L_{n-1}$ for $n \in \{3, 4, 5, \cdots\}$ where $L_1 = 1$, $L_2 = 3$ [7]. Fibonacci and Lucas sequences have been widely studied by many researchers. A *Fibonacci string* A_n of length n is a binary string $b_1b_2b_3...b_n$ containing no two consecutive $1s$. A *Lucas string* B_n of length n is a binary string $b_1b_2b_3 \cdots b_n$ containing no two consecutive $1s$ and no 1 in the first and in the last positions simultaneously [3, 4].

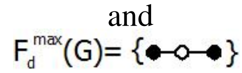
In graph theory, Oystein Ore [5] introduced the concept of a domination set in a graph. The concept of the domination in graphs provides several applications especially in protection strategies and business networking [1]. Let $G = (V(G), E(G))$ be a graph. Let $v \in V(G)$. Then *neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $Z \subseteq V(G)$, then the *open neighborhood* of Z is the set $N_G(Z) = N(Z) = \bigcup_{v \in Z} N_G(v)$. The *closed neighborhood* of Z is $N_G[Z] = N[Z] = Z \cup N(Z)$. A subset D of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus D$ there exists $u \in D$ such that $uv \in E(G)$, i.e., $N[D] = V(G)$. A dominating set $I \subseteq V(G)$ is called an *independent dominating set* of G if no two dominating vertices in the set are adjacent. The *independent domination number* of G, denoted by $i(G)$, is the smallest cardinality of an independent dominating set of G. An independent dominating set with the cardinality $i(G)$ is called a $i - set$ [2]. For example, let $G = P_5$, then we have $i(G) = 2$.

The degree of $v \in V(G)$, denoted by $deg(v)$, is the number of edges incident with v in G. A *walk* is a sequence $u_1, u_2, u_3, \cdots, u_n$ of vertices of graph G such that $\{u_i, u_{i+1}\} \in E(G)$ for each $i = 1, 2, ..., n$. Vertices u_1 and u_n are the endpoints of the walk while the vertices $u_2, u_3, \cdots, u_{n-1}$ are internal vertices of the walk. The *length* of walk is the number of edges on the walk, i.e., the walk $u_1, u_2, u_3, ..., u_n$ has length $n - 1$. A *path* is a walk that does not repeat edges and does end where it starts, i.e., $u_1 \to u_2 \to ... \to u_n, u_1 \neq u_n$. A path of order n and length $n - 1$ is denoted by P_n [5, 1, 6].

Let G be a path and $f_m(G)$ be a path with m (≥ 0) independent dominating vertices which follows a Fibonacci string of binary numbers where 1 is the dominating vertex. A set $F(G)$ contains all possible $f_m(G)$, $m \geq 0$, having the cardinality of the Fibonacci number F_{n+2}. Thus, a set $F(G)$ contains paths of order n with no dominating vertex up to the maximum independent dominating vertices. Let $F_d(G)$ be a set of $f_m(G)$ where $m = i(G)$ and $F_d^{max}(G)$ be a set of paths with maximum independent dominating vertices. For example, let $G = P_3$, then we have

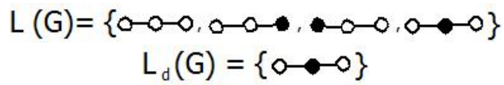


and

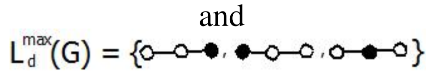


Thus, we obtain $|F(G)| = F_5 = 5$, $|F_d(G)| = 1$ and $|F_d^{max}(G)| = 1$.

Let $l_m(G)$ be a path with m (≥ 0) independent dominating vertices which follows a *Lucas string* of binary numbers where 1 is the *dominating vertex*. The set $L(G)$ contains all possible $l_m(G)$, $m \geq 0$, having the cardinality of the Lucas number L_n. Thus, the set $L(G)$ contains paths of order n with no dominating vertex up to the maximum independent dominating vertices and no two dominating vertices with degree 1. Let $L_d(G)$ be a set of $l_m(G)$ where $m = i(G)$ and $L_d^{max}(G)$ be a set of paths with maximum independent dominating vertices. For example, let $G = P_3$, then we have



and



Thus, we obtain $|L(G)| = L_3 = 4$, $|L_d(G)| = 1$ and $|L_d^{max}(G)| = 3$.

The *sum* of paths P_n and P_m is a path of $n + m$ vertices by connecting the last vertex of P_n to the first vertex of P_m, and it is denoted as $P_n \Phi P_m = P_{n+m}$. A path with order n and with one dominating vertex at i^{th} vertex is denoted by P_n^i.

2. Results

From the above definitions, the following Remark is immediate.

Remark 2.1. Let $G = P_{n-1}$ and $H = P_{n-2}$ with order n. If $n \geq 3$ and $m \geq 0$, then
$F_d(P_n) = \{P_1 \Phi f_m(G)\} \cup \{P_2^1 \Phi f_m(H)\}$.

The next theorem is a direct consequence of Remark 2.1.

Theorem 2.1. *Let G be a path of order $n \geq 1$. Then, $|F(G)| = F_{n+2}$.*

Proof. Suppose G is a path of order $n \geq 1$ and $f_m(G) \in F(G)$ where $m \geq 0$. Then, consider the following cases.

Case 1. Let G be a path of order 1. Then, the element of $F(G)$ is a trivial graph of either dominating vertex P_1^1 or non-dominating vertex P_1, i.e., $|F(G)| = 2 = F_3$.

Case 2. Now consider a graph $H = P_2$. Since $f_m(H) \in F(H)$, there exist $v \in F$ such that $N(v)$ is non-dominating vertex. This implies that the element of $F(H)$ are P_2, P_2^1 and P_2^2.

In Remark 2.1 and by cases (1) and (2), $F(P_3) = \{P_1 \Phi f_m(G)\} \cup \{P_2^1 \Phi f_m(H)\}$ where $|\{P_1 \Phi f_m(G)\}| = 2 = F_3$ and $|\{P_2^1 \Phi f_m(H)\}| = 3 = F_4$. This implies that $|F(P_3)| = 5 = F_5$. By definition we have $F_n = F_{n-2} + F_{n-1}$ for $n \geq 3$ where $F_1 = F_2 = 1$. Then, it follows that if $G = P_n$, then we obtain $|F(G)| = F_{n+2}$. This completes the proof. □

Let $G = P_n$. The maximum independent domination number m in path of order $n \geq 1$ is a path with with maximum number of no two consecutive dominating vertices satisfying Fibonacci string is denoted by $f_m^{max}(G)$, where $m \geq 0$. A set that contains $f_m^{max}(G)$, $m \geq 0$, is denoted by $F_d^{max}(G)$.

Theorem 2.2. *Let G be a path of order $n \geq 1$ and m be a maximum independent domination number that follows Fibonacci string in G. Then,*

$$m = \begin{cases} \frac{n+1}{2}, & n \equiv 1(\textit{mod } 2), \\ \frac{n}{2}, & n \equiv 0(\textit{mod } 2). \end{cases}$$

Proof. Let G be a graph of order n. First, consider if n is odd integer, then we have the following cases.

Case 1. If $n = 1$, then it follows that $v \in V(G)$ is a dominating vertex. Thus, it satisfies that $m = 1 = \frac{n+1}{2}$.

Case 2. If $n \geq 3$, then the leaves of G are dominating vertices. Since no two consecutive dominating vertices, it follows that the arrangement of $u_i \in I$ and $v_i \in V(G) \setminus I$ for all i, are alternate. So, we have $|V(G) \setminus I| = |I| + 1$ and $deg(v_i) = 2$ for all i. This implies that $|V(G) \setminus I| = \frac{n-1}{2}$ and $|I| = m = \frac{n+1}{2}$.

Now, consider the following cases for n is even.

Case 1. If n is even, then there exist $u \in F$ such that $deg(u) = 1$ with no dominating vertex in the first and last in G simultaneously. Also, since the arrangement is alternate, it follows that $|V(G) \setminus F| = |F|$. Thus, this implies $m = \frac{n}{2}$.

Case 2. If there exist two dominating vertices u and v such that $deg(u) = deg(v) = 1$, then there exists one edge $e \in E(G)$ such that e is incident with two non-dominating vertices a and b. This also follows that $N(a)$ and $N(b)$ are sets with two elements, dominating and non-dominating vertices. So, this implies that $|V(G) \setminus F| = |F| = m = \frac{n}{2}$. This completes the proof. □

The next Corollary determines the cardinality of $F_d^{max}(G)$.

Corollary 2.1. *Let G be a path of order $n \geq 1$. Then,*

$$|F_d^{max}(G)| = \begin{cases} 1, & n \equiv 1(\textit{mod } 2), \\ \frac{n+2}{2}, & n \equiv 0(\textit{mod } 2). \end{cases}$$

Proof. (i) Suppose that the order of G is odd, then consider the following cases.

Case 1. If $n = 1$, then it follows that $v \in V(G)$ is a dominating vertex.

Case 2. If $n \geq 3$, then by Theorem 2.2, $m = \frac{n+1}{2}$. This implies that there is only one possibility for this arrangement. Thus, it follows that $|F_d^{max}(G)| = 1$.

Now, suppose that the order of G is even. Then consider the following cases.

Case 1. Suppose that $u_i \in F$ and $v_i \in F$. For positive integer i, are alternate, there are 2 arrangement of this form with no dominating vertex in first and the last in G simultaneously.

Case 2. If there exists dominating vertex in the first and last in G such that the degree are 1 and there exists 1 edge $e \in E(G)$ such that e is incident with 2 non-dominating vertex a and b, then by Theorem 2.2 we obtained $|F| = m = \frac{n}{2}$. This implies that there are $\frac{n-2}{2}$ possible arrangement of this form.

Combining Case (1) and (2), this follows that there are $2 + \frac{n-2}{2}$ possible distinct arrangement. Thus, we have $|F_d^{max}(G)| = \frac{n+2}{2}$. This completes the proof. □

The next results are immediate from above definitions of Lucas numbers.

Remark 2.2. Let $G = P_{n-1}$ and $H = P_{n-3}$ with order n. If $n \geq 4$ and $m \geq 0$, then
$L_d(P_n) = \{P_1 \Phi f_m(G)\} \cup \{P_2^1 \Phi f_m(H) \Phi P_1\}$.

Remark 2.3. [7] If F_n is a Fibonacci number and L_n is a Lucas number, then $F_{n+1} + F_{n-1} = L_n$.

Theorem 2.3. *Let G be a path of order $n \geq 1$. Then, $L(G) = L_n$.*

Proof. Consider the following cases.
Case 1. Suppose G is a path of order 1, then there is one element in the set $|L(G)|$, i.e., P_1. This implies that $|L(G)| = 1 = L_1$.

Case 2. Let $H = P_2$. Then, there exist $v \in V(G) \setminus L$ such that $N(v)$ is a dominating vertex set. This implies that there are 3 elements including the path P_2, i.e., $|L(G)| = 3 = L_2$.

Case 3. Let $G = P_1$, $H = P_2$, and $J = P_3$. Then,
$|L_d(J)| = |\{P_1 \Phi l_m(H)\} \cup \{P_2^1 \Phi l_m(G)\}| = L_1 + L_2 = 4 = L_3$.

Case 4. Let $G = P_n$, $G_1 = P_{n-1}$ and $G_2 = P_{n-3}$. Then, by Remark 2.2 and 2.3,
$|L(G)| = |\{P_1 \Phi f_m(G_1)\} \cup \{P_2^1 \Phi f_m(G_2) \Phi P_1\}| = F_{n+1} + F_{n-1} = L_n$.

Thus, combining the four cases we have $|L(G)| = L_n$ whenever $G = P_n$. This completes the proof. □

Let $G = P_n$. The maximum independent domination number m in path of order $n \geq 1$ is a path with with maximum number of no two consecutive dominating vertices satisfying Lucas string is denoted by $l_m^{max}(G)$ where $m \geq 0$. A set that contains $l_m^{max}(G)$, $m \geq 0$, is denoted by $L_d^{max}(G)$.

Theorem 2.4. *Let G be a path of order $n \geq 1$ and m be a maximum independent domination number that follows Lucas string in G. Then,*

$$m = \begin{cases} \frac{n-1}{2}, & n \equiv 1 (mod\ 2), \\ \frac{n}{2}, & n \equiv 0 (mod\ 2). \end{cases}$$

Proof. Let G be a path of order n. If n is odd then consider the following cases.

Case 1. If $n = 1$, then the only element in the set $L_d(G)$ is $l_0(G)$, i.e., P_1. Thus, $m = 0 = \frac{n-1}{2}$.

Case 2. If $n \geq 3$, then by Theorem 2.2, i.e., one of the leaf vertex is a dominating vertex. Thus, $m = \frac{n-1}{2}$.

Now, consider if n is even. Then by Theorem 2.3 it follows that $m = \frac{n}{2}$. This completes the proof. □

The next Corollary determines the cardinality of $L_d^{max}(G)$.

Corollary 2.2. *Let G be a path of order $n \geq 2$. Then,*

$$|L_d^{max}(G)| = \begin{cases} n, & n \equiv 1 (mod\ 2), \\ 2, & n \equiv 0 (mod\ 2). \end{cases}$$

Proof. Let G be a path of order n. If n is odd, then consider the following cases.

Case 1. If $n = 3$, then by Theorem 2.4, $m = 1$. Since there are three ways to assign the dominating vertex in P_3, it follows that $|L_d^{max}(G)| = 3 = n$.

Case 2. If $n \geq 5$, then consider the following subcases.

Sub case 2.1. Let $u_i \in I$ and $v_i \in V(G) \setminus I$. Then, $N(u_i) \subseteq V(G) \setminus I$ implies that there is one arrangement can be form such that $u_i \in I$ and $v_i \in V(G) \setminus I$ are alternate.

Sub case 2.2. Consider that there exists $u \in I$ such that $deg(u) = 1$ and there exists $v \in V(G) \setminus I$ such that $deg(v) = 1$ and $N(v) \in V(G) \setminus I$. Then, there are 2 distinct arrangement of this form.

Sub case 2.3. Suppose that there exists $v_1, v_2 \in V(G) \setminus I$ and $e \in E(G)$ such that e is an internal edge and incident with v_1 and v_2. Then, there are $n-3$ distinct possible arrangement can be formed.

Combining Sub cases (2.1), (2.2) and (2.3), it follows that $|L_d^{max}(G)| = n$.

Now, consider if n is even. Then, by Theorem 2.8 it implies that $u_i \in I$ and $v_i \in V(G) \setminus I$ must be alternate. Thus, there are only two possible ways, i.e., $|L_d^{max}(G)| = n$. This completes the proof. □

The following results are immediate from the definition of an independent dominating set in a path.

Remark 2.4. Let G be a path of order n. Then,

$$i(G) = \begin{cases} \dfrac{n}{3}, & n \equiv 0(\text{mod } 3), \\ \dfrac{n+2}{3}, & n \equiv 1(\text{mod } 3), \\ \dfrac{n+1}{3}, & n \equiv 2(\text{mod } 3). \end{cases}$$

Theorems 2.5 and 2.6 are direct consequences of Remark 2.4.

Theorem 2.5. *Let G be a path of order $n \geq 3$ and $n \neq 4$. If $f_m(G)$ and $l_m(G)$ are independent domination in G and for each $u \in I$ satisfies $deg(u) = 2$, then*

$$|F_d(G)| = |L_d(G)| = \begin{cases} 1, & n \equiv 0(\textit{mod 3}), \\ \dfrac{n^2 - 5n + 4}{18}, & n \equiv 1(\textit{mod 3}), \\ \dfrac{n-2}{3}, & n \equiv 2(\textit{mod 3}). \end{cases}$$

Proof. Let G be a path of order n. If $n \equiv 0(\text{mod } 3)$ then by Remark 2.10 we have G has an independent domination, i.e., $i(G) = \frac{n}{3}$. Let $u_1, u_2, \ldots, u_{n/3} \in I$. Then, $N(u_1) \cap N(u_2) \cap \cdots \cap N(u_{n/3}) = \phi$ where each $u_i \in I$ is adjacent with two non-dominating vertices which is the only arrangement in a path of order divisible by 3. Thus, $|F_d(G)| = |L_d(G)| = 1$ whenever $n \equiv 0(\text{mod } 3)$.

Now, let $n \geq 7$ and $n \equiv 1(\text{mod } 3)$. Then, by Remark 2.4, $i(G) = \frac{n+2}{3}$ and there exist u, $v \in V(G) \setminus I$ such that $deg(u) = 1 = deg(v)$ and $N(u) \cup N(v) \subseteq I$. Moreover, there exists edges such that incident of two non-dominating vertices when G has order $n \geq 10$ and let e_i be that edges. Also, P_2^2 and P_2^1 are in the first and last in G, respectively. Hence, the order of $G - P_2^2 \cup P_2^1$ is $n-4$ and the dominating vertex remaining is $\frac{n-4}{3}$. This implies that there are $\frac{n-7}{3}$ edges incident of two non-dominating vertices, i.e., $e_1, e_2, \ldots, e_{(n-7)/3}$. It follows that the number of ways of two vertices u and v and edges $e_1, e_2, \ldots, e_{(n-7)/3}$ can be arranged is $(\frac{n-4}{3})^2$. But since vertices u and v and edges $e_1, e_2, \ldots, e_{(n-7)/3}$ are not distinct, then the possible distinct arrangement is given by $\sum_{i=1}^{\frac{n-4}{3}} i$. Thus, $|F_d(G)| = |L_d(G)| = \frac{n^2-5n+4}{18}$.

Let $n \geq 5$ and $n \equiv 2(\text{mod } 3)$. Then, by Remark 2.4, $i(G) = \frac{n+1}{3}$ and there exists $u \in V(G) \setminus I$ with $deg(u) = 2$ such that $N(u) \subseteq I$. Furthermore, there exists $e_i \in E(G)$ such that the incident vertices are dominating vertices. So, P_2^2 and P_2^1 are in the first and the last in G, respectively. Hence, $G - P_2^2 \cup P_2^1$ has order $n-4$ with $\frac{n-5}{3}$ dominating vertex remaining. It follows that there are $\frac{n-5}{3}$ non-distinct edges namely $e_1, e_2, \ldots, e_{(n-5)/3}$. Thus, there are $\frac{n-5}{3} + 1$ ways of distinct arrangement for vertex u and edges $e_1, e_2, \ldots, e_{(n-5)/3}$, i.e., $|F_d(G)| = |L_d(G)| = \frac{n-2}{3}$. This completes the proof. □

Theorem 2.6. *Let G be a path of order $n \geq 1$. If $f_m(G)$ and $l_m(G)$ are independent domination in G, then the following holds.*

(i) $|F_d(G)| = |L_d(G)| = 1$ *whenever* $n \equiv 0$(*mod 3*),

(ii) $|F_d(G)| = \frac{n^2+7n+10}{18}$ *and* $|L_d(G)| = \frac{n^2+7n-8}{18}$ *whenever* $n \equiv 1$(*mod 3*), *and*

(iii) $|F_d(G)| = |L_d(G)| = \frac{n+4}{3}$ *whenever* $n \equiv 2$(*mod 3*).

Proof. Let G be a path of order n. If $n \equiv 0(\text{mod } 3)$, then by Remark 2.4, $i(G) = \frac{n}{3}$. This follows that there is only one arrangement in which $u_i \in I$ and $N(u_1) \cap N(u_2) \cap \cdots \cap N(u_{n/3}) = \phi$. Now, consider the following cases.

Case 1. Suppose that there exists $u, v \in I$ such that $deg(u) = deg(v) = 1$, then this implies that only one arrangement can be formed.

Case 2. By Theorem 2.5, $|F_d(G)| = |L_d(G)| = \frac{n^2-5n+4}{18}$ where $u_i \in I$ are independent and for each of them has degree 2.

Case 3. Suppose that there exists $u \in I$ and $v \in V(G) \setminus I$ such that $deg(u) = deg(v) = 1$ and $N(v) \subseteq I$, then $\frac{2(n-1)}{3}$ independent domination in G can be formed.

Combining Cases (1), (2) and (3), it implies that there are $1 + \frac{n^2-5n+4}{18} + \frac{2(n-1)}{3}$ distinct arrangements can be formed as an independent domination in $f_m(G)$. It follows that $|F_d(G)| = \frac{n^2+7n+10}{18}$. Combining cases (2) and (3) implies that there are $\frac{n^2-5n+4}{18} + \frac{2(n-1)}{3}$ distinct arrangement for $l_m(G)$. Thus, $|L_d(G)| = \frac{n^2+7n-8}{18}$.

Now, consider $n \equiv 2(\text{mod } 3)$. Then, there are two possible arrangements where there exists $u \in I$ and $v \in V(G) \setminus I$ such that $deg(u) = deg(v) = 1$ and $N(v) \subseteq V(G) \setminus I$. Also, by Theorem 2.11 there are $\frac{n-2}{3}$ arrangements for independent domination where $u_i \in I$ are independent dominating vertices and for each of them has degree 2. This implies that there are $2 + \frac{n-2}{3}$ distinct arrangements can be formed to be independent dominations for both $f_m(G)$ and $l_m(G)$, i.e., $|F_d(G)| = |L_d(G)| = \frac{n+4}{3}$. This completes the proof. □

The following Remark is immediate from the definition of $F_d(G)$, $L_d(G)$ and triangular numbers.

Remark 2.5. Let G be a path of order $n \geq 7$ and $n \equiv 1(\text{mod } 3)$. If $f_m(G)$ and $l_m(G)$ is an independent domination in G and for each $u \in I$ satisfies $deg(u) = 2$, then $|F_d(G)|$ and $|L_d(G)|$ are $(\frac{n-4}{3})^{th}$ triangular numbers.

3. Acknowledgement

The author would like to acknowledge and thank the referees for the rigorous review and giving significant comments for the improvement of this paper.

References

[1] E.J. Cockayne, and S.T. Hedetniemi, Towards a Theory of Domination in Graph, *Networks Advanced Topics*, (1977), 247–261.

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[3] S. Klavzar and P. Zigert, Fibonacci Cubes are the Resonance Graphs of Fibonaccenes, *Fibonacci Quarterly*, **43** (3), 2005.

[4] E. Munarini, C.P. Cippo, and N.Z. Salvi, On the Lucas Cubes, *Fibonacci Quarterly* **39** (1), 2001.

[5] O. Ore, *Theory of Graphs.* American Mathematical Society, Provedence, R. I., 1962.

[6] J. Tarr, *Domination in Graphs*, University of South Florida, (2010), 34–36.

[7] S. Vajda. *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Publications, 2007.