On an edge partition and root graphs of some classes of line graphs

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Abstract

The Gallai and the anti-Gallai graphs of a graph $G$ are complementary pairs of spanning subgraphs of the line graph of $G$. In this paper we find some structural relations between these graph classes by finding a partition of the edge set of the line graph of a graph $G$ into the edge sets of the Gallai and anti-Gallai graphs of $G$. Based on this, an optimal algorithm to find the root graph of a line graph is obtained. Moreover, root graphs of diameter-maximal, distance-hereditary, Ptolemaic and chordal graphs are also discussed.

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1. Introduction

The line graph $L(G)$ of a graph $G$ has as its vertices the edges of $G$, and any two vertices are adjacent in $L(G)$ if the corresponding edges are incident in $G$. The Gallai graph $\text{Gal}(G)$ \cite{10, 15} of a graph $G$ has as its vertices the edges of $G$, and any two vertices are adjacent in $\text{Gal}(G)$ if the corresponding edges are incident in $G$, but do not span a triangle in $G$. The anti-Gallai graph $\text{antiGal}(G)\cite{13}$ of a graph $G$ has as its vertices the edges of $G$, and any two vertices of $G$ are adjacent in $\text{antiGal}(G)$ if the corresponding edges are incident in $G$ and lie on a triangle in $G$.

In \cite{13} it is shown that the four color theorem can be equivalently stated in terms of anti-Gallai graphs. The problems of determining the clique number and the chromatic number of $\text{Gal}(G)$ are
NP-Complete[13]. In [3] it is shown that there are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs and anti-Gallai graphs. In [2] it is shown that the complexity of recognizing anti-Gallai graphs is NP-complete.

A graph \( H \) is forbidden in a graph family \( \mathcal{G} \), if \( H \) is not an induced subgraph of any \( G \in \mathcal{G} \). For any finite graph \( H \), there exist a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be \( H \)-free [3]. However, both Gallai graphs and anti-Gallai graphs cannot be characterized using forbidden subgraphs [13].

The Gallai and the anti-Gallai graphs are spanning subgraphs of line graphs. In fact, they are complement to each other in \( L(G) \). Therefore a natural question arises: is it possible to identify the edges of \( Gal(G) \) and \( antiGal(G) \) from \( L(G) \)? A positive answer to this is given in this paper by introducing an algorithm to partition the edge set of a line graph into the edges of Gallai and anti-Gallai graphs, using the adjacency properties of common neighbors of the edges of a line graph in a hanging [8].

A graph \( G \) is a root graph of the line graph \( H \) if \( L(G) \cong H \). The root graph of a line graph is unique, except for the triangle and \( K_{1,3} \) [16]. In this paper, using the edge-partition, an algorithm is obtained to find the root graph of a line graph. Also, the root graphs of diameter-maximal, distance-hereditary, Ptolemaic and chordal graphs are obtained.

Let \( H = (V, E) \) be a graph with vertex set \( V = V(H) \) and edge set \( E = E(H) \). Let \( N(v) \) denote the set of all vertices adjacent to \( v \) and \( N_M(v) = M \cap N(v) \), where \( M \subseteq V \). The edge joining \( u \) and \( v \) is denoted by \( uv \). The common neighbors of \( uv \) is \( N(u) \cap N(v) \) and \( N(uv) = N(u) \cup N(v) \). The subgraph induced by \( \{v_1, v_2, ..., v_k\} \subseteq V \) is denoted by \(< v_1, v_2, ..., v_k >\). A clique is a complete subgraph of a graph. An edge clique cover of \( H \) is a family of cliques \( \mathcal{E} = \{q_1, q_2, ..., q_k\} \) such that each edge of \( H \) is in at least one of \( E(q_1), E(q_2), ..., E(q_k) \).

A path on \( n \) vertices \( P_n \) is the graph with vertex set \( \{v_1, v_2, ..., v_n\} \) and \( v_i v_{i+1} \) for \( i = 1, 2, ..., n-1 \) are the only edges. The distance between two vertices \( u \) and \( v \), denoted by \( d(u, v) \), is the length of a shortest \( u-v \) path in \( H \). The diameter of \( H \), denoted by \( d(H) \), is the maximum length of a shortest path in \( H \).

The join of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \lor G_2 \), is the graph with vertex set \( V(G_1) \cup V(G_2) \) and \( E(G_1 \lor G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\} \).

All graphs mentioned in this paper are simple and connected, unless otherwise specified. Also, all other basic concepts and notations not mentioned in this paper are from [4].

2. Adjacency properties of edges of \( L(G) \)

The hanging [8] of a graph \( H = (V, E) \), with \( |V| = n \) and \( |E| = m \), by a vertex \( z \) is the function \( h_z(x) \) that assigns to each vertex \( x \) of \( H \) the value \( d(z, x) \). The \( i \)-th level of \( H \) in a hanging \( h_z \) is defined as \( L_i = \{x \in H : h_z(x) = i\} \). A hanging can be obtained using a breadth first search(BFS) [1], which has a time complexity of \( O(m+n) \).

For a vertex \( v \) in \( L_i \), a supporter of \( v \) is a vertex in \( L_{i-1} \), which is adjacent to \( v \). A vertex in \( L_i \) is an ending vertex if it has no neighbors in \( L_{i+1} \). An arbitrary supporter of \( v \) is denoted by \( S(v) \).

It is clear that any vertex \( v \) in the level \( L_i \) for \( i \geq 1 \) has at least one supporter.

We use the following, well known, forbidden subgraph characterization of a line graph.
Theorem 2.1. [6] A graph $H$ is a line graph if and only if the nine graphs in Fig 1 are forbidden subgraphs for $H$.

Figure 1. Forbidden Subgraphs of line graph

Theorem 2.2. Consider a hanging of a line graph $H$ by an arbitrary vertex in $H$ and let $uv$ denote the edge joining $u$ and $v$ in the same level $L_i$. Then, the following statements hold.

1. All common neighbors of $uv$ in $L_{i-1}$ are adjacent to each other.
2. All common neighbors of $uv$ in $L_{i+1}$ are adjacent to each other.
3. If $uv$ has no common neighbor in $L_{i-1}$, then all the common neighbors of $uv$ in $L_i$ which are adjacent to all other neighbors of $uv$ are adjacent to each other.
4. There is at most one common neighbor of $uv$ in $L_i$, which is adjacent to all the neighbors of $uv$ but not adjacent to the common neighbors of $uv$ in $L_{i-1}$ and $L_i$.

Proof.

1. Let $x$ and $x'$ be two (distinct) common neighbors of an edge $uv$ in $L_{i-1}$, then $i \geq 2$. Assume that $x$ and $x'$ are not adjacent. Now, if $x$ and $x'$ have a common neighbor $w$ in $L_{i-2}$, then
Lemma 3.1. Consider a line graph $G$ have a vertex in common.

Remark 3.1

Let $w, x, x', u, v > \equiv F_2$ in Fig 1 which contradicts the fact that $H$ is a line graph. So, let $w$ and $w'$ be any two vertices in $L_{i-2}$ adjacent to $x$ and $x'$ respectively. Then $<w, w', x, x', u, v > \equiv F_7$ or $F_4$ according as, $w$ and $w'$ are adjacent or not.

2. Let $w$ and $x$ be two common neighbors of an edge $uv$ in $L_{i+1}$. Assume that $x$ and $w$ are not adjacent. Now, if $z$ is a supporter of $u$ in $L_{i-1}$, then $<z, u, w, x > \equiv K_{1,3}$, which is a contradiction.

3. Let $uv$ has no common neighbor in the level $L_{i-1}$ and hence $i \geq 2$. Let $x$ and $w$ be two common neighbors of $uv$ in $L_i$ which are adjacent to all the neighbors of $uv$. Assume that $x$ and $w$ are not adjacent. Now $u$ and $v$ cannot have a common supporter. So let $z_1$ and $z_2$ be two supporters of $u$ and $v$ respectively. Since $z_1$ and $z_2$ are neighbors of $uv$, both $x$ and $w$ are adjacent to them. Now, the vertices $z_1, x, w$ and $S(z_1)$ induce a $K_{1,3}$ which is a contradiction.

4. Assume that $x$ and $w$ are two nonadjacent common neighbors of $uv$ in $L_i$ which are not adjacent to the common neighbors of $uv$ but adjacent to all the other neighbors of $uv$ in $L_{i-1}$ and $L_i$. So, it is clear that $i \geq 2$. Let $z$ be a common neighbor of $uv$ in $L_{i-1}$. Now $u$ must have at least one neighbor in $L_{i-1}$ other than the common neighbors of $uv$ in $L_{i-1}$, for otherwise, the vertices $u, x, w$ and $z$ induce a $K_{1,3}$ which is a contradiction. Similar is the case for the vertex $v$. So let $z_1$ and $z_2$ be two neighbors (not common neighbors) of $u$ and $v$ in $L_{i-1}$ respectively. But, we have, $<S(z_1), z_1, x, w > \equiv K_{1,3}$, which is also a contradiction.

Remark 2.1. In fact the above theorem is applicable to a larger class of graphs than line graphs as only some of the forbidden sub graphs of line graphs are used in the proof.

3. Anti-Gallai triangles in $L(G)$

Let $uvw$ be a triangle in $L(G)$ and let $\bar{u}, \bar{v}$ and $\bar{w}$ be the edges in $G$ representing the vertices $u, v$ and $w$ respectively in $L(G)$. If the edges $\bar{u}, \bar{v}$ and $\bar{w}$ induce a triangle in $G$ then the triangle $uvw$ in $L(G)$ is referred to as an anti-Gallai triangle. All the triangles in $antiGal(G)$ need not be an anti-Gallai triangle and the number of anti-Gallai triangles in $L(G)$ is equal to the number of triangles in $G$. Since each edge of an anti-Gallai graph belongs to some anti-Gallai triangle, the set of all anti-Gallai triangles in $L(G)$ induces $antiGal(G)$.

Remark 3.1. When a triangle $uvw$ in $L(G)$ is not an anti-Gallai triangle, the edges $\bar{u}, \bar{v}$ and $\bar{w}$ in $G$ have a vertex in common.

Lemma 3.1. Consider a line graph $H \not\equiv K_3$. If a triangle $uvw$ in $H$ is an anti-Gallai triangle, then for all $x \in \{u, v, w\}$, one of the following holds.

a) $<u, v, w, x > \equiv K_4 - e$

b) $<u, v, w, x >$ is disconnected.
Theorem 3.1. Consider a line graph $\overline{H}$ and assume that the triangle $uvw$ is an anti-Gallai triangle in $H$. Then the edges $\overline{u}, \overline{v}$ and $\overline{w}$ in $G$ induce a triangle in $G$. Now corresponding to any vertex $x$ in $H$, there is an edge $\overline{x}$ in $G$. If $\overline{x}$ is adjacent to the triangle $\overline{uvw}$, then $\overline{x}$ is adjacent to exactly two of the edges of $\overline{uvw}$ and hence $<u, v, w, x> \cong K_4 - e$ in $H$. If $\overline{x}$ is not adjacent to the triangle $\overline{uvw}$, then $<u, v, w, x>$ is disconnected. □

Lemma 3.2. If a triangle $uvw$ is not an anti-Gallai triangle in a line graph $H \cong L(G)$, then there is at most one common neighbor $z$ for an edge of $uvw$ in $H$ such that $<u, v, w, z> \cong K_4 - e$.

Proof. Let $\overline{u}, \overline{v}$ and $\overline{w}$ be the edges in $G$, representing the vertices $u, v$ and $w$ respectively in $H$. Let $z$ be such that $<u, v, w, z> \cong K_4 - e$ in $L(G)$ and let it be a common neighbor of $uv$. Then the edge $\overline{z}$ in $G$ is adjacent to both the edges $\overline{u}$ and $\overline{v}$ and not adjacent to $\overline{w}$. Clearly $\overline{u}, \overline{v}$ and $\overline{w}$ induce a triangle in $G$ and hence $uvz$ is an anti-Gallai triangle in $L(G)$. Now assume that $z'$ is a vertex different from $z$ such that it is a common neighbor of $uv$ and $<u, v, w, z'> \cong K_4 - e$. Then the vertices $z$ and $z'$ cannot be adjacent, otherwise $<u, v, w, z'> \cong K_4$ and by Lemma 3.1 it will contradict the fact that $uvz$ is an anti-Gallai triangle. But, we have, $<u, v, w, z'> \cong K_{1,3}$ and hence $H$ cannot be a line graph by Theorem 2.1. □

Theorem 3.1. Consider a line graph $H \cong K_3, K_4 - e, C_4 \lor K_1$ and $C_4 \lor 2K_1$. A triangle $uvw$ in $H$ is an anti-Gallai triangle if and only if $<u, v, w, x> \cong K_4 - e$ or disconnected for all $x \in V(H) \setminus \{u, v, w\}$.

Proof. Let $G$ be the graph such that $L(G) \cong H$. The necessary part of the theorem follows from Lemma 3.1.

Conversely, assume that $uvw$ is a triangle in $H$ such that $<u, v, w, x> \cong K_4 - e$ or disconnected for all $x \in V(H)$ and that $uvw$ is not an anti-Gallai triangle. Then the edges $\overline{u}, \overline{v}$ and $\overline{w}$ induce a $K_{1,3}$ in $G$. Note that any vertex which induces a $K_4 - e$ with the triangle $uvw$ is adjacent to exactly two vertices among $u, v$ and $w$. Now, since $H$ is connected and not a $K_3$, there is a vertex $x$ adjacent to the triangle $uvw$. Assume that $x$ is adjacent to $u$ and $w$. Then in $G$, $\overline{u}, \overline{v}$ and $\overline{x}$ induce a triangle so that $uwx$ is an anti-Gallai triangle. Since $H \cong K_4 - e$ and also connected, there is a vertex $y$ adjacent to at least one of the vertices $u, v$ and $w$. If there is no vertex adjacent to the triangle $uvw$, then it must be adjacent to $x$ alone, which is a contradiction to the fact that $uwx$ is anti-Gallai triangle. So let $y$ be adjacent to $uvw$. By Lemma 3.2 $y$ cannot be adjacent to $u$ and $w$. So let $y$ be adjacent to $v$ and $w$. Now we have $vwy$ is also an anti-Gallai triangle. But, since $H \cong C_4 \lor K_1$ and connected, using the same arguments as before, we have a vertex $z$ adjacent to the triangle $uvw$ again. The only possibility then is that $z$ is adjacent to the vertices $u$ and $v$. Now we show that there are no more vertices possible in $H$. If not, let $p$ be a vertex in $H$ different from $u, v, w, x, y$ and $z$. But, by Lemma 3.2, the vertex $p$ cannot be adjacent to $uvw$. Now if $p$ is adjacent to $x$, it must be adjacent to $u$ or $w$ as $uwx$ is an anti-Gallai triangle, which again is not possible. Similarly, $p$ cannot be adjacent to $y$ and $z$. Hence no such vertex $p$ can be adjacent to any of the vertices $u, v, w, x, y$ and $z$. So such a vertex does not exist in $H$, as $H$ is a connected graph. Now we have $H \cong <u, v, w, x, y, z> \cong C_4 \lor 2K_1$, which is a contradiction. □

We observe that it is possible to suitably re-label the edges in the root graph of $C_4 \lor K_1$ so that no triangles in $C_4 \lor K_1$ can be claimed to be an anti-Gallai triangle, see Figure 2. It can be seen
that $K_4 - e$ and $C_4 \lor 2K_1$ also have this property. Theorem 3.1 shows that these three graphs are the only exceptions (the graph $K_3$ is excluded as it is a trivial case with 3 vertices). Hence, the graphs $K_4 - e$, $C_4 \lor K_1$ and $C_4 \lor 2K_1$ are excluded in the following discussions.

**Definition 1.** A triangle in a hanging of a line graph is an $L\triangle (M\triangle, R\triangle)$ if it is an anti-Gallai triangle and it is induced by two vertices in one level and one vertex from the lower (same, higher) level of the ordering.

We can see that any anti-Gallai triangle is either an $L\triangle$, $M\triangle$ or $R\triangle$ in a hanging of $L(G)$

**Theorem 3.2.** Let $uv$ be an edge in any level of a hanging of $H \cong L(G)$ by an arbitrary vertex in $H$, then

1. $uv$ cannot be an edge of an $L\triangle$ in any level $L_i$ for $i > 1$.
2. $uv$ cannot be an edge of an $M\triangle$ in $L_1$.
3. If $uv$ is an edge in an $M\triangle$ then $uv$ cannot be an edge of an $L\triangle$.
4. If $uv$ is an edge in an $M\triangle$ then $uv$ cannot be an edge of an $R\triangle$.
5. If $uv$ is an edge in an $L\triangle$ then $uv$ cannot be an edge of an $R\triangle$.
6. $uv$ can be an edge of at most one $L\triangle$ or $R\triangle$ or $M\triangle$.

**Proof.**
1. Let $uv$ be an edge in an $L_i$ for $i > 1$ and let it belong to an $L\triangle uvx$, where $x \in L_{i-1}$. Let $w$ be the vertex in $L_{i-2}$ which is adjacent to $x$. Then $<w, x, u, v>$ induces a subgraph which is neither a $K_4 - e$ nor disconnected, which is a contradiction.

2. Let $uvx$ be an $M\triangle$ in $L_1$ and $z$ be the vertex, from where the hanging of $H$ being considered. Then $d(z) \geq 3$ and $<z, x, u, v>$ induce a $K_4$ and hence $uvx$ cannot be an anti-Gallai triangle, which is a contradiction.

3. Let $uv$ be an edge in $L\triangle$ then $uv$ is in $L_1$ by (1) and hence $uv$ cannot be an edge of an $M\triangle$ by (2).

From (3) and Theorem 3.1, it follows that anti-Gallai triangles of a graph cannot share an edge in a line graph. Hence the proof of (4) to (6) follows.

Now, Lemma 3.3 follows.

**Lemma 3.3.** Exactly one triangle of a $K_4 - e$ in a line graph is an anti-Gallai triangle.

From Theorems 2.2 and 3.1, we have the following propositions.

**Proposition 3.1.** The edge $uv$ is in an $L\triangle$, with both its ends in the same level of a hanging of a line graph if and only if it satisfies the following conditions

1. Each vertex in $L_1$ is either adjacent to $u$ or $v$ but not to both.

2. Each neighbor of $uv$ in $L_2$ is a common neighbor of $uv$.

**Proposition 3.2.** The edge $uv$ is in an $M\triangle$ in a hanging of a line graph if and only if it satisfies the following conditions

1. The edge $uv$ has a common neighbor $x$ in $L_i$ which is not adjacent to the other common neighbors of $uv$ in $L_{i-1}$ and $L_i$.

2. Either $u$ or $v$ is adjacent to each neighbor of $x$.

3. Each non neighbor of $x$ is either a common neighbor of $uv$ or not a neighbor of $uv$.

**Proposition 3.3.** The edge $uv$ is in an $R\triangle$ with both its ends in the $i^{th}$ level of a hanging of a line graph if and only if it satisfies the following conditions

1. The edge $uv$ has exactly one common neighbor $x$ in $L_{i+1}$.

2. The vertex $x$ is an ending vertex.

3. Either $u$ or $v$ is adjacent to each neighbor of $x$.

4. Each non neighbor of $x$ in $L_{i-1} \cup L_i$ is either a common neighbor of $uv$ or not a neighbor of $uv$. 

123
4. Partitioning the edges of a line graph

We now provide an algorithm to partition the edge set of a line graph into edge sets of its Gallai and anti-Gallai graphs. The following three tests checks whether an edge $uv \in L_i$ belongs to an $L\triangle$, $M\triangle$ or $R\triangle$.

**Algorithm 1.** $L\triangle$ test

1. If $i \neq 1$ go to step 7.
2. Find $N(u)$ and $N(v)$.
3. If $N_{L_i}(u) \cup N_{L_i}(v) \neq L_i$ then go to step 7.
4. If $N_{L_i}(u) \cap N_{L_i}(v) \neq \emptyset$ then go to step 7.
5. If $N_{L_{i+1}}(u) \neq N_{L_{i+1}}(v)$ then go to step 7.
6. Triangle $uvz$ is an $L\triangle$.
7. The edge $uv$ is not in $L\triangle$.

**Algorithm 2.** $M\triangle$ test

1. If $i = 1$ go to step 9.
2. Find the set $C$ of common neighbors $w_j$ of $uv$ in $L_i$. If $C = \emptyset$, go to step 9.
3. Find the set $B$ of common neighbors $x_j$ of $uv$ in $L_{i-1}$ and $L_{i+1}$.
4. For each $x_j \in B$, delete the members of the set $N_C(x_j)$ from C. If $C = \emptyset$ go to step 9.
5. For each $w_j$, if $|N_C[w_j]| > 1$, delete the members of the set $N_C[w_j]$. If $|C| \neq 1$ go to step 9.
6. Find the set $N(uv)$ in $H$.
7. If $|N_C(y_j)| = 1$, for each $y_j \in N(uv) \setminus (B \cup C)$, go to step 8. Else go to step 9.
8. Triangle $uvx$ is an $M\triangle$.
9. The edge $uv$ is not in $M\triangle$.

**Algorithm 3.** $R\triangle$ test

1. Find the set $C_R$ of common neighbors of $uv$ in $L_{i+1}$.
2. If $|C_R| \neq 1$ go to step 7. Else choose the common neighbor of $uv$ in $L_{i+1}$ as $x$.
3. If the vertex $x$ is not an ending vertex, go to step 7.
4. Either \( u \) or \( v \) is adjacent to each neighbor of \( x \). Else go to step 7.

5. Each non neighbor of \( x \) is either a common neighbor of \( uv \) or not a neighbor of \( uv \). Else go to step 7.

6. Triangle \( uvx \) is an \( R\Delta \).

7. The edge \( uv \) is not in \( R\Delta \).

Given a line graph \( H \cong L(G) \), obtain a hanging \( h_z \) by an arbitrary vertex \( z \). Consider all the edges starting from a vertex \( u \) in \( L_1 \). For each edge of the form \( uv \) for some \( v \in L_1 \), apply tests 1, 2 and 3 one by one. Choose another edge whenever an anti-Gallai triangle is found or when all the tests fail. When all the edges in a level are considered, go to the next level and repeat the procedure. This algorithm ends when all the edges in the last level of the hanging are considered and uses a time complexity of \( O(m) \).

We now observe that in a line graph \( L(G) \), any edge that is in the edge set of \( antiGal(G) \) belongs to some anti-Gallai triangle. Hence the set of all the edges of the anti-Gallai triangles gives the edge set of \( antiGal(G) \) and the remaining edges of the \( L(G) \) corresponds to the edge set of \( Gal(G) \).

5. An algorithm to find the root graph of a line graph

An optimal algorithm to recognize a line graph and output its root graph can be seen in [14], the time complexity of which is \( O(n) + m \). Using the above edge partition, an algorithm, which uses a time complexity of \( O(m) + O(n) \), is provided to find the root graph of a line graph \( H \). The same algorithm can be used as a recognition algorithm for line graphs. For this, applying the above three tests for the edges in an arbitrary graph, we call a triangle type I if it belongs to the category of anti-Gallai triangles and type II otherwise.

Algorithm 4. Root graph of a line graph

Consider a connected graph \( H = (V, E) \) with \( |V| = n, |E| = m \) and its hanging \( h_z \), by an arbitrary vertex \( z \).

Let \( M = \{z, u\} \), where \( u \) is a neighbor of \( z \). Let \( G \) be a path on three vertices with \( V(G) = \{\{z\}, \{z, u\}, \{u\}\} \) and \( E(G) = \{(\{z\}, \{z, u\}), (\{z, u\}, \{u\})\} \). Here the labels of vertices of \( G \) are represented as sets which can be re-labeled, in the steps of the following algorithm, using set operations.

1. Choose a vertex \( v \) from \( V(H) \setminus M \) with \( N_M(v) \neq \emptyset \).

2. If \( v \) induces a clique in \( N_M(v) \) and does not induce a type I triangle go to step 3. Else go to step 4.

3. Make \( V(G) = V(G) \cup \{v\} \), and join \( \{v\} \) with a vertex \( C \in V(G) \), where \( C = N_M(v) \), and make \( M = M \cup \{v\} \) and \( C = C \cup \{v\} \). If no such vertex \( C \) exists, go to step 4.
4. Find two vertices \( A \) and \( B \) in \( V(G) \) such that \( A \cup B = N_{M}(v) \) and make \( M = M \cup \{v\} \), \( A = A \cup \{v\} \) and \( B = B \cup \{v\} \). Go to step 1.

The algorithm ends whenever \( M = V(H) \) or there does not exist \( C \) or \( A \) and \( B \) as required. Here the graph \( G \) represents the root graph of the line graph \( H \) and in the latter case it can be concluded that the graph \( H \) is not a line graph of any graph.

The correctness of the algorithm can be verified with the help of the following theorem due to Krausz [12].

**Theorem 5.1.** A graph \( H \) is a line graph if and only if it has an edge clique cover \( E \) such that both the following conditions hold:

1. Every vertex of \( H \) is in exactly two members of \( E \).
2. Every edge of \( H \) is in exactly one member of \( E \).

Since the vertex labels of \( G \) are represented as sets, a vertex in \( <M> \) is an element of some vertex label(set), of \( G \). Here the elements of each vertex label in \( V(G) \) induce a clique in \( <M> \) of \( H \), since \( x, y \) are in a vertex label of \( G \) if and only if \( x \) and \( y \) are adjacent in \( <M> \) of \( H \). Now from the construction of \( G \), each vertex of \( <M> \) is an element of exactly two vertex labels of \( G \) and also any adjacent vertices in \( <M> \) belong to a vertex label of \( G \). Now \( V(G) \) gives an edge clique cover of \( <M> \) which satisfies the two conditions given in Krausz’s theorem. Hence the algorithm obtains a graph \( G \) with \( L(G) \cong H \) if and only if \( M = V(H) \).

We now provide the difference between our algorithm and the algorithm in [14].

Given a graph \( H \), the algorithm in [14] assumes that \( H \) is a line graph and defines a graph \( G \) such that \( H \) is necessarily the line graph of \( G \). A comparison of \( L(G) \) and \( H \) is then made to check whether the given graph is actually a line graph. The algorithm starts with two adjacent basic nodes, labeled 1-2 and 2-3, and labels the vertices in \( H \), on the go, depending on their adjacency. The algorithm proceeds to determine all connections in \( G \) corresponding to a clique, containing the basic nodes in \( H \), simultaneously finding an anti-Gallai triangle \( \{1-2, 2-3, 1-3\} \), if it exists. In each step, the cliques sharing the vertices, which are already worked out, are considered and the algorithm finally outputs a labeled graph \( G \).

In our algorithm, the types of triangles are found using the first three algorithms, the time complexity of which is calculated as follows. We can see that a hanging of the graph \( H \) can be obtained in \( O(m + n) \) steps. In each of the algorithms 1, 2 and 3 only a subset of \( E(H) \) are considered (as edges between the levels are not included) and the algorithm 4, which assumes that algorithms 1, 2 and 3 are already done, finishes in \( O(n) \) steps. Hence using these algorithms the root graph of a line graph can be obtained in \( O(m) + O(n) \) steps. It can be noted, as a consequence of Theorem 3.1, that irrespective of the starting set \( M \) of nodes, any pre-labeled line graph \( H \) with more than four vertices gives a uniquely labeled root graph \( G \).

6. **Root graphs of diameter-maximal line graphs**

A graph \( G \) is diameter-maximal [7], if for any edge \( e \in E(G) \), \( d(G + e) < d(G) \).
Theorem 6.1. [7] A connected graph $G$ is diameter-maximal if and only if

1. $G$ has a unique pair of vertices $u$ and $v$ such that $d(u, v) = d(G)$.
2. The set of nodes at distance $k$ from $u$ induce a complete sub graph.
3. Every node at distance $k$ from $u$ is adjacent to every node at distance $k + 1$ from $u$.

Lemma 6.1. Let $G$ be a diameter-maximal line graph and $u, v$ be two vertices of $G$ with $d(u, v) = d(G)$. Let $L^* = (|L_0|, |L_1|, \ldots, |L_d|)$ be the sequence generated from the hanging $h_u$. Then, $|L_i| \leq 2$ for $i = 0, 1, \ldots, d$.

Proof. Clearly $|L_0| = |L_d| = 1$ in $L^*$. If possible, let $u, v$ and $w$ be three vertices in $L_i$ for some $i$ for $0 < i < d$. By Theorem 6.1, $<u, v, w> \simeq K_3$ and there exist vertices $x$ in $L_{i-1}$ and $y$ in $L_{i+1}$ such that $u, v$ and $w$ are adjacent to both $x$ and $y$. But, then, $<x, u, v, w, y> \simeq F_3$ which is a contradiction.

A sequence $S$ is forbidden in $L^*$ if the consecutive terms of $S$ do not appear consecutively in $L^*$.

Theorem 6.2. For every $d \geq 3$, there exists three diameter-maximal line graphs with diameter $d$.

Proof. First, we show that the sequence $(a_1, a_2, 2, a_3, a_4)$, where $a_i \in \{1, 2\}$, is forbidden in $L^*$. For, assuming the contrary, let $|L_i| = 2$ for some $i$, $2 \leq i \leq d - 2$, and $L_i = \{v_1, v_2\}$. Let $v_3, v_4, v_5$ and $v_6$ be arbitrary vertices in $L_j$, for $j = i - 2, i - 1, i + 1$ and $i + 2$ respectively. But $<v_1, \ldots, v_6> \simeq F_4$ which is a contradiction.

Applying the same argument, we see that the sequences $(a_1, a_2, 2, 2, a_3, a_4)$ and $(2, 2, a_1, a_2)$ and $(2, 2, 2)$ are also forbidden in $L^*$, so that the integer 2 appears at most twice in $L^*$ and hence either $|L_1| = |L_{d-1}| = 2$, $(ii) |L_1| = 2$ or $(iii)$ all the entries of $L^*$ are 1. Note that the case when $L^*$ has $|L_{d-1}| = 2$ is not considered, as it is similar to $(ii)$. Hence there are only three possible sequences of $L^*$ when $d \geq 3$. As the three sequences are different and the pair $(u, v)$ in Theorem 6.1 is unique, there exist exactly three diameter-maximal line graphs.

Corollary 6.1. The root graphs of diameter-maximal line graphs with diameter $d$ are of the form $G$ in Table 1.

<table>
<thead>
<tr>
<th>Diameter of $L(G)$</th>
<th>$d = 1$</th>
<th>$d = 2$</th>
<th>$d \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Table 1. Graph $G$, for Corollary 6.1
7. Root graphs of DHL graphs

A graph $G$ is distance-hereditary if for any connected induced subgraph $H$, $d_H(u, v) = d_G(u, v)$, for any $u, v \in V(H)$. A detailed study can be seen in [5]. A graph $G$ is chordal if every cycle of length at least four in $G$ has an edge(chord) joining two non-adjacent vertices of the cycle [4]. A graph is Ptolemaic if it is both distance-hereditary and chordal [11].

In this section, the family of root graphs of distance-hereditary line (DHL) graphs is obtained. The root graphs of chordal and Ptolemaic graphs are also discussed.

**Theorem 7.1.** [5] Let $G$ be a connected graph. Then $G$ is distance-hereditary if and only if the graphs of Fig 4 and the cycles $C_n$ with $n \geq 5$ are forbidden subgraphs of $G$.

![Figure 4. The graphs for Theorem 7.1: house, domino and gem graphs](image)

**Theorem 7.2.** [11] Let $G$ be a graph. The following conditions are equivalent

1. $G$ is a Ptolemaic graph
2. $G$ is distance-hereditary and chordal
3. $G$ is chordal and does not contain an induced gem

A vertex $v$ is simplicial if $N(v)$ is a clique. The ordering $\{v_1, \ldots, v_n\}$ of the vertices of $H$ is a perfect elimination ordering if, for all $i \in \{1, \ldots, n\}$, the vertex $v_i$ is simplicial in $H_i = < v_1, \ldots, v_n >$.

**Theorem 7.3.** [9] Let $G$ be a graph. The following statements are equivalent:

1. $G$ is a chordal graph.
2. $G$ has a perfect elimination ordering. Moreover, any simplicial vertex can start a perfect elimination ordering.

**Theorem 7.4.** In a DHL graph if a vertex is adjacent to at least one vertex in a $C_4$ then it must be adjacent to all the vertices of that $C_4$ and to no other vertices in the graph.

**Proof.** Let $H$ be a DHL graph which contains a $C_4$ and let a vertex $u$ be adjacent to at least one vertex of the $C_4$. If $u$ is adjacent to exactly one vertex of $C_4$ then a $K_{1,3}$ is formed in $H$, which is a contradiction. Let $u$ be adjacent to exactly two vertices of $C_4$. Then either a house, when $u$ is adjacent to two adjacent vertices of $C_4$, or a $K_{1,3}$, when $u$ adjacent to two non-adjacent vertices of $C_4$. Therefore, $u$ must be adjacent to all vertices of $C_4$. 

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$C_4$ is formed, which is also a contradiction. Since an $F_2$ is obtained when $u$ is adjacent to three vertices of a $C_4$, $u$ must be adjacent to all the four vertices of the $C_4$.

Next we show that two adjacent vertices can not be made adjacent to a $C_4$ in $H$. For, otherwise each of the two vertices must be adjacent to all the vertices of $C_4$ and hence induces $C_4 \lor K_2$. But a copy of $F_3$ is induced in $C_4 \lor K_2$, which is a contradiction. If only one vertex of two adjacent vertices is adjacent to $C_4$, a $K_{1,3}$ is induced in $H$ which is also a contradiction.

**Corollary 7.1.** A DHL graph contains at most one $C_4$.

**Corollary 7.2.** The root graphs of DHL graphs which contain a $C_4$ are $K_4$, $K_4 - e$ and $C_4$.

**Proof.** The proof is complete as we see from Corollary 7.1 that the only DHL graphs which contain a $C_4$ are $C_4 \lor 2K_1$, $C_4 \lor K_1$ and itself.

As there are only three DHL graphs containing a $C_4$, we restrict our discussion in the following sections to DHL graphs not containing $C_4$’s.

If $H$ is a DHL graph containing no anti-Gallai triangle then its root graph contains no triangles. Also, a DHL graph is $C_n$-free, $n \geq 5$. Now, together with Corollary 7.2, we have the following result.

**Theorem 7.5.** Let $H \not\cong C_4$ be a DHL graph not containing an anti-Gallai triangle, then $H$ is a line graph of a tree.

**Lemma 7.1.** An anti-Gallai triangle in a DHL graph has a vertex of degree two.

**Proof.** Let $uvx$ be an anti-Gallai triangle in a DHL graph $H \not\cong K_3$. Then $uvx$ is in some $K_4 - e$ in $H$. Let $uvy$ be a triangle such that $u, x, y, w \cong K_4 - e$. We now show that degree of the vertex $x$ is two. Consider $h_x$; we just need to show that $L_1$ contains no vertices other than $u$ and $v$. For, let $w$ be a vertex in $L_1$. Then $wx$ is an edge and, by Theorem 3.1, either $u$ or $v$ is adjacent to $w$. Then $y$ cannot be adjacent to $w$ as $N(w) \cap \{u, v, x, y\}$ together with $w$ induce $C_4 \lor K_1$. But, $<u, v, w, x, y>$ is a gem, a contradiction.

By lemma 7.1, it now follows that each triangle in the root graph of a DHL graph is attached to the graph by sharing at the most one vertex. Let $T$ be the family of trees. Let $T_\Delta$ be the family of graphs obtained by attaching some triangles to some vertices in a tree $T$, for each $T \in T$.

**Theorem 7.6.** A graph $G$ is a root graph of a $C_4$-free DHL graph if and only if $G \in T_\Delta$.

**Proof.** The proof is by induction on the number of edges in a $T \in T_\Delta$. It can be verified that the root graphs of distance-hereditary graphs of size $\leq 3$ are in $T_\Delta$ and hence the theorem is true for all $m \leq 3$.

Let $T \in T_\Delta$ has $m$ edges and $T$ is a root graph of a DHL graph. Let $T'$ be a graph in $T_\Delta$ with $E(T') = E(T) \cup \{e\}$. Since $T'$ must be connected, there can be two cases: either (i) the edge $e$ is added as a pendant edge to $T$ or (ii) the edge $e$ is formed by joining two vertices in $T$.

Let $l_e$ be the vertex in $L(T')$ corresponding to the edge $e$ in $T'$. In case(i), since $e$ is a pendant edge in $T'$, $l_e$ is simplicial in $L(T')$. We can now show that $L(T')$ is gem-free. If possible let a gem
is there in $L(T')$. Since $L(T)$ is distance-hereditary and $C_4$-free, it is chordal. By Theorem 7.2 $L(T)$ is gem-free, $l_e$ must be a vertex in the induced gem. But, $N(l_e)$ is complete so that $l_e$ is one of the degree two vertices in the gem. Now $l_e$ is in a $K_4 - e$. By Lemma 7.1, one of the two triangles in the $K_4 - e$ must be an anti-Gallai triangle. But the triangle containing $l_e$ cannot be so, as $e$ is a pendant edge in $T'$. But the other triangle has no vertex of degree 2 in the induced gem. This is a contradiction, by Lemma 7.1, to the assumption that $L(T')$ contains a gem.

In case(ii), as $T$ is connected, adding an edge $e$ joining two vertices of $T$ makes a cycle in $T'$. But $T \in T_\Delta$ is $C_n$-free, $n \geq 4$, and contains no $K_4 - e$. Hence $e$ joins two pendant vertices of $T$, forming a triangle and has end vertices of degree two. Therefore in $L(T')$, the corresponding vertex $l_e$ is in an anti-Gallai triangle and has degree two. It now follows that $l_e$ is simplicial. If $L(T')$ contains a gem, $l_e$ must be one of the degree two vertices in the induced gem. But in this case the anti-Gallai triangle containing $l_e$ do not satisfy Theorem 3.1 with the other vertex of degree two in the induced gem, which is again a contradiction.

In both the cases we have a one-vertex extension $L(T')$ of a gem-free chordal graph $L(T)$ and hence $L(T')$ is a DHL graph.

**Corollary 7.3.** A graph $L(G)$ is Ptolemaic if and only if $G \in T_\Delta$

**Corollary 7.4.** Let $T_\Delta^c$ be the family of graphs obtained by attaching some triangles to some vertices in a tree $T$ and identifying each edge of $T$ by an edge of at most one triangle, for each $T \in T$. Then $L(G)$ is a chordal graph if and only if $G \in T_\Delta^c$

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**References**


