The eccentric-distance sum of some graphs

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Abstract

Let \( G = (V, E) \) be a simple connected graph. The eccentric-distance sum of \( G \) is defined as
\[
\xi_{ds}(G) = \sum_{\{u,v\} \subseteq V(G)} [e(u) + e(v)]d(u, v),
\]
where \( e(u) \) is the eccentricity of the vertex \( u \) in \( G \) and \( d(u, v) \) is the distance between \( u \) and \( v \). In this paper, we establish formulae to calculate the eccentric-distance sum for some graphs, namely wheel, star, broom, lollipop, double star, friendship, multi-star graph and the join of \( P_{n-2} \) and \( P_2 \).

Keywords: eccentricity, star, path, broom, lollipop graph, double star, complete \( k \)-partite

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1. Introduction

Let \( G \) be a simple connected graph with the vertex set \( V(G) \) and the edge set \( E(G) \). The degree of a vertex \( u \in V(G) \) is denoted by \( d(u) \) and is the number of vertices adjacent to \( u \). For vertices \( u, v \in V(G) \), the distance \( d(u, v) \) is defined as the length of any shortest path connecting \( u \) and \( v \) in \( G \) and \( D(u) \) denotes the sum of distances between \( u \) and all other vertices of \( G \). The eccentricity \( e(u) \) of a vertex \( u \) is the largest distance between \( u \) and any other vertex \( v \) of \( G \), i.e.,
\[
e(u) = \max \{d(u, v); v \in V(G)\}.
\]
Let \( K_n, P_n, W_n, C_n \) and \( K_{1,n-1} \) denote a complete graph, path, wheel, cycle and star on \( n \) vertices, respectively [14].
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The **Wiener index** is defined as the sum of all distances between unordered pairs of vertices

\[ W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v). \]

It is considered as one of the most used topological index with high correlation with many physical and chemical indices of molecular compounds (for the recent survey on Wiener index see [4, 5]).

The parameter \( DD(G) \) called the degree distance of \( G \) was introduced by Dobrynin and Kochetova [6] and Gutman [13] as a graph-theoretical descriptor for characterizing alkanes; it can be considered as a weighted version of the Wiener index and is defined as

\[ DD(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]d(u,v) = \sum_{u \in V(G)} d(u)D(u) \]

where the summation goes over all pairs of vertices in \( G \). In fact, when \( G \) is a tree on \( n \) vertices, it has been demonstrated that Wiener index and degree distance are closely related by (see [15, 19])

\[ DD(G) = 4W(G) - n(n - 1). \]

Sharma, Goswami and Madan [26] introduced a distance-based molecular structure descriptor, **eccentric connectivity index (ECI)** defined as

\[ \xi^c(G) = \sum_{v \in V(G)} e(v)d(v). \]

The index \( \xi^c(G) \) was successfully used for mathematical models of biological activities of diverse nature [9, 11, 12, 22, 25, 26]. The investigation of its mathematical properties started only recently (for a survey on eccentric connectivity index see [17]). In [8, 18, 23, 28], the extremal graphs in various class of graphs with maximal or minimal ECI are determined. In [1, 2, 7] the authors determined the closed formulae for the eccentric connectivity index of nanotubes and nanotori.

Recently, a novel graph invariant for predicting biological and physical properties eccentric-distance sum was introduced by S.Gupta, M.Singh and A.K.Madan [12]. This topological index has vast potential application in structure activity/property relationships of molecules and it also displays high discriminating power with respect to both biological activities and physical properties; see[12]. The authors in [12] have shown that some structure activity and quantitative structure property studies using eccentric-distance sum were better than the corresponding values obtained using the Wiener index. It is also interesting to study the mathematical property of this topological index. The **eccentric-distance sum (EDS)** of \( G \) is defined as

\[ \xi^{ds}(G) = \sum_{u \in V(G)} e(u)D(u). \]

The eccentric-distance sum can be defined alternatively as

\[ \xi^{ds}(G) = \sum_{\{u,v\} \subseteq V(G)} [e(u) + e(v)]d(u,v). \]
Yu, Feng and Ilić [27] identified the extremal unicyclic graphs of given girth having the minimal and second minimal EDS; they also characterized trees with minimal EDS among the n-vertex trees of a given diameter. Hua, Xu and Shu[16] obtained the sharp lower bound on EDS of n-vertex cacti. Ilić, Yu and Feng [20] studied various lower and upper bounds for the EDS in terms of other graph invariant including the Wiener index, the degree distance index, the eccentric connectivity index and so on.

In this paper we establish formulae to calculate the eccentric-distance sum for some graphs, namely wheel, star, broom graph, lollipop, double star, friendship, multi-star graph and $P_{ln}$ graph.

2. Main Results

M. J. Morgan et al. calculated the eccentric-distance sum for complete graph, cycle graph and path graph.

**Proposition 2.1.** [3]

1. $\xi_{ds}(K_n) = n(n - 1)$

2. $\xi_{ds}(C_n) = \begin{cases} 
\frac{n^4}{8}, & \text{if } n \text{ is even} \\
\frac{n(n - 1)^2(n + 1)}{8}, & \text{if } n \text{ is odd.}
\end{cases}$

3. $\xi_{ds}(P_n) = \begin{cases} 
\frac{25n^4 - 16n^3 - 28n^2 + 16n}{96}, & \text{if } n \text{ is even} \\
\frac{25n^4 - 16n^3 - 34n^2 + 16n + 9}{96}, & \text{if } n \text{ is odd.}
\end{cases}$

**Proposition 2.2.** $\xi_{ds}(W_n) = (n - 1)(4n - 9)$

*Proof.* $e(v_1) = 1$ and $e(v_i) = 2$ where, $i = 2, 3, \ldots, n$. Then,

$$\xi_{ds}(W_n) = \sum_{\{v_i, v_j\} \subseteq V(W_n)} [e(v_i) + e(v_j)]d(v_i, v_j)$$

$$= (1 + 2)1(n - 1) + (2 + 2)1(2) + (2 + 2)2(n - 4) + (2 + 2)1 + (2 + 2)2(n - 5) + \cdots + (2 + 2)1 + (2 + 2)2(1) + (2 + 2)1$$

$$= (n - 1)(4n - 9).$$

\[\square\]
Proposition 2.3. $\xi^{ds}(K_{1,n-1}) = (n-1)(4n-5)$

Proof. $e(v_0) = 1$ and $e(v_i) = 2$ where, $i = 1, 2, 3, \ldots, n-1$. Then,

$\xi^{ds}(K_{1,n-1}) = (1+2)1(n-1) + (2+2)2(n-2) + (2+2)2(n-3) + \cdots + (2+2)2(1)$

$= (n-1)(4n-5)$.

\[\Box\]

Definition 2.1. [23] The broom graph $B_{n,d}$ is a graph consisting of a path $P_d$, together with $(n-d)$ end vertices all adjacent to the same end vertex of $P_d$.

![Figure 1. Broom graph $B_{n,d}$](image)

Theorem 2.1. The eccentric-distance sum of a broom graph $B_{n,d}$ is

$$
\xi^{ds}(P_{d+1}) + (n-d-1) \left[ \frac{d}{2}(d^2 - 3d + 4n) + \sum_{k=d}^{d-1} k^2 + \sum_{k=1}^{d} (d-k)k \right],
$$

when $d$ is even,

$$
\xi^{ds}(P_{d+1}) + (n-d-1) \left[ \frac{d}{2}(d^2 - 3d + 4n) + \sum_{k=d+1}^{d} k^2 + \sum_{k=1}^{d} (d-k)k \right],
$$

when $d$ is odd.

Proof. Let $\{v_1, v_2, \ldots, v_d, u_1, u_2, \ldots, u_{n-d}\}$ be the set of $n$ vertices of the broom graph $B_{n,d}$. We consider the following cases.

Case(i): $d$ is even.

$e(v_1) = d$, $e(v_2) = e(v_d) = d - 1$, $e(v_3) = e(v_{d-1}) = d - 2$, $\ldots$, $e(v_{\frac{d}{2}}) = e(v_{\frac{d}{2}+2}) = \frac{d}{2} + 1$,

$e(v_{\frac{d}{2}+1}) = \frac{d}{2}$, and $e(u_1) = e(u_2) = \cdots = e(u_{n-d}) = d$. Then,
\[\xi_{ds}(B_{n,d}) = \xi_{ds}(P_{d+1}) + \{[d + d] + [(d - 1) + d] (d - 1) + [(d - 2) + d] (d - 2) + \cdots + [\left(\frac{d}{2} - 1\right) + 2] \left(\frac{d}{2} + 1\right) + \left(\frac{d}{2} + d\right) \left(\frac{d}{2} + 1\right) + d\} (n - d - 1) + (d + d) ([n - d - 1] + (n - d - 2) + \cdots + 2 + 1)\]

\[= \xi_{ds}(P_{d+1}) + (n - d - 1) \left[\frac{d}{2}(d^2 - 3d + 4n) + \sum_{k=\frac{d}{2}+1}^{d} k^2 + \sum_{k=1}^{\frac{d-1}{2}} (d - k)k\right].\]

Case(ii): \(d\) is odd.
\(e(v_1) = d, e(v_2) = e(v_d) = d - 1, e(v_3) = e(v_{d-1}) = d - 2, \ldots, e(v_{\frac{d+1}{2} - 1}) = e(v_{\frac{d+1}{2} + 1}) = \frac{d-1}{2} + 2, e(v_{\frac{d+1}{2}}) = e(v_{\frac{d+1}{2} + 1}) = \frac{d-1}{2} + 1,\) and \(e(u_1) = e(u_2) = \cdots = e(u_{n-d}) = d.\) Then,

\[\xi_{ds}(B_{n,d}) = \xi_{ds}(P_{d+1}) + \{[d + d] + [(d - 1) + d] (d - 1) + [(d - 2) + d] (d - 2) + \cdots + [\left(\frac{d}{2} - 1\right) + 1] \left(\frac{d}{2} + 1\right) + \left(\frac{d}{2} + d\right) \left(\frac{d}{2} + 1\right) + \left(\frac{d}{2} + 1\right) + d\} (n - d - 1) + (d + d) ([n - d - 1] + (n - d - 2) + \cdots + 2 + 1)\]

\[= \xi_{ds}(P_{d+1}) + (n - d - 1) \left[\frac{d}{2}(d^2 - 3d + 4n) + \sum_{k=\frac{d}{2}+1}^{d} k^2 + \sum_{k=1}^{\frac{d-1}{2}} (d - k)k\right].\]

**Definition 2.2.** [23] The lollipop graph \(L_{n,d}\) is a graph obtained from a complete graph \(K_{n-d}\) and a path \(P_d\), by joining one of the end vertices of \(P_d\) to all the vertices of \(K_{n-d}\).
Theorem 2.2. The eccentric-distance sum of a lollipop graph $L_{n,d}$ is

$$\xi^{ds}(P_{d+1}) + (n - d - 1) \left[ \frac{d^2}{2} (d + 1) + d(n - d) + \sum_{k=\frac{d}{2}}^{d} k^2 + \sum_{k=1}^{d-1} (d - k)k \right],$$

when $d$ is even,

$$\xi^{ds}(P_{d+1}) + (n - d - 1) \left[ \frac{d^2}{2} (d + 1) + d(n - d) + \sum_{k=\frac{d+1}{2}+1}^{d} k^2 + \sum_{k=1}^{d-1} (d - k)k \right],$$

when $d$ is odd.

Proof. Let $\{v_1, v_2, \ldots, v_d, u_1, u_2, \ldots, u_{n-d}\}$ be the set of $n$ vertices of the lollipop graph $L_{n,d}$. We consider the following cases.

Case(i): $d$ is even.

$$\xi^{ds}(L_{n,d}) = \xi^{ds}(P_{d+1}) + \{[d + d] d + [(d - 1) + d] (d - 1) + [(d - 2) + d] (d - 2) + \cdots \}
+ \left[ \left( \frac{d}{2} + 2 \right) + d \right] \left( \frac{d}{2} + 2 \right)
+ \left[ \left( \frac{d}{2} + 1 \right) + d \right] \left( \frac{d}{2} + 1 \right)
+ \left[ \frac{d}{2} + d \right] \frac{d}{2}
+ \left( \frac{d}{2} + 1 \right) + d \left( \frac{d}{2} - 1 \right) + \cdots + [(d - 2) + d] 2 + [(d - 1) + d] 1 \} (n - d - 1)
+ d (n - d - 1) (n - d)
= \xi^{ds}(P_{d+1}) + (n - d - 1) \left[ \frac{d^2}{2} (d + 1) + d(n - d) + \sum_{k=\frac{d}{2}}^{d} k^2 + \sum_{k=1}^{d-1} (d - k)k \right].$$

Case(ii): $d$ is odd.

$$\xi^{ds}(L_{n,d}) = \xi^{ds}(P_{d+1}) + \{[d + d] d + [(d - 1) + d] (d - 1) + [(d - 2) + d] (d - 2) + \cdots \}
+ \left[ \left( \frac{d-1}{2} + 2 \right) + d \right] \left( \frac{d-1}{2} + 2 \right)
+ \left[ \left( \frac{d-1}{2} + 1 \right) + d \right] \left( \frac{d-1}{2} + 1 \right)
+ \left[ \frac{d-1}{2} + d \right] \frac{d-1}{2}
+ \left( \frac{d-1}{2} + 1 \right) + d \left( \frac{d-1}{2} - 1 \right) + \cdots + [(d - 2) + d] 2 + [(d - 1) + d] 1 \} (n - d - 1) + d (n - d - 1) (n - d)
= \xi^{ds}(P_{d+1}) + (n - d - 1) \left[ \frac{d^2}{2} (d + 1) + d(n - d) + \sum_{k=\frac{d+1}{2}+1}^{d} k^2 + \sum_{k=1}^{d-1} (d - k)k \right].$$

$\square$
Definition 2.3. [21] A double star graph $S_{n,m}$ is a graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers $v_0$ and $u_0$. The vertex-set $V(S_{n,m})$ is $V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, \ldots, v_{n-1}, u_0, u_1, \ldots, u_{m-1}\}$ and the edge-set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j | 1 \leq i \leq (n - 1); 1 \leq j \leq (m - 1)\}$. Therefore, a double star graph is bipartite.

Theorem 2.3. The eccentric-distance sum of a double star graph $S_{n,m}$ is $3[(n - 2)(2n - 1) + m(2m + 5) + 6(n - 1)(m - 1) - 5]$.

Proof. $e(v_0) = 2 = e(u_0)$ and $e(v_i) = 3 = e(u_j)$ where, $i = 1, 2, \ldots, n - 1$ and $j = 1, 2, \ldots, m - 1$ Then,

\[
\xi_{ds}(S_{n,m}) = [(3 + 2)1 + (3 + 2)2 + (3 + 3)3(m - 1)](n - 1) \\
+ (2 + 2)1 + (2 + 3)2(m - 1) + (2 + 3)1(m - 1) \\
+ (3 + 3)2(n - 2) + (3 + 3)2(n - 3) + \cdots + (3 + 3)2(1) \\
+ (3 + 3)2(m - 2) + (3 + 3)2(m - 3) + \cdots + (3 + 3)2(1) \\
= 3[(n - 2)(2n - 1) + m(2m + 5) + 6(n - 1)(m - 1) - 5].
\]

Definition 2.4. [13] The friendship (or Dutch windmill or fan) graph $F_n$ is a graph constructed by joining $n$ copies of the cycle graph $C_3$ with a common vertex.

Figure 3. Double star graph $S_{n,m}$.

Figure 4. Friendship graph $F_n$.
Theorem 2.4. The eccentric-distance sum of friendship graph $F_n$ is $2n(8n - 3)$.

Proof. $e(v_0) = 1$ and $e(v_i) = 2$ where, $i = 1, 2, \ldots, 2n$. Then,

$$
\xi^{ds}(F_n) = (2 + 2)1 + (2 + 2)2(2n - 2) + (2 + 2)2(2n - 2) + (2 + 2)1 + (2 + 2)2(2n - 4) + (2 + 2)2(2n - 4) + (2 + 2)1 + (2 + 2)2(2n - 6) + (2 + 2)2(2n - 6) + \cdots + (2 + 2)1 + (2 + 2)2(2) + (2 + 2)2(2) + (2 + 2)1 + (1 + 2)1(2n) = 2n(8n - 3).
$$

Definition 2.5. [24] Consider the star graph $K_{1,n}$ with vertex set $\{v_0, v_1, v_2, \ldots, v_n\}$, introduce an edge to each of the pendant vertices $v_1, v_2, \ldots, v_n$ to get the resulting graph $K_{1,n,n}$ with vertices $\{v_0, v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$, again introduce an edge to each of the pendant vertices $v_{n+1}, \ldots, v_{2n}$, to get the graph $K_{1,n,n,n}$. Repeating this $(m - 1)$ times we get a graph $K_{1,n,n,\ldots,n}$ called multi-star graph with $(mn + 1)$ vertices $v_0, v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1}, \ldots, v_{3n}, \ldots, v_{(m-1)n+1}, \ldots, v_{mn}$ and $mn$ edges, as shown in Figure 5.

![Figure 5. Multi-star graph $K_{1,n,n,\ldots,n}$](image)

Theorem 2.5. The eccentric-distance sum of a multi-star graph $K_{1,n,n,\ldots,n}$ is

$$
\frac{nm}{6} (m + 1)(8m + 1) + n(n + 1) \left\{ m \left[ \sum_{k=1}^{m} (k+1)k + \sum_{k=1}^{m-1} (m + 1 + k)(m - k) \right] \right\}
$$

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\[
\begin{align*}
+ \sum_{k=1}^{m} k + \frac{1}{2} \left[ \sum_{k=1}^{m} k(k+1)^2 + \sum_{k=1}^{m-1} (k+1)(k+2) \right] + 1 \right] + \frac{n}{2} \left[ \sum_{k=1}^{m-1} (m+k+1)k(k+1) \\
+ \sum_{k=1}^{m} (m+k)(m-k)(m-k+1) \right].
\end{align*}
\]

Proof.

\[
\xi_{ds}\left(K_{i,n,n,\ldots,n}^{m-1,mes}\right)
= \left\{ [m + (m + 1)]1 + [m + (m + 2)]2 + \cdots + [m + (m + m)]m \right\} n \\
+ \left\{ [(m + 1) + (m + 1)]2 + [(m + 1) + (m + 2)]3 + \cdots + [(m + 1) + (m + m)](m + 1) \right\} \\
[\left( n - 1 \right) + (n - 2) + (n - 3) + \cdots + 2 + 1] \\
+ \left\{ [(m + 2) + (m + 1)]3 + [(m + 2) + (m + 2)]4 + \cdots + [(m + 2) + (m + m)](m + 2) \right\} \\
[\left( n - 1 \right) + (n - 2) + (n - 3) + \cdots + 2 + 1] \\
\vdots \\
+ \left\{ [(m + m) + (m + 1)](m + 1) + [(m + m) + (m + 2)](m + 2) + \cdots \\
+ [(m + m) + (m + m)](m + m) \right\} [(n - 1) + (n - 2) + (n - 3) + \cdots + 2 + 1] \\
+ \left\{ [(m + 1) + (m + 2)]1 + [(m + 1) + (m + 3)]2 + \cdots + [(m + 1) + (m + m)](m - 1) \right\} n \\
+ \left\{ [(m + 2) + (m + 3)]1 + [(m + 2) + (m + 4)]2 + \cdots + [(m + 2) + (m + m)](m - 2) \right\} n \\
\vdots \\
+ \left\{ [(m + (m - 1)) + (m + m)]1 \right\} n \\
\right.
\]

\[
= \frac{nm}{6}(m + 1)(8m + 1) + n(n + 1) \left\{ m \left[ 2 \sum_{k=1}^{m} (k+1)k + \sum_{k=1}^{m-1} (m + 1 + k)(m - k) \right] \\
+ \sum_{k=1}^{m} k + \frac{1}{2} \left[ \sum_{k=1}^{m} k(k+1)^2 + \sum_{k=1}^{m-1} (k+1)(k+2) \right] + 1 \right\} \\
+ \frac{n}{2} \left[ \sum_{k=1}^{m-1} (m + k + 1)k(k+1) + \sum_{k=1}^{m-1} (m + k)(m-k)(m-k+1) \right].
\]

\[\square\]

Definition 2.6. [24] \(P_{l_n} (n \geq 3)\) is a graph obtained by the join of \(P_{n-2}\) and \(P_2\), as shown in Figure 6.
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Theorem 2.6. The eccentric-distance sum of $P_l_n$ ($n \geq 6$) graph is $4n^2 + 19n - 29$.

Proof. $e(u_1) = 1 = e(u_2)$ and $e(u_i) = 2$ where, $i = 3, 4, \ldots, n$. Then,

$$
\xi_{ds}(P_l_n) = (1 + 2)1(n - 2) + (1 + 1)1 + (1 + 2)1(n - 2) \\
+ (2 + 2)1 + (2 + 2)2(n - 4) \\
+ (2 + 2)1 + (2 + 2)2(n - 5) \\
\vdots \\
+ (2 + 2)1 + (2 + 2)2(1) \\
+ (2 + 2)1 \\
= 4n^2 + 19n - 29.
$$

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