On the spectrum of a class of distance-transitive graphs

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Abstract
Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic additive group $\mathbb{Z}_n$ ($n \geq 4$), where $S_1 = \{1, n-1\}$, ..., $S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. In this paper, we will show that $\chi(\Gamma) = \omega(\Gamma) = k + 1$ if and only if $k + 1 | n$. Also, we will show that if $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \wr_1 \text{Sym}(k + 1)$ where $I = \{1, \ldots, k + 1\}$ and in this case, we show that $\Gamma$ is an integral graph.

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1. Introduction

In this paper, a graph $\Gamma = (V, E)$ always means a simple connected graph with $n$ vertices (without loops, multiple edges and isolated vertices), where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. The size of the largest clique in the graph $\Gamma$ is denoted by $\chi(\Gamma)$ and the size of the largest independent sets of vertices by $\alpha(\Gamma)$. A graph $\Gamma$ is called a vertex-transitive graph if for any $x, y \in V$ there is some $\pi$ in $\text{Aut}(\Gamma)$, the automorphism group of $\Gamma$, such that $\pi(x) = y$. Let $\Gamma$ be a graph, the complement $\Gamma'$ of $\Gamma$ is the graph whose vertex set is $V(\Gamma)$ and whose edges are the pairs of nonadjacent vertices of $\Gamma$. It is well known that for any graph $\Gamma$, $\text{Aut}(\Gamma') = \text{Aut}(\Gamma)$
[8]. If $\Gamma$ is a connected graph and $\partial(u, v)$ denotes the distance in $\Gamma$ between the vertices $u$ and $v$, then for any automorphism $\pi$ in $\text{Aut}(\Gamma)$ we have $\partial(u, v) = \partial(\pi(u), \pi(v))$.

Let $k$ be a positive integer, a $k$-colouring of a graph $\Gamma$ is a mapping $f : V(\Gamma) \rightarrow \{1, \ldots, k\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices $x$ and $y$ in $\Gamma$, and if such a mapping exists we say that $\Gamma$ is $k$-colorable. The chromatic number $\chi(\Gamma)$ of $\Gamma$ is the minimum number $k$ such that $\Gamma$ is $k$-colorable. Let $\Gamma$ be a graph and $\mathcal{I}(\Gamma)$ denote the set of all independent sets of the graph $\Gamma$. A fractional colouring of a graph $\Gamma$ is a weight function $\mu : \mathcal{I}(\Gamma) \rightarrow [0, 1]$ such that for any vertex $x$ of $\Gamma$, $\sum_{x \in I \in \mathcal{I}(\Gamma)} \mu(I) = 1$, and if such a weight function exists we say that $\Gamma$ is fractional colouring. The fractional chromatic number of a graph $\Gamma$ is denoted by $\chi_f(\Gamma)$ and defined in [9, Page 134]. Also a fractional clique of a graph $\Gamma$ is denoted by $\psi_f(\Gamma)$ and defined in [9, Page 134].

Let $\Upsilon = \{\gamma_1, \ldots, \gamma_{k+1}\}$ be a set and $K$ be a group then we write $\text{Fun}(\Upsilon, K)$ to denote the set of all functions from $\Upsilon$ into $K$, we can turn $\text{Fun}(\Upsilon, K)$ into a group by defining a product:

$$(fg)(\gamma) = f(\gamma)g(\gamma) \text{ for all } f, g \in \text{Fun}(\Upsilon, K) \text{ and } \gamma \in \Upsilon,$$

where the product on the right is in $K$. Since $\Upsilon$ is finite, the group $\text{Fun}(\Upsilon, K)$ is isomorphic to $K^{k+1}$ (a direct product of $k + 1$ copies of $K$) via the isomorphism $f \rightarrow (f(\gamma_1), \ldots, f(\gamma_{k+1}))$. Let $H$ and $K$ be groups and suppose $H$ acts on the nonempty set $\Upsilon$. Then the wreath product of $K$ by $H$ with respect to this action is defined to be the semidirect product $\text{Fun}(\Upsilon, K) \rtimes H$ where $H$ acts on the group $\text{Fun}(\Upsilon, K)$ via

$$f^x(\gamma) = f(\gamma^x) \text{ for all } f \in \text{Fun}(\Upsilon, K), \gamma \in \Upsilon \text{ and } x \in H.$$  

We denote this group by $K\text{wr}_\Upsilon H$. Consider the wreath product $G = K\text{wr}_\Upsilon H$. If $K$ acts on a set $\Delta$ then we can define an action of $G$ on $\Delta \times \Upsilon$ by

$$(\delta, \gamma)^{(f, h)} = (\delta^{f(\gamma)}, \gamma^h) \text{ for all } (\delta, \gamma) \in \Delta \times \Upsilon,$$

where $(f, h) \in \text{Fun}(\Upsilon, K) \rtimes H = K\text{wr}_\Upsilon H$ [6].

Eigenvalues of an undirected graph $\Gamma$ are the eigenvalues of an arbitrary adjacency matrix of $\Gamma$. Harary and Schwenk [10] defined $\Gamma$ to be integral, if all of its eigenvalues are integers. For a survey of integral graphs see [3]. In [2] the number of integral graphs on $n$ vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes, see [1].

Let $G$ be a finite group and $S$ a subset of $G$ that is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = \text{Cay}(G, S)$ is a graph whose vertex-set and edge-set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} | x^{-1}y \in S\}.$$ 

It is well known that every Cayley graph is vertex-transitive.

For any graph $\Gamma$, $\omega(\Gamma) \leq \chi(\Gamma)$ [8]. Also it is well known that for bipartite graphs $\omega(\Gamma) = \chi(\Gamma) = 2$. Let $\Gamma$ be the $\text{Cay}(\mathbb{Z}_n, S_k)$ where $\mathbb{Z}_n (n \geq 4)$, is the cyclic additive group with identity $\{0\}$, and for any $k \in \mathbb{N}$, $1 \leq k \leq \left[\frac{n}{2}\right] - 1$, $S_1 = \{1, n-1\}, \ldots, S_{k-1} = \{k, n-k\}$ are inverse-closed subsets of $\mathbb{Z}_n - \{0\}$. In this paper we will show that $\chi(\Gamma) = \omega(\Gamma) = k + 1$ if and only if $k + 1 | n$, also we show that if $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\text{Aut}(\Gamma) \cong \mathbb{Z}_2\text{wr}_\Gamma\text{Sym}(k+1), \quad$ where $I = \{1, \ldots, k + 1\}$. 

64
2. Definitions and Preliminaries

Proposition 2.1. [11] For any graph $\Gamma$ we have

$$\omega(\Gamma) \leq \omega_f(\Gamma) \leq \chi_f(\Gamma) \leq \chi(\Gamma).$$

Proposition 2.2. [8] If $\Gamma$ is vertex transitive graph, then we have

$$\omega_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha(\Gamma)}.$$ 

Definition 1. [4] Let $\Gamma$ be a graph with automorphism group $\text{Aut}(\Gamma)$. We say that $\Gamma$ is symmetric if, for all vertices $u, v, x, y$ of $\Gamma$ such that $u$ and $v$ are adjacent, also, $x$ and $y$ are adjacent, there is an automorphism $\pi$ such that $\pi(u) = x$ and $\pi(v) = y$. We say that $\Gamma$ is distance-transitive if, for all vertices $u, v, x, y$ of $\Gamma$ such that $\partial(u, v) = \partial(x, y)$, there is an automorphism $\pi$ such that $\pi(u) = x$ and $\pi(v) = y$.

Remark 2.1. [4] Let $\Gamma$ be a graph. It is clear that we have a hierarchy of conditions:

distance-transitive $\Rightarrow$ symmetric $\Rightarrow$ vertex-transitive

Definition 2. [4], [5] For any vertex $v$ of a connected graph $\Gamma$ we define

$$\Gamma_r(v) = \{u \in V(\Gamma) | \partial(u, v) = r\},$$

where $r$ is a non-negative integer not exceeding $d$, the diameter of $\Gamma$. It is clear that $\Gamma_0(v) = \{v\}$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(v), \ldots, \Gamma_d(v)$, for each $v$ in $V(\Gamma)$. The graph $\Gamma$ is called distance-regular with diameter $d$ and intersection array $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$, if it is regular of valency $k$ and for any two vertices $u$ and $v$ in $\Gamma$ at distance $r$ we have $|\Gamma_{r+1}(v) \cap \Gamma_1(u)| = b_r$, and $|\Gamma_{r-1}(v) \cap \Gamma_1(u)| = c_r$ $(0 \leq r \leq d)$. The numbers $c_r, b_r$ and $a_r$, where

$$a_r = k - b_r - c_r \quad (0 \leq r \leq d),$$

is the number of neighbours of $u$ in $\Gamma_r(v)$ for $\partial(u, v) = r$, are called the intersection numbers of $\Gamma$. Clearly $b_0 = k$, $b_d = c_0 = 0$ and $c_1 = 1$.

Remark 2.2. [4] It is clear that if $\Gamma$ is distance-transitive graph then $\Gamma$ is distance-regular.

Lemma 2.1. [4] A connected graph $\Gamma$ with diameter $d$ and automorphism group $G = \text{Aut}(\Gamma)$ is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer $G_v$ is transitive on the set $\Gamma_r(v)$, for each $r \in \{0, 1, \ldots, d\}$, and $v \in V(\Gamma)$.

Theorem 2.1. [5] Let $\Gamma$ be a distance-regular graph which the valency of each vertex as $k$, with diameter $d$, adjacency matrix $A$ and intersection array,

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}.$$ 

Then the tridiagonal $(d + 1) \times (d + 1)$ matrix
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Seyed Morteza Mirafzal, Ali Zafari

\[
J(\Gamma) = \begin{bmatrix}
a_0 & b_0 & 0 & 0 & \cdots \\
c_1 & a_1 & b_1 & 0 & \cdots \\
0 & c_2 & a_2 & b_2 & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
c_{d-2} & a_{d-2} & b_{d-2} & 0 & \\
\cdots & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\
\cdots & 0 & 0 & c_d & a_d
\end{bmatrix},
\]
determines all the eigenvalues of \( \Gamma \).

**Theorem 2.2.** [7] Let \( \Gamma \) be a graph such that contains \( k + 1 \) components \( \Gamma_1, \ldots, \Gamma_{k+1} \). If for any \( i \in I = \{1, \ldots, k+1\} \), \( \Gamma_i \cong \Gamma_1 \) then \( \text{Aut}(\Gamma) \cong \text{Aut}(\Gamma_1) \wr I \text{Sym}(k+1) \).

### 3. Main Results

**Proposition 3.1.** Let \( \Gamma = \text{Cay}(\mathbb{Z}_n, S_k) \) be the Cayley graph on the cyclic group \( \mathbb{Z}_n (n \geq 4) \), where \( S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\} \) are the inverse-closed subsets of \( \mathbb{Z}_n - \{0\} \) for any \( k \in \mathbb{N}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1 \). Then \( \chi(\Gamma) = \omega(\Gamma) = k + 1 \) if and only if \( k + 1 \mid n \).

**Proof.** By definition of \( S_i \), \( 1 \leq i \leq k \) clearly \( |S_i| = 2i \), hence \( |S_k| = 2k \). Let \( \Gamma = \text{Cay}(\mathbb{Z}_n, S_k) \) be the Cayley graph on the cyclic group \( \mathbb{Z}_n \) and \( S_k \) be the set of inverse-closed subset of \( \mathbb{Z}_n - \{0\} \) which is defined as before. By definition of \( \Gamma \) clearly \( \omega(\Gamma) = k + 1 \). So, if \( \chi(\Gamma) = \omega(\Gamma) = k + 1 \) then by Proposition 2.1, \( \chi_f(\Gamma) = \omega_f(\Gamma) = k + 1 \). Also we know that \( \Gamma \) is a vertex transitive graph, so by Proposition 2.2, \( k + 1 = \omega_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha(\Gamma)} \) therefore \( k + 1 \mid n \). Conversely, if \( k + 1 \mid n \) then \( \chi(\Gamma) = k + 1 \), because \( \Gamma \) is a vertex transitive graph and the size of every clique in the graph \( \Gamma \) is \( k + 1 \), therefore \( \chi(\Gamma) = \omega(\Gamma) = k + 1 \).

**Example 1.** Suppose \( \Gamma_1 = \text{Cay}(\mathbb{Z}_{12}, S_2) \) and \( \Gamma_2 = \text{Cay}(\mathbb{Z}_{12}, S_3) \) are two Cayley graphs, then \( \chi(\Gamma_1) = \omega(\Gamma_1) = 3 \) and \( \chi(\Gamma_2) = \omega(\Gamma_2) = 4 \).

![Figure 1](image1.png)

Figure 1: \( \chi(\Gamma_1) = \omega(\Gamma_1) = 3 \)  
![Figure 2](image2.png)

Figure 2: \( \chi(\Gamma_2) = \omega(\Gamma_2) = 4 \)
**Proposition 3.2.** Let $\Gamma = Cay(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 4$), where $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. If $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \wr r_1 \text{Sym}(k+1)$, where $I = \{1, \ldots, k+1\}$.

**Proof.** Let $V(\Gamma) = \{1, \ldots, n\}$ be the vertex set of $\Gamma$. By assumptions and Proposition 2.2, the size of the largest independent set of vertices in the $\Gamma$ is 2, because $\Gamma$ is a vertex transitive graph and the size of every clique in the graph $\Gamma$ is $k+1$. Thus, the size of the every independent set of vertices in the $\Gamma$ is 2. Therefore for any $x \in V(\Gamma)$, there is exactly one $y \in V(\Gamma)$ such that $x^{-1}y = k + 1$. Hence, if $x^{-1}y = k + 1$ then two vertices $x$ and $y$ adjacent in the complement $\overline{\Gamma}$ of $\Gamma$, so $\overline{\Gamma}$ contains $k+1$ components $\Gamma_1, \ldots, \Gamma_{k+1}$ such that for any $i \in I = \{1, \ldots, k+1\}$, $\Gamma_i \cong \Gamma_1 \cong K_2$, where $K_2$ is the complete graph of 2 vertices. Therefore $\overline{\Gamma} \cong (k+1)K_2$, hence by Theorem 2.2, $\text{Aut}(\overline{\Gamma}) \cong \text{Aut}(K_2) \wr r_1 \text{Sym}(k+1) = \mathbb{Z}_2 \wr r_1 \text{Sym}(k+1)$, so $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \wr r_1 \text{Sym}(k+1)$.

**Example 2.** Let $\Gamma = Cay(\mathbb{Z}_{12}, S_5)$ be the Cayley graph on the cyclic group $\mathbb{Z}_{12}$, then $\chi(\Gamma) = \omega(\Gamma) = 6$, and $\text{Aut}(\Gamma) = \mathbb{Z}_2 \wr r_1 \text{Sym}(6)$, where $I = \{1, \ldots, 6\}$.

![Figure 3: $\chi(\Gamma) = \omega(\Gamma) = 6$](image)

**Proposition 3.3.** Let $\Gamma = Cay(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 4$), where $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. If $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\Gamma$ is a distance-transitive graph.

**Proof.** By Lemma 2.1, it is sufficient to show that vertex-stabilizer $G_v$ is transitive on the set $\Gamma_r(v)$ for every $r \in \{0, 1, 2\}$ and $v \in V(\Gamma)$, because of $\Gamma$ is a vertex-transitive graph. We know $V(\Gamma) = \{1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 1, \ldots, n\}$ is the vertex set of $\Gamma$. Let $G = \text{Aut}(\Gamma)$. Consider the vertex $v = 1$ in the $V(\Gamma)$, then $\Gamma_0(v) = \{1\}$, $\Gamma_1(v) = \{2, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n\}$ and $\Gamma_2(v) = \{\frac{n}{2} + 1\}$. Let $\rho = (2, 3, \ldots, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n)$ be the cyclic permutation of the vertex set of $\Gamma$. It is an easy task to show that $\rho$ is an automorphism of $\Gamma$. We can show that $H = \langle (2, 3, \ldots, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n) \rangle$ acts transitively on the set $\Gamma_r(v)$ for each $r \in \{0, 1, 2\}$, because $H$ is a cyclic group. Note that if $1 \neq v \in V(\Gamma)$ then, we can show that vertex-stabilizer $G_v$ is transitive on the set $\Gamma_r(v)$ for each $r \in \{0, 1, 2\}$, because $\Gamma$ is a vertex-transitive graph.
**Proposition 3.4.** Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 4$), where $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. If $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\Gamma$ is an integral graph.

**Proof.** By Remark 2.2, it is clear that $\Gamma$ is distance-regular, because $\Gamma$ is a distance-transitive graph. Let $V(\Gamma) = \{1, 2, \ldots, n\}$ be the vertex set of $\Gamma$. Consider the vertex $v = 1$ in the $V(\Gamma)$, then $\Gamma_0(v) = \{1\}$, $\Gamma_1(v) = \{2, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n\}$ and $\Gamma_2(v) = \{\frac{n}{2} + 1\}$. Let be $u$ in the $V(\Gamma)$ such that $\partial(u, v) = 0$ then $u = v = 1$ and $|\Gamma_1(v) \cap \Gamma_1(u)| = 2k$, hence $b_0 = 2k$ and by Definition 2, $a_0 = 2k - b_0 = 0$. Also, if $u$ in the $V(\Gamma)$ and $\partial(u, v) = 1$ then two vertices $u, v$ are adjacent in $\Gamma$, so $|\Gamma_0(v) \cap \Gamma_1(u)| = 1$ and $|\Gamma_2(v) \cap \Gamma_1(u)| = 1$, hence $c_1 = 1, b_1 = 1$ and $a_1 = 2k - b_1 - c_1 = 2k - 2$. Finally, if $u$ in the $V(\Gamma)$ and $\partial(u, v) = 2$ then two vertices $u, v$ are not adjacent in $\Gamma$, so $|\Gamma_1(v) \cap \Gamma_1(u)| = 2k$, hence $c_2 = 2k$ and $a_2 = 2k - c_2 = 0$. So the intersection array of $\Gamma$ is $\{2k, 1; 1, 2k\}$. Therefore by Theorem 2.1, the tridiagonal $(3) \times (3)$ matrix,

$$
\begin{bmatrix}
 a_0 & b_0 & 0 \\
 c_1 & a_1 & b_1 \\
 0 & c_2 & a_2
\end{bmatrix}
= \begin{bmatrix}
 0 & 2k & 0 \\
 1 & 2k - 2 & 1 \\
 0 & 2k & 0
\end{bmatrix},
$$

determines all the eigenvalues of $\Gamma$. It is clear that all the eigenvalues of $\Gamma$ are $2k, 0, -2$ and their multiplicities are $1, k + 1, k$, respectively. So $\Gamma$ is an integral graph. \hfill \square

**References**


